Curves And Surfaces.
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Module-III.
Surfaces-2: First Fundamental Form.
Lecture-15.
Euler's Theorem.

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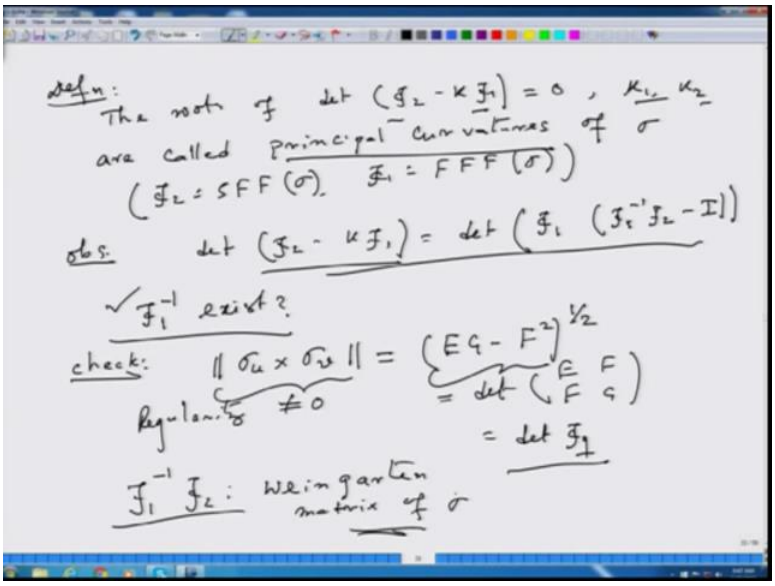

Okay, let us continue where we stopped in the last lecture. So, we have defined principal vectors and principal curvature. So, remember K1 and K2, these were the, yes, roots, K1 and K2, they are called the principal curvatures of the this matrix, the roots of this matrix, K1 and K2 and this is between the Eigen values of Weingarten matrix, that is what we derived.

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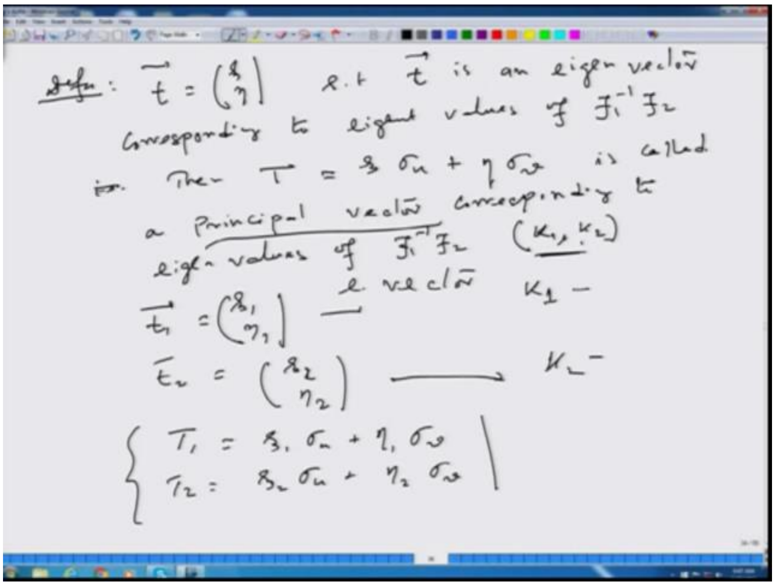

And with that we derived that we have defined principal vector. So, what we do with these quantities, that is what I want to, what to do one more proposition today and then you will see the geometric interpretation of this of all these quantities.

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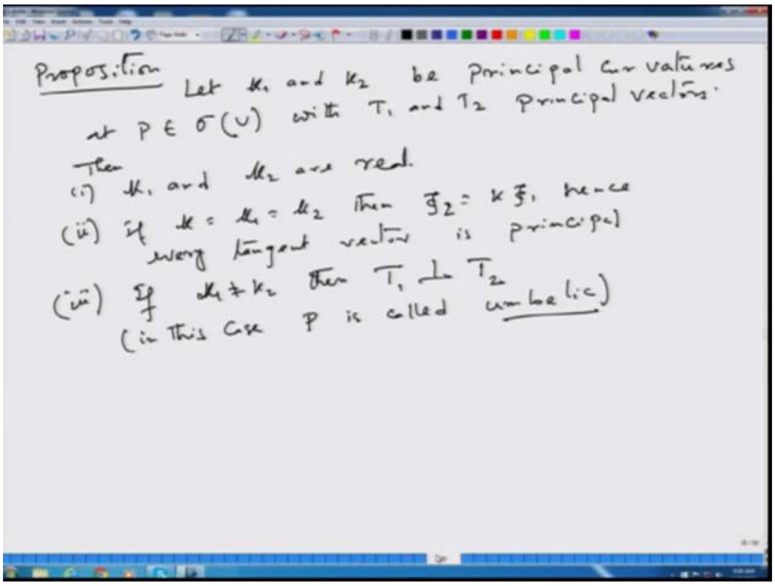

So, let me write proposition 1st. So, let K1 and K2 be principal curvature at P of the surface, regular surface patch Sigma with T1 and T2 principal vectors are then… 1st, what are they, K1 and K2, principal curvature means they are root of the matrix F1 inverse F2. That is a 2 cross 2 matrix, but no one nobody guarantees me that K1 and K2 will be real. So, 1st part of the proposition is that that K1 and K2 are real, it can be complex, right. And 2nd, if K1 and K2 are same, that is this only, K1 and K2 is same, then F2 equal to K F1 and hence every tangent vector is principal, because F1 - K F2 is 0.

And 3rd, what happens is that if they are not equal, then T1 is actually perpendicular to T2 and in this case, in this case, in 3rd case is called, this P is called umbelic, when we will come to the geometric part we will see what is this umbelic word means. Okay. So, I want to prove this proposition today. Once again, this is the proposition, so let us start with the 1st part. Proof is little, it is just some calculations, okay, just follow the calculations.

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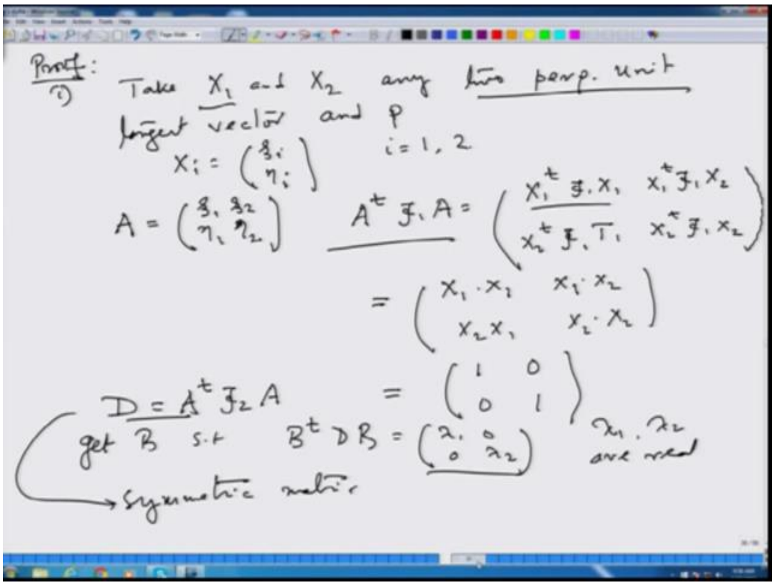

I will go I am going slow. So, 1st part, K1 and K2 real, okay. So, I have let us take any 2 perpendicular unit tangent vector at P, this is tangent space is two-dimensional, so, I will have 2 perpendicular vectors sparing that. Let us take XI equal to zhi I eta I, I equal to 1 and 2. So, let us form this matrix A. Zhi 1 eta 1 Zhi 2 eta 2, then you observe A transpose F1 A, let us write what it is, as this will be X1 transpose F1 X1, X1 transpose F1 X2, X2 transpose F1 T1 and X2 transpose F1 X2, simple calculations, okay. Very good. But let us go back, what is this quantity? We had an exercise, if I have 2 tangent vectors, X and Y, any 2, and T1 dot T2, T so this will be T1 dot T1. So, we use that exercise, you will get this is X1 transpose dot X1, X1 dot X2, X1 dot X1, X1 dot X2, X2 dot X1, X2 dot X2.

Okay. What is it? X1 is a unit vector, so X1 dot X1 is 1. X1 and X2 are perpendicular unit vectors, so this is 0, so this is 0, this is one. So, this is identity matrix. Okay. Now let us take D equal to A transpose F to A. Okay, so D is diagonal, so get B such that B transpose D B equal to lambda 10 lambda 2. What I have done? So look, T is a self adjust symmetric matrix, right, this is symmetric matrix 2 cross 2. So, I can diagonalize it because symmetric matrix, then I will have Eigen values real and I can get some matrix B, so this is a spectral theorem I have used, so there can be, you will get some 2 cross 2 matrix as in B transpose D B is actually diagonal lambda 1 lambda 2 and lambda 1 lambda 2 are real for this matrix because this is a symmetric matrix.

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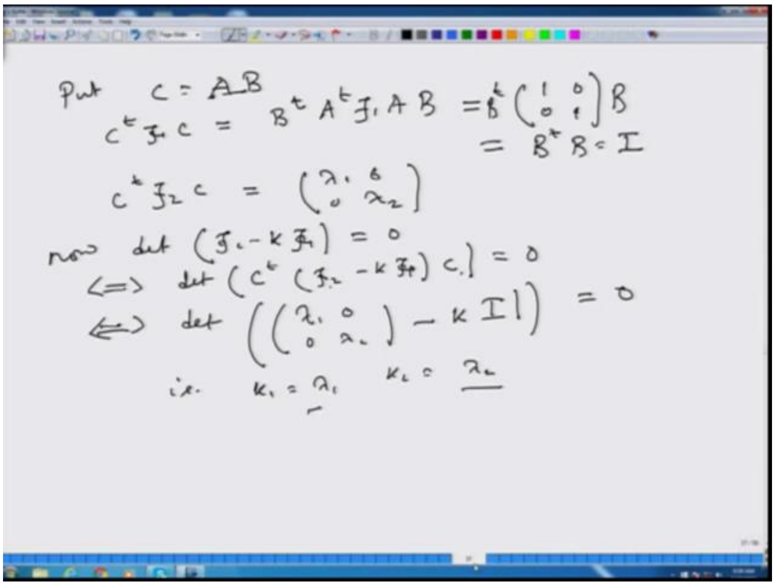

Then put C equal to A B, so there is some trick calculations here. Then observe C transpose F1 C which is B transpose A transpose F1 F B which is equal to again B transpose identity 0 identity B, but then B transpose B is identity, because that is the way we get such spectral theorem. And similarly C transpose F2 C, this will give lambda 1 lambda 2 are real. Now, determinant of F2 - K F1 equal to 0, I have to see the roots, K1 and K2 are roots, so I have 2 shows they are real. This is if and only if determinant C transpose F2 - K F1 C, this is 0. Okay.

Why is so, why is C invert… Why is so because you look at A and B, A is invertible and B is invertible and C is AB, so I just put some invertible matrix on left and right, C transpose see, so this is determinant, this determinant will be is 0 if and only if this determinant is 0. But that is equal to, if and only if, determinant of, now we use C2 F2 C, this is this, lambda 10, 0 lambda 2 and C transpose, K comes out K, C transpose F1 C, C transpose F1 C is identity, this is 0. That this K1 is to be lambda 1, K2 has to be lambda 2. But lambda 1 and lambda 2 are already real because they are Eigen values of asymmetric matrix.

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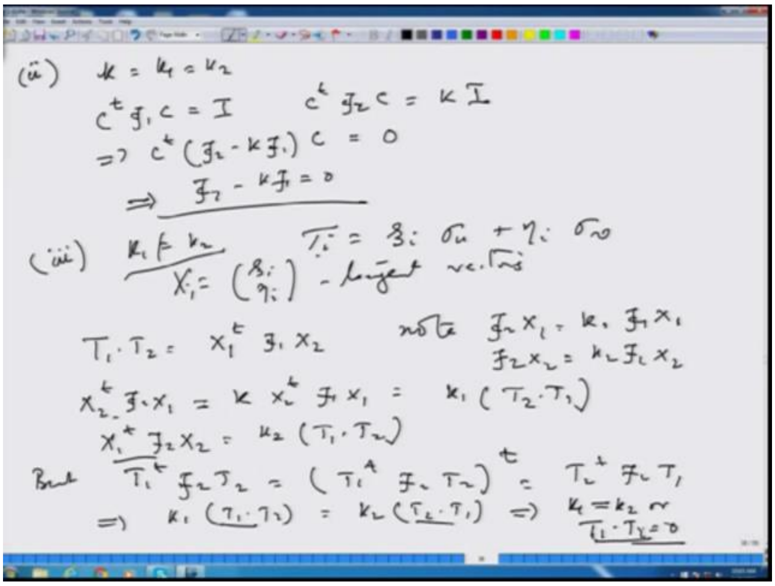

So, this is the 1st part. Let us go to the 2nd part. What was it, that if K equal to K1 equal to K2, then the principal vectors T1 and T2 are same, right. T1 and T2 are, sorry, then F2 equal to K into F1, any tangent vector is principal. But then what happens if this is true? Again C 2 transpose, C is identity, C transpose F2 C is KI. This will give C transpose F2 - K F1 C, this is equal to 0. But C, C is invertible matrix, so I can cancel C on both sides and that is what you wanted, right… Okay.

3rd one, K1 not equal to K2 and my TI, what was that, Zhi I Sigma U + eta I Sigma V, where Zhi I and eta I are tangent vectors. So, TI is the principal vector. Then, let us do it. T1 dot T2, this will be X1 transpose F1 X2, we have done it before with X and Y. Note, F2 X1 is K1 F1 X1 and F2 X2 equal to K2 F1 X2 by definition, right they are eigenvectors. So, X2 transpose F2 X1, this will be K X2 transpose F1 X1 which is K1 T2 dot T1. And X1 transpose F2 X2 similarly we will see K2 T1 dot T2, similar equation. But then T1 transpose F2 T2, this is equal to T1 transpose F2 T2 transpose, this is a scalar, so this is T2 transpose F2 T1, so that shows from these 2 that K1 T1 dot T2 equal to K2 T2 dot T1. But these 2 quantities are equal, T1 dot T2 equal to T2 dot T1, dot product is commutative. But that enforces K1 equal to K2, this or T1 is perpendicular to T2 or T1 dot T2 equal to 0.

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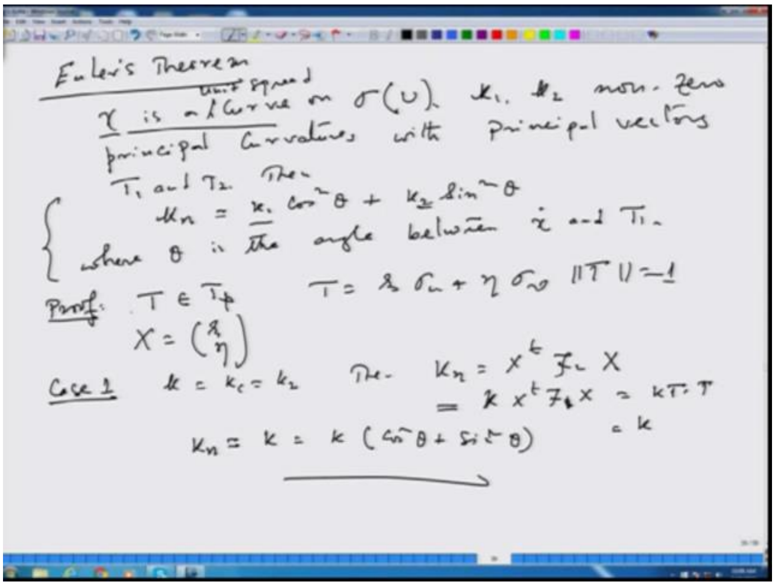

If it is 0, you cannot… But I now we started with K1 not equal to K2, so I must have T1 dot T2 equal to 0. So, look at the proof, just simple matrix manipulation, nothing there. Okay. We are converging to geometric which one more theorem. That is the famous Euler's theorem which connects, which connects normal curvature to principal curvature. So, everything same, gamma is a curve on a surface patch, regular surface patch, K1 K2 nonzero, I have to assume nonzero principal, we will see what happens if these are 0.

Curvatures with visible vectors T1 and T2, then conclusion, very nice, normal curvature is K1 cos square Theta + K2 Sin square Theta, where Theta is the angle between gamma dot and T1. There is nothing special about T 1, if you take T2, angle between T2 and T1, then it will be K2 cos square, so K1 and K2 will be interchanged. Okay. Remember the State, I will prove it, do not worry, proof is again matrix manipulation and understanding the which vector dot product with which vector is 0. But remember this conclusion of Euler's theorem, this is very important while doing calculations here on. Okay.

And we assume, actually I should have, I can here but I have called in the beginning of the 1st lecture that we will always consider unit speed, so gamma is we should have mentioned in the theorem that is unit speed. Maybe I add it here because of the theorem. But, anyways, in this, next series of lectures, whatever curve we consider, we will take unit speed for calculation of curvatures and related quantities. So, let us try to see the proof of this one, proof is very easy as I said.

So, let us take any tangent vector, right. Then T I can write as we know, correct. And I take this vector Zhi N. Suppose in the case one, there will be 2 cases, that K1 and K2 are same. Then, just verify K eta is X transpose F2 X which is equal to K X transpose F2 X, sorry F1 X which is K T dot T, okay, should have taken unit vector is equal to K and therefore K normal curvature is equal to K But K is anyway, this, so this relation is true because K1 is equal to K2. So, this case is very easy, right, whatever Theta you take, does not matter.

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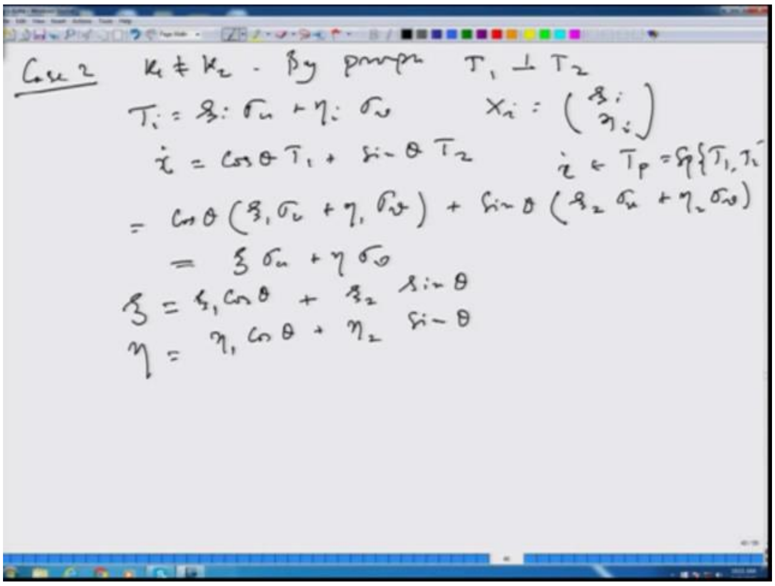

Nontrivial case is case 2 where K1 not equal to K2. In this case we know from previous proposition, by proposition T1 is perpendicular to T2. So, let us take TI equal to again Zhi I Sigma U eta I Sigma V X I equal to, so same setup as in the proposition. Now, gamma dot, I write it as cos Theta T1 + sin Theta T2, what is that, T1 and T2 perpendicular vectors, gamma is a gamma dot is a vector in the plane, gamma dot is in TP which is panned by T1 and T2 of course, perpendicular vectors. So now we write it as cos Theta Zhi 1 Sigma U eta 1 Sigma V + sin Theta Zhi 2 Sigma U eta 2 Sigma V, no problem. Okay.

So, let us write this as, so let us collect the term Zhi Sigma U + Theta Sigma V. What will be Zhi, Zhi will be cos Theta is Zhi 1+ Zhi 2 sin Theta, right. And eta is eta 1 cos Theta + eta 2 Sin Theta, right.

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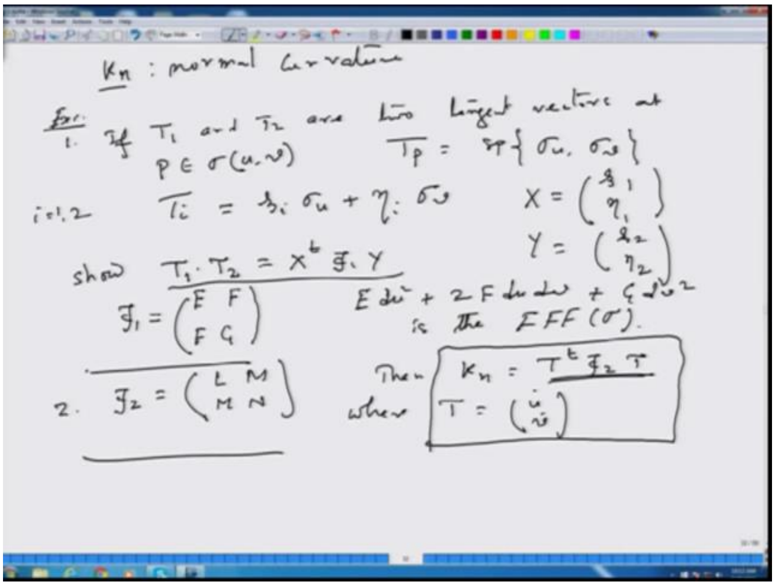

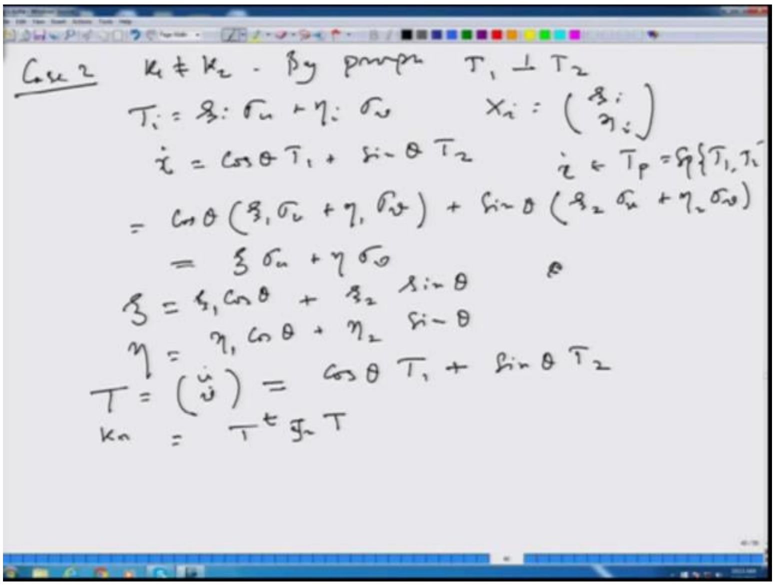

So, now I calculate K, normal curvature. Normal curvature is, you remember we defined the normal curvature, that is given by, what does it give, the 2nd fundamental form… This was the formula, right. This is cos Theta T1 sin Theta T2, as I know K normal curvature from that formula is T transpose 2nd fundamental form T. That was the formula, recall that, which page it was?

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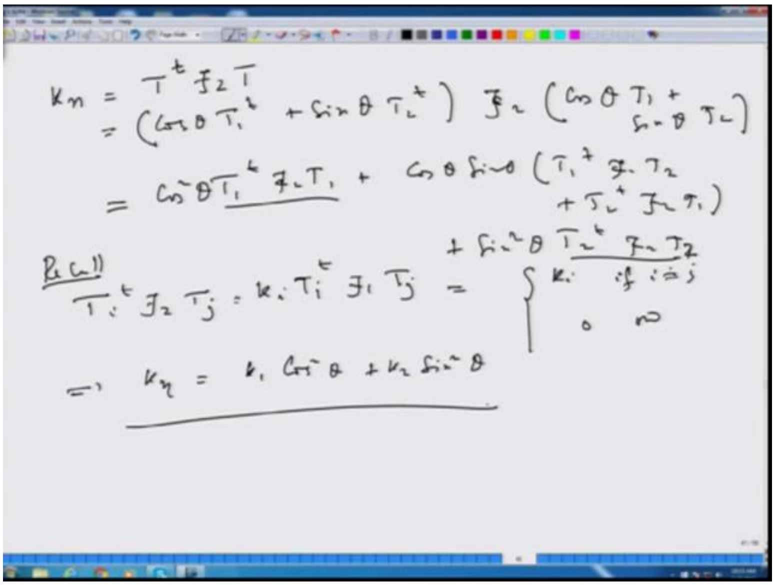

T transpose F2 T which is now I write cos Theta T1 transpose sin Theta T2 transpose F2 cos Theta T1 + sin Theta T2. Which now, you just break it up cos square Theta T1 transpose F2 T1 cos Theta sin Theta T1 transpose F2 T2 + T2 transpose F2 T1 + sin square Theta T2 transpose F2 T1. Now what is this, recall, TI transpose F2 TJ is KI, T1 transpose, TI transpose F1 TJ, because K has the roots of F2 - KT Phi. So this is equal to K I if I equal to J, 0 otherwise. Correct, we have done it before.

So, that will imply K eta equal to K1 cos square Theta + K2 sin square Theta, so that was proof of Euler’s theorem. Next day we will have all the calculations, all the quantities ready, we will talk about the geometric, what Geometry it implies. What happens to the geometry of the surface… thank you.