Curves And Surfaces.
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Module-I.
Curves In R^2 And R^3.
Lecture-02.
How Much A Curve Is 'Curved', Signed Unit Normal And Signed Curvature, Rigid Motions, Constant Curvature.

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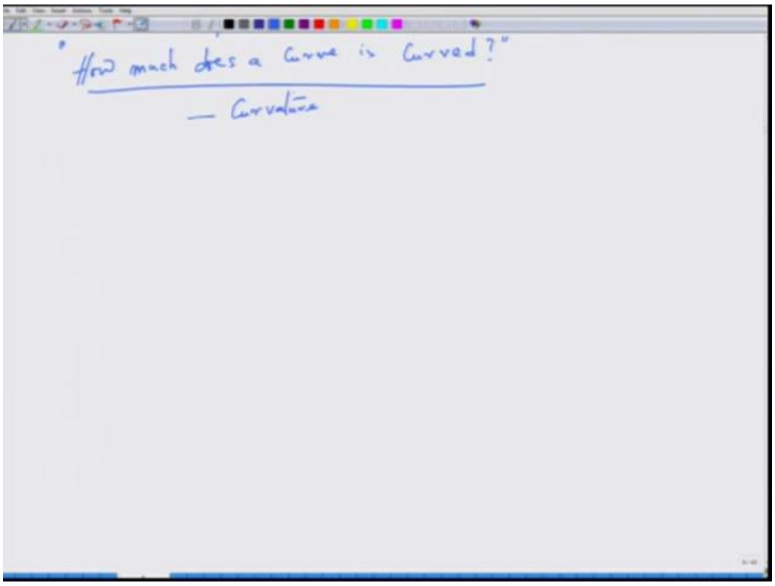

Okay, in the 2nd lecture we will be concerned about this question which sounds very qualitative but we will make it quantitative. How much does a curve is curved? What does it mean? That we have to define, we have to assign to a curve, a quantity called curvature. And all of you have some physical notion of curvature, that is how much you have to bend if you move along the curve. And our intuition tells us that curvature if at all I can define such a quantity, which should be…

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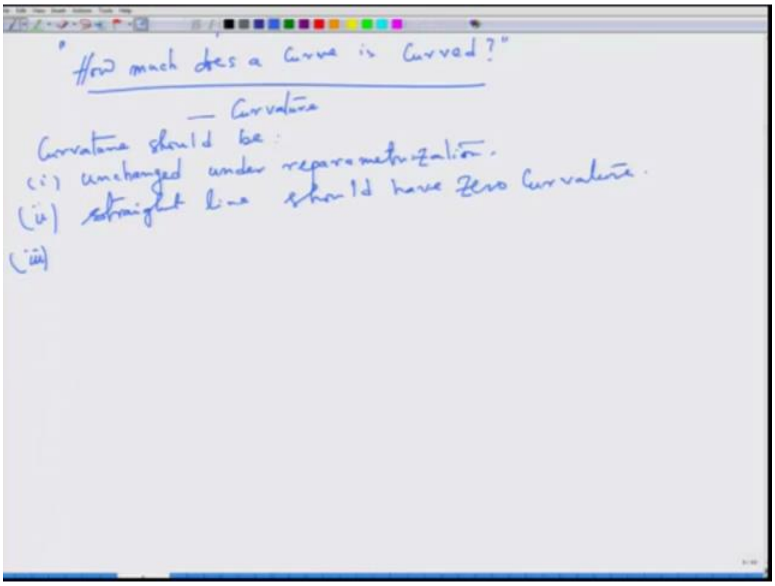

Okay 1st one unchanged under reparameterisation. This is very natural demand. Suppose I am moving in a circle and you are also moving on this circle and you are moving in the speed which is double than me.

But I should be bending as much from the point as you should be however the speed be. And reparameterisation is actually saying the how much actually and how much speed you are moving in. And all of you, should agree that I, it is not very bad to propose this condition also that a straight line should have 0 curvature. That if I move in a straight line, I do not bend at all, so straight line should have 0 curvature. And 3rd, 3rd I will explain you pictorially.

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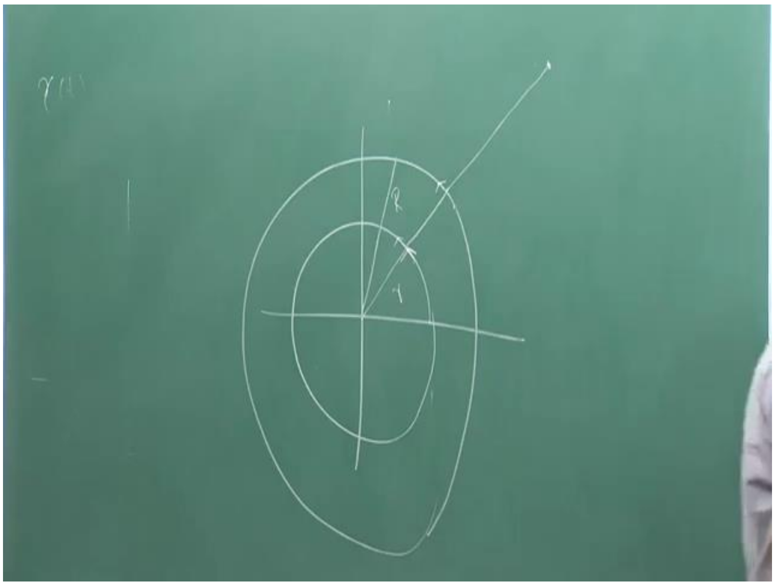

So, suppose on the plane, you are moving on this circle, and I am moving on this circle and you are moving on this circle. And the radius here is r and radius here is R, so suppose at this point where somebody is seeing us on this line. When I am moving, let us say in a cycle, in a bike, then obviously you should see me bending more than how much you bend here at the same point when you are seeing that. Or you can imagine it that when in a bicycle race, there are different circular tracks, the person who is on the inner circle bends more than the person who is on the outer circle. This happens while people are taking athletic run or something.

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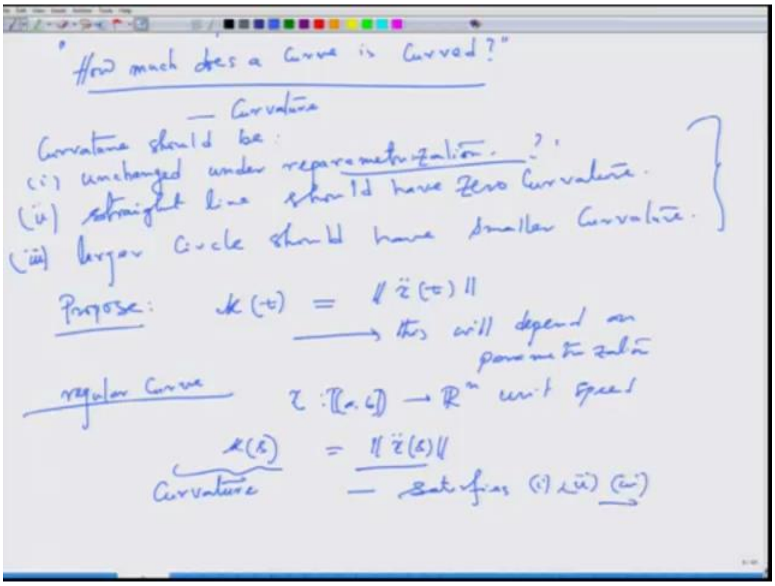

So, I put that as a condition that larger circle should have smaller curvature, right. Now stop, suppose I start thinking about a quantity about a curve which satisfies all these traits, then perhaps I should be proposing for curvature a quantity KT which is the rate of change of speed and its absolute value. But the problem is, this will depend on parameterisation, so the 1st condition will be a problem. But I know if I working on a regular curve, then I have a unit speed parameterisation. So, let us say gamma is unit speed, then I can propose curvature to be rate of change of speed and it is norm and you can easily verify this satisfies 1, 2, 3.

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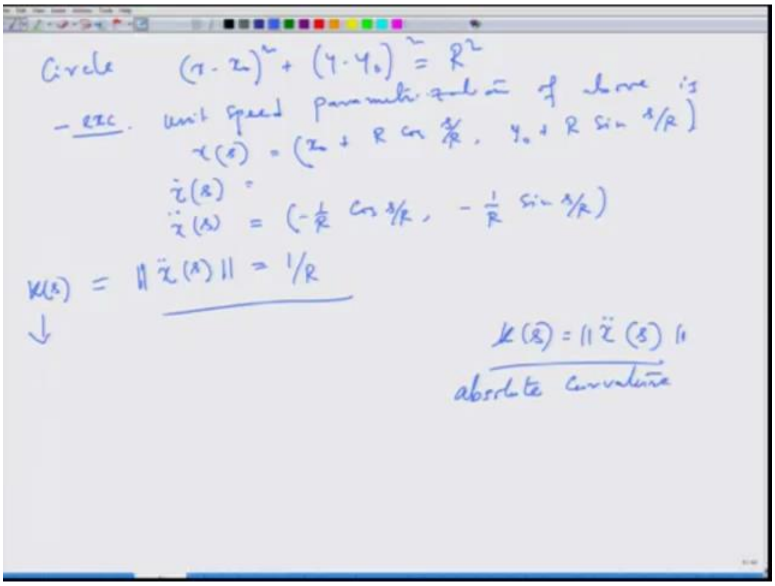

For example, let me show for a circle, a circle of radius R, Centre at X nought, Y nought. Well, this is an exercise I put that the unit speed parameterisation of above is X nought + R cos S by R and Y nought + R sin S by R. Now if you calculate gamma S and then gamma double dot S, that will be, you can calculate very easily and I write directly gamma double dot S is equal to - R S cos R - 1 by R sin S by R. So, gamma double dot S is equal to the curvature is equal to 1 by R.

So, if R is more, curvature is less and that is what I wanted. So, this quantity KS equal to gamma double dot S and its norm we will refer as absolute curvature. Because later on very soon I am going to introduce what is called sign curvature.

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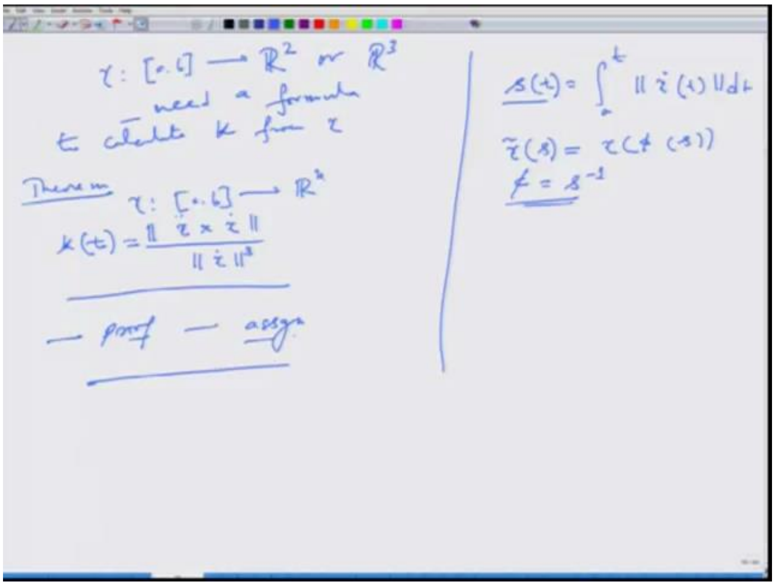

However there is a problem, suppose that I am given a curve 2 from some interval a b to R, let us say plain curve R2 or R3. Now, finding its unit speed parameterisation maybe problem in itself, in the sense that how do you find unit speed parameterisation? Okay.

1st you can consider the arc length, okay then you consider S as a function of T and then you write gamma tilde S is equal to gamma phi S and in that case phi has to be S inverse so I have to have invert this function. Inverting a function in general is not an easy task to do. So, finding a unit speed parameterisation for a curve is in itself is a difficult task. So, need a formula, so what I mean here, need a formula to calculate curvature from gamma itself. And the formula, I write here the theorem that suppose gamma is a curve from a b to Rn then KT is equal to gamma double dot cross product gamma dot norm divided by gamma dot cube.

Okay, I will not prove this, this group is available in any standard book. So, proof is a part of assignment. It is just… We will look at the unit speed parameterisation of gamma and calculate KS in terms of KT. So, to a curve, I got a another quantity which is called curvature. Which has an absolute value, I mean you say a positive number.

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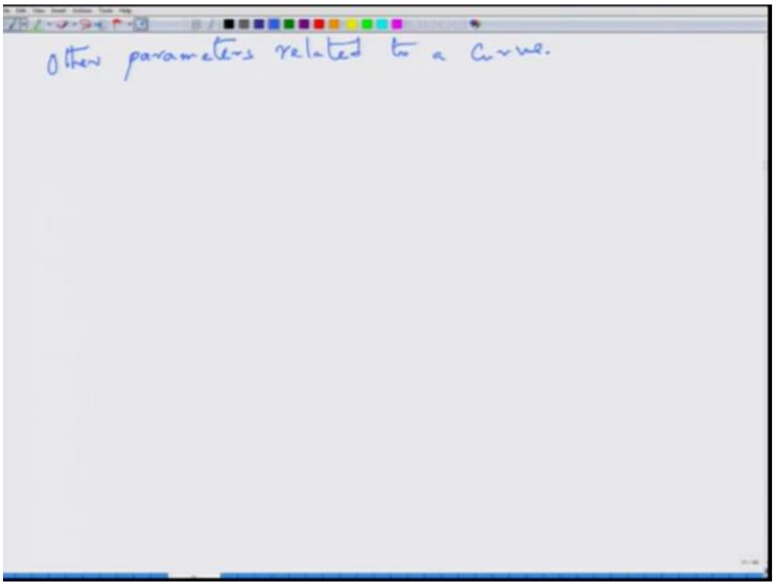

So, but to determine a curve, I often need to consider other parameters related to a curve. Okay.

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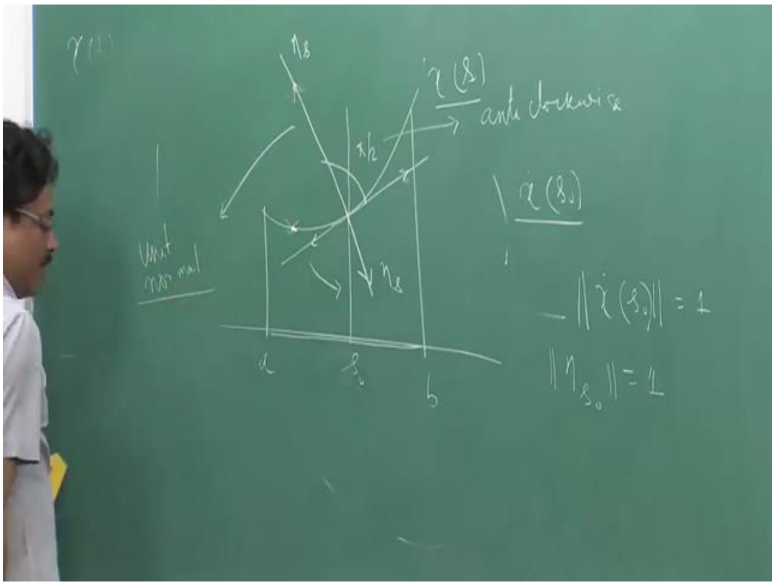

One is this, you, suppose I have a curve like this, we have a curve out here, gamma T, there is a point T nought and let us say it has unit speed S, I have a point S nought here. So, this is gamma dot S nought is that, you can also interpret as a tangent vector. This is a tangent vector.

If I rotate the tangent vector by angle pie by 2 anticlockwise, so rotate this fellow anticlockwise by pie by 2, then I get another vector. Suppose I take unit speed parameterisation, then gamma dot S nought has length 1 and therefore this Eta S nought will also have norm on because I am just rotating a vector of, rotating a vector does not change its length. And so this fellow is called unit normal. You see for the same curve if I am moving from let us say point a to b, then I moving this way, right?

So, the direction of the tangent is the one what I have drawn. But suppose I from b to a, the direction of the curve will change and the direction of the tangent will also change and so does the unit normal because I am moving anticlockwise. In that case this will be the unit normal. So, depends on how I move from a to b or b to a. So, this unit normal fellow, direction of that depends on how I move and that will give rise to a quantity called orientation, we will soon come to it. Okay.

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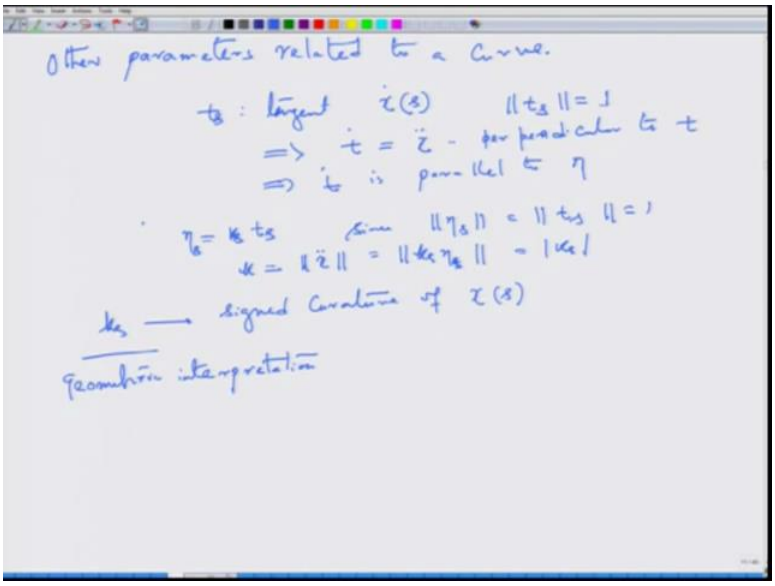

Now see, if suppose T is a tangent vector, that is gamma S T S. What is the length? 1, right. That would tell you that T dot which is gamma double dot, this is perpendicular to T. So, that will imply T dot is parallel to eta. Suppose I have 2 parallel vectors Eta and T dot and those both of them has unit length, so Eta will be some multiple, so Eta S at a point will be multiple, so multiple KS TS. Since I have K which I defined as absolute curvature equal to… So this quantity KS, this is called the signed curvature of gamma x. Okay.

So this is signed curvature, absolute value is curvature. But this fellow has nice geometric interpretation. I will do it in the next page.

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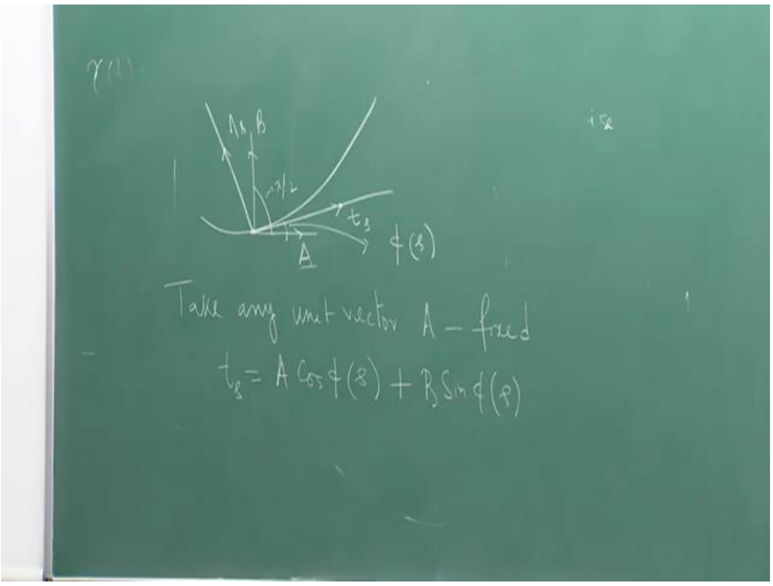

Okay, let me go to the board again. Suppose this is my curve, here is a point, and this is a tangent vector, unit tangent vector TS and this is Eta S, so what is the geometric interpretation of KS? So, what I do, so take any unit vector, unit vector let us select A. Consider the angle between A and tangent vector, let us call it phi S. A is fixed, once you take it is fixed. Okay.

Now what I do, I rotate this vector A by an angle pie by 2, so I get a vector B. So, my TS now, since this angle is phi S, will be resoluted A cos phi S + B sin phi S, simple geometry. Keep the picture in mind.

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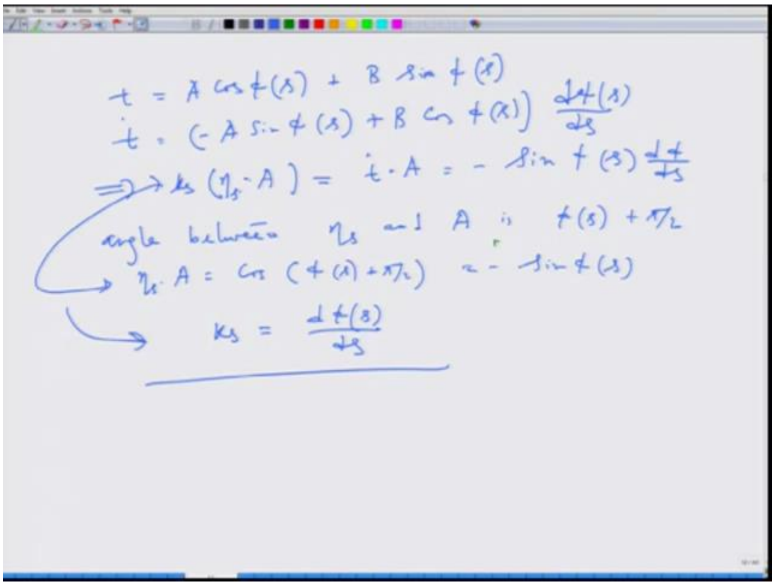

So, your T is A cos phi + B sin phi S. Now you calculate T dot which is A sin phi S B cos phi S d phi S dS. This will give us from the previous consideration Ks into Eta S dot A dot dot product is equal to T dot dot product A is equal to - sin phi d phi dS. Now angle between Eta S and A is how much? Phi S + pie by 2. So, Eta S into A, both are unit vector has cos phi S + pie by 2 equal to - sin phi S.

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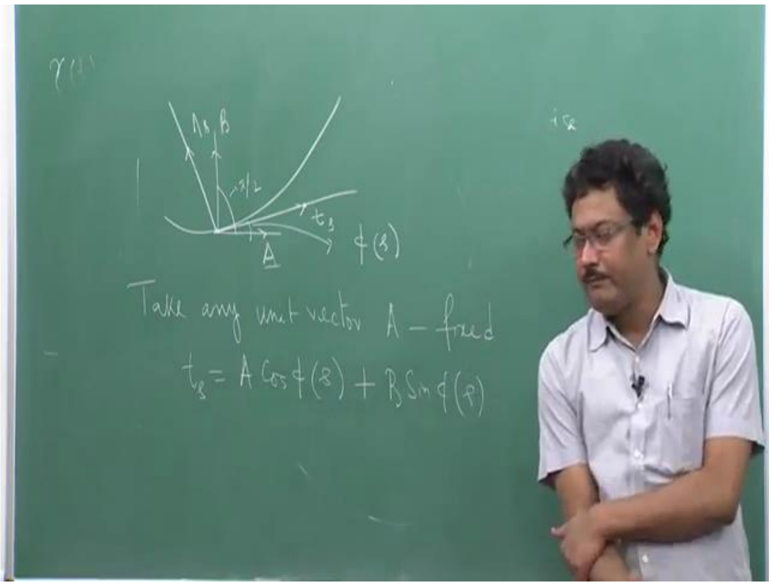

So, if you combine these 2 equations, what you get is KS equal to d phi S dS. What does it mean? It means if you take any unit vector, sign curvature is the angle or rate of change of angle, sorry, rate of change of angle phi, to which you must move A to make it parallel with TS. Now we will, I will ask all of you to think of a physical interpretation of this mathematical explanation, that KS equal to this quantity so what should be the physical interpretation? I am not telling you, you ask yourself.

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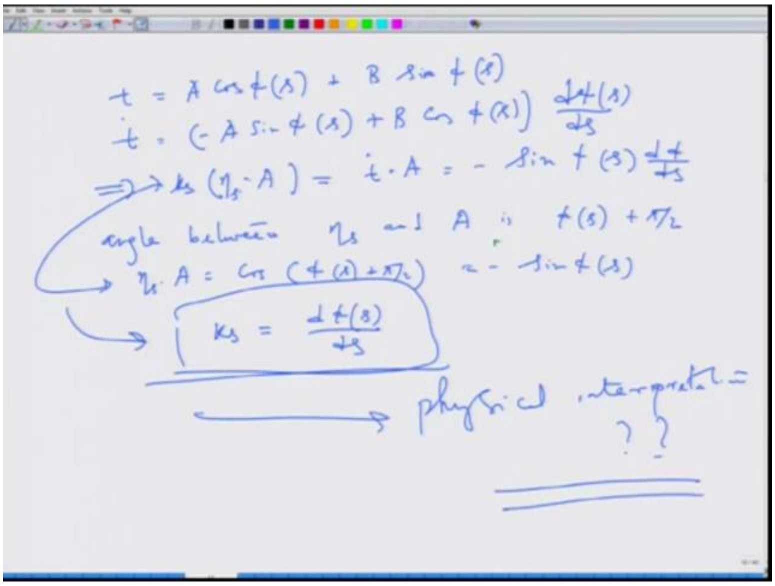

So, think of a physical interpretation of this quantity, that suppose somebody is moving in a curve and you are an observer standing whatever, what is the sign curvature is. I am standing and… You think of this situation, that somebody is moving in a cycle in some curve, you are standing somewhere and somebody is standing just opposite to you in a straight line. So and both of you are observing, what will be the sign curvature for you and your friend. In the last part, maybe I will continue this in the next lecture also. We will be concerned about determination of unit speed planar curves.

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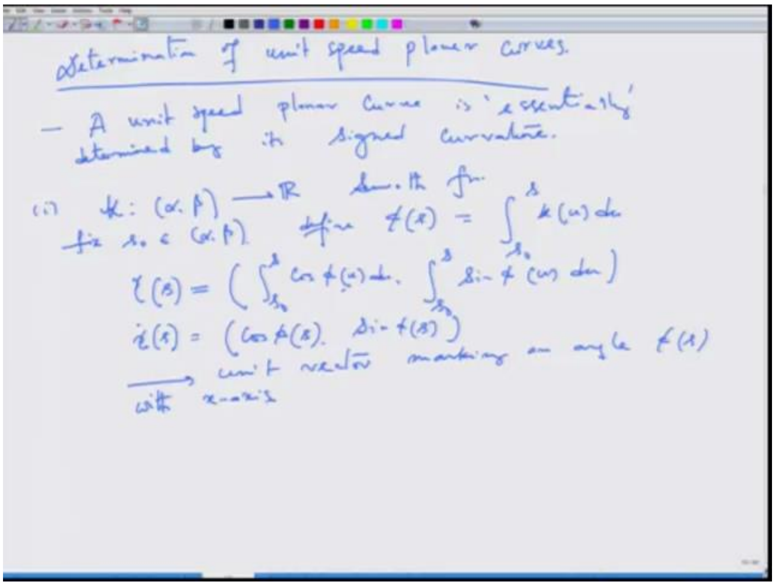

That means I have curves in R2 and what I want to say here that I want to make a statement here, unit speed planar curves is essentially, essentially you have to make it mathematical, determined by S sign curvature. I will not prove this as a theorem because in the next lecture we will prove some general theorem about special curves, curves in R3, from which this will follow. But I want to see, to tell you right now that that is how the sign curvature determines a curve.

So, 1st let us see, suppose I have a function like this, it is a smooth function. Smooth function means that always the derivative exists. Fix any S nought in Alpha, beta and define this quantity phi S equal to S nought to S K u du. And consider gamma S, interval S nought to S, cos phi u du, interval S nought to sin phi u du. So, what will happen, what is gamma dot? What is phi dot S. This is cos phi S sin phi S, so this is a unit vector making an angle phi S with x-axis. Right.

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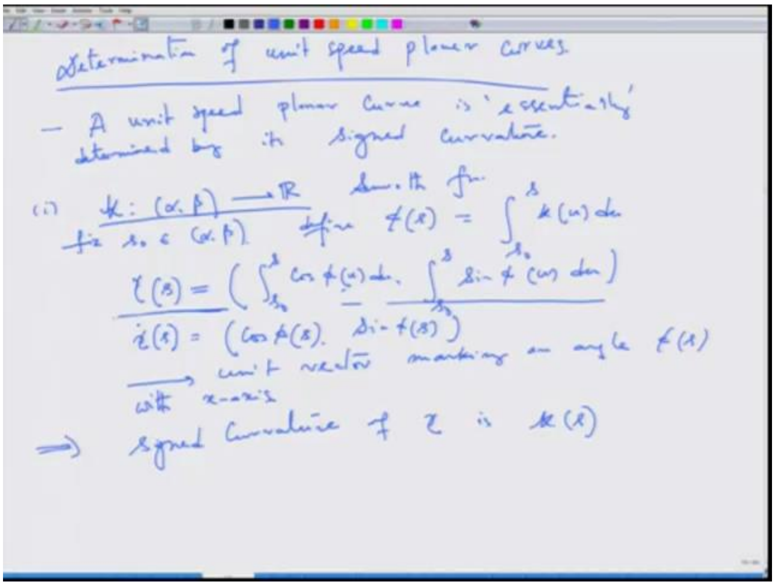

Now, what we just observed from here, from the previous page that means here that the sign curvature of this implies from the previous page, sign curvature of gamma is KS because this is a vector phi S, cos phi S sin phi S. If I rotate it by angle phi S and angle phi S, it will be along the x-axis. And I know precisely there is a sign curvature, so sign curvature of gamma is KS.

So, what it says? If I am given a smooth function on any interval, I can, given a smooth function K on an interval, I can determine a curve whose sign curvature, I can find out a curve this one, whose sign curvature is KS. And its converse is also true.

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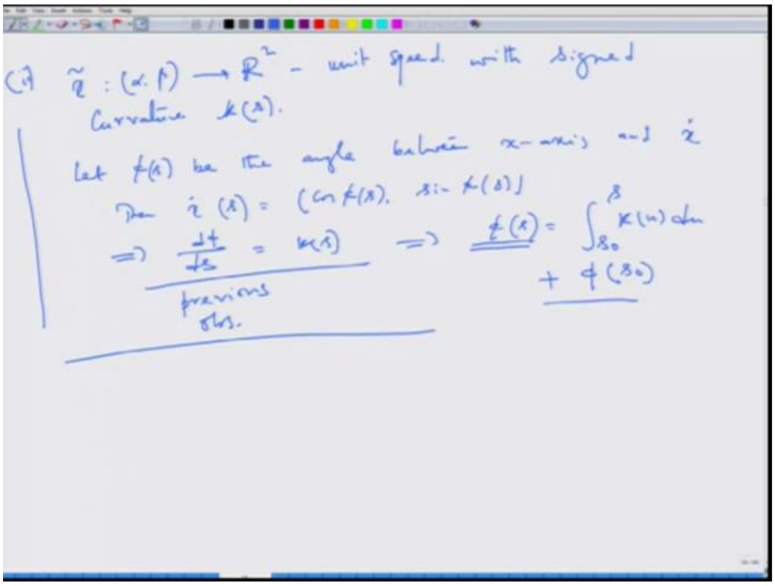

In the sense that, suppose that I have a curve gamma tilde from Alpha, beta to R2 which is unit speed, it is sign curvature KS KS, let phi S be the angle between x-axis and gamma dot. Then from the discussion above I know gamma dot S, okay, I mean, phi S is the angle, so gamma dot can be written as cos phi S sin phi S, that will imply.

Now, d phi ds equal to KS that I know from the previous discussion. This will give us that phi S is equal to S nought to S K u du. Okay. So I can find this angle Phi S, even unit speed, curve and sign curvature case. So, essentially, a curve in R2 in determined by its sign curvature. Now, when you move to special curve or curve in R3, we will show this is not true and in that case also we will observed something about planar curves, that planar curves are actually, I will write it down as a theorem that it is exactly determined by its sign curvature up to some rigid motion of the plane. Okay, next lecture.