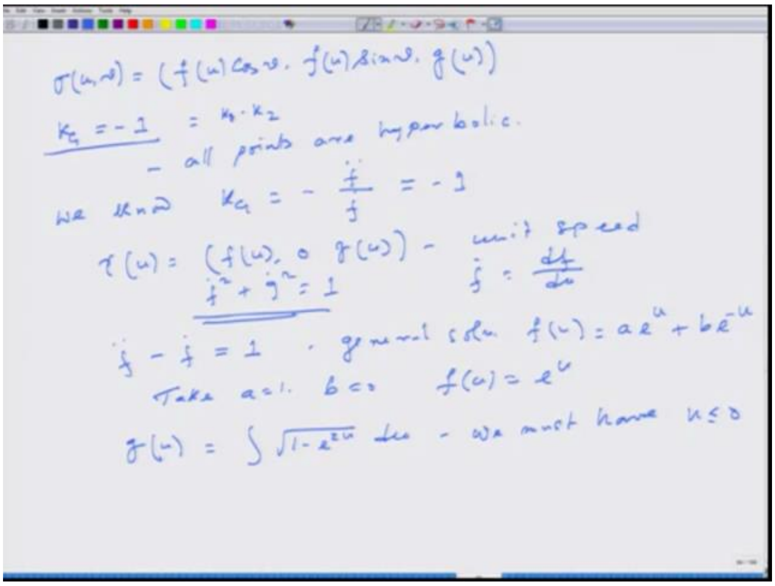
Curves And Surfaces.
Proffessor Sudipta Dutta.
Department Of Mathematics And Statistics, Indian Institute Of Technology Kanpur.
Module-IV.
Surfaces-3: Curvature and Geodesics.
Lecture-20.
Pseudosphere.

So, today is the last lecture for this course and as we had decided in the last lecture, today we will try to find geodesic on a surface which is a very special kind of surface.

(Refer Slide Time: 0:28)

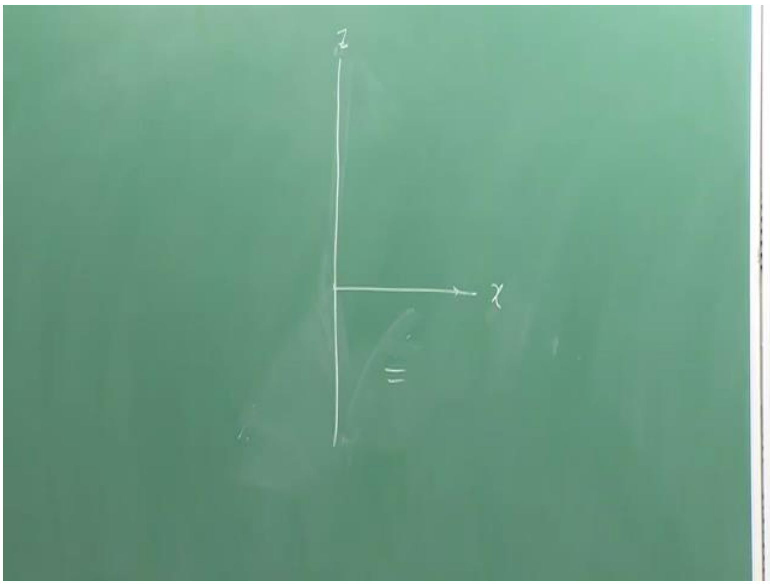


I mean it is again a surface of revolution, but, okay let me write the equation 1st. What we want, the Gaussian curvature to be constant -1. In the last lecture we had seen Gaussian curvature 0, then what are the possible surfaces, then we had seen Gaussian curvature constant +1, that only part of the sphere. And today we will see if such a if such a surface exists or not and what happens to their geodesic.

It is very interesting phenomena starts happening here that our a regular sense of geodesic, that fails in this kind of surfaces, these are typically called hyperbolic surfaces, because Gaussian curvature is -1, so that is you remember K product of the 2 principal curvatures and such equal to constant -1 that all the points are hyperbolic. So, such surfaces are called hyperbolic surfaces. So, in such surfaces, all points are hyperbolic. Now, we know that for surface of rotation, the Gaussian curvature is given by F dot double dot by F dot and we consider unit speed curve, so, so this is for our gamma U, gamma U was FU 0 0 unit speed and that means that F dot square + G dot square equal to1.

Dot F dot is with respect to df du, right. Well, now this is, if this is equal to -1, this gives rise to a differential equation, correct. This is a linear differential equation, whose general solution is FU equal to AU + B E power EU. Okay, I can take any curve for that matter. So, take A equal to 1, B equal to 0. So, FU equal to EU and what will be GU then, from this equation GU will be integral 1 - E to the power 2 U du.

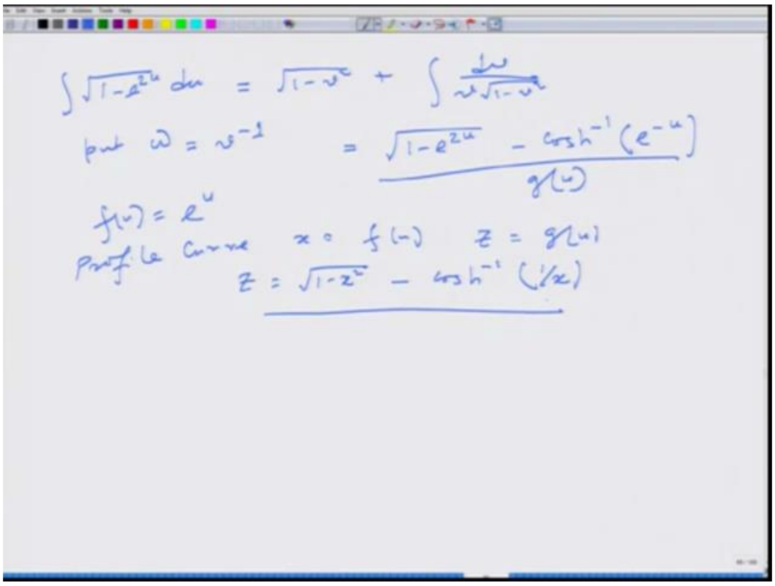
(Refer Slide Time: 4:31)



Now, see, to integral to make sense, we must have U less than equal to 0 so we must have, so I draw the picture little bit, so, instead of taking, this is Z, this is X, so my curve must lie, okay, I am not drawing the curve right now curve must lie in this region, here X has to be U, has to be negative.

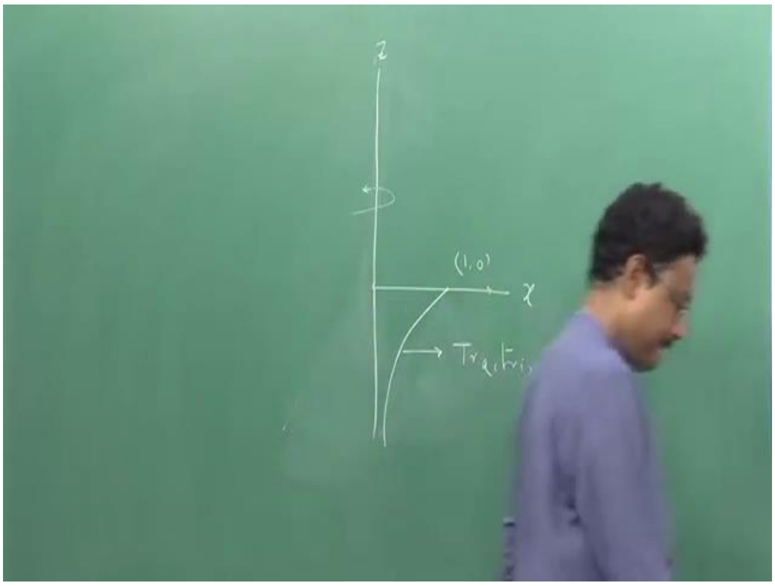
So, I will draw the curve but let us see if we can do little bit more about it before we can actually draw the curve. Let us try to evaluate this integral.

(Refer Slide Time: 5:03)



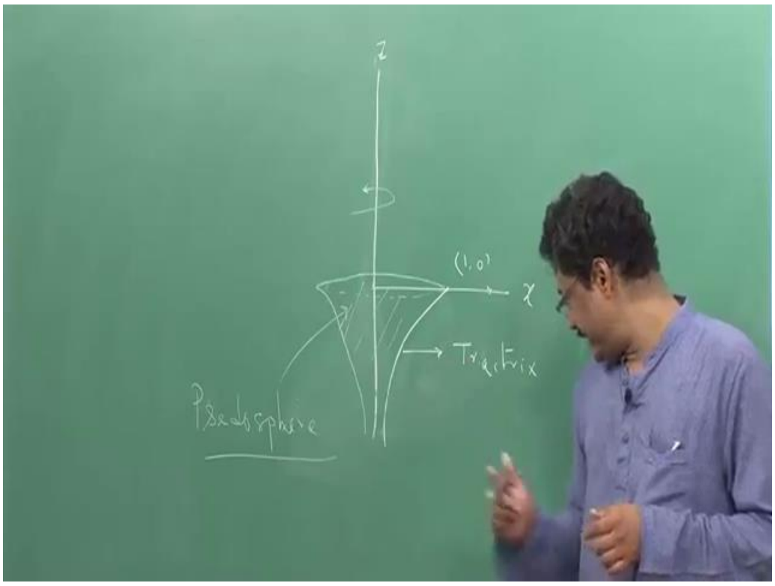
This integral, if you evaluate, this comes out to be root over 1 - V square +… okay. Then if you put W equal to V inverse, then this will come out to be 1 - E power 2U - cos hyperbolic E power U. So, there will be some constants but I omit the constants because we did only one curve. So, what I get? My FU is E power U and GU is this, this is GU and FU is E power U. So, the profile curve, in the profile curve, if I put X equal to FU, then Z and Z equal to GU, then Z is actually root over 1 - X square - cos hyperbolic inverse 1 upon X.

(Refer Slide Time: 6:35)



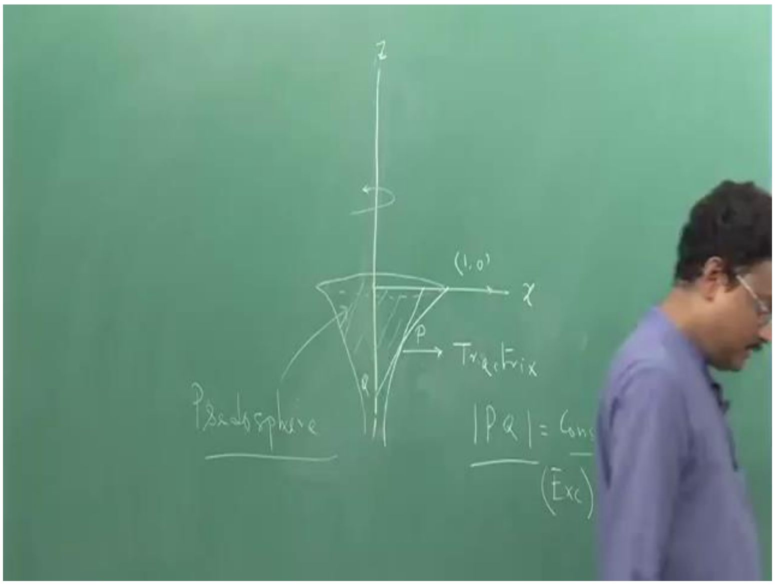
So, this curve looks like passes through 1, 0 in XZ plane, this curve looks like, so this is point 1, 0 and such a curve as a name, it is called Trectrix.

(Refer Slide Time: 7:17)



Now, we will rotate it around Z axis, Y axis is your, I am not drawing the Y axis here, this drawing becomes difficult in that case. So, it is called Trectrix. This curve is very, so coil will rotate around Z axis to get a figure like this and this surface, this is known as pseudo-sphere. Why pseudo-sphere? Maybe I will tell, when you this, okay, just see why it is called pseudo-sphere, maybe I will explain. But this curve Trectrix has a very interesting property,

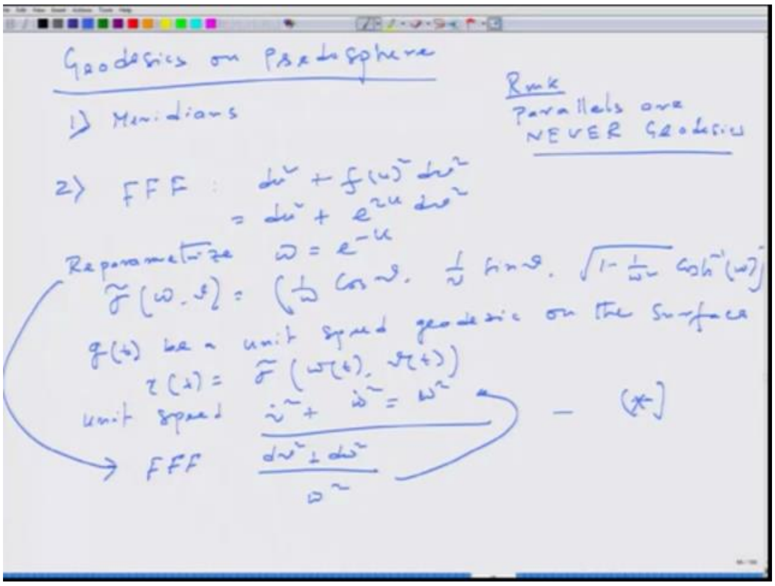
(Refer Slide Time: 8:07)



what happens that if you take any point on the Trectrix, let us say P and you do the draw the tangent through P and that it hit the Z axis at a point Q.

Then the length of PQ, this is always constant. What does it mean, wherever to take your P and you draw the tangent line, let it cut the Z axis at Q, some point Q, the length of PQ is always a constant, it does not depend on the parameterisation of the curve. So, I put it this is as an exercise, nice exercise to solve. The surface generated as pseudo-sphere.

(Refer Slide Time: 8:56)

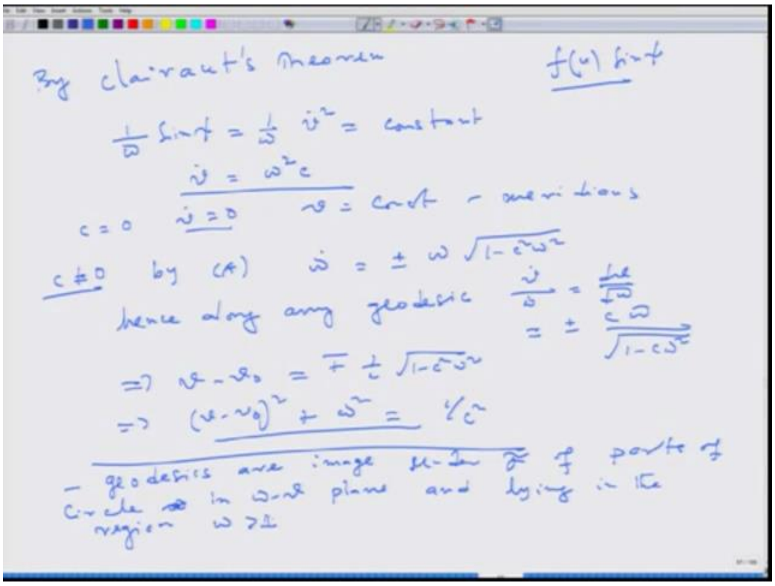


So, let us find out geodesics on pseudo-sphere. Okay, one 1st I know, all meridians are geodesics. All meridians are geodesics, while changing signs of equations, we have proved in general that all meridians are geodesics and what are the meridians? They are the parallel lines of the profile curve.

Now to find and parallels are never geodesics here because to be parallel geodesic I must have FU constant but here FU is E power U, never constant, so parallels are never geodesics. So, there is a remark I make here, parallels are never geodesics here. Okay. To find the other geodesics, so we will be using the Clairot's theorem. Okay, I know the 1st fundamental form of this surface is how much? This is always given by du square + FU square dv square. So, in this case it is du square + E power 2 U dv square. I reparameterise this curve. By putting W equal to E power - U again. Then the surface looks like Sigma W V equal to 1 by W, you check it, it is very easy cos V 1 by V Sin V 1 -1 by W square cos hyperbolic inverse W, right.

So, let us take gamma T be a GT unit speed geodesic on the surface, on the pseudo-sphere. So, let us take gamma T to be Sigma tilde omega T VT. Unit speed will give us V dot square + W dot square equal to W square. How do I get it because, see, if I if I reparameterise, after reparameterise, the 1st fundamental form becomes dv square + dw square, this thing. And the speed of the, and if I take and integrate the 1st fundamental form, I get the length of the curve and unit speed, I am taking unit speed curve. So, this will give us that V dot square + W dot square W square. So, let us put mark on this equation star.

(Refer Slide Time: 12:40)



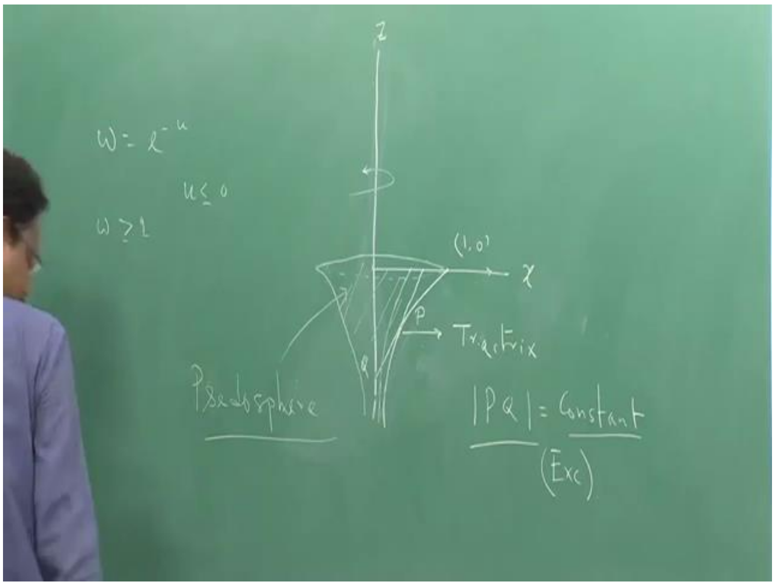
Now, what I know, I apply Clairot's theorem now, what is constant, distance from, so distance, so FU, now it is FW, not FW I mean, recall what happens in Clairot's theorem, if FU Sin psi or Sin psi is the angle between gamma dot and the meridian, gamma dot and T, then this remains constant. So, by Clairot's theorem what I will have, now one, now distance is 1 by W because I have put W to be FU is E power U and I have put W equal to E power U, E power - U, so this distance is 1 by W sin psi equal to 1 by W angular momentum V square, that must be constant.

So, I get V dot equal to W square C, C is a constant. Now if I put C equal to 0, then V dot equal to 0, so V is constant again, so that is a meridian, we get back the meridians. Okay. Now, C not equal to 0, then by star, what was star, this fellow, I get W dot equal to + - W 1 - C square W square. Hence along any geodesics, I must have V dot W dot, but then I can write it.

I can remove T part dv dw, that is equal to + - C into W 1 - C W square. If you solve it you get V - V0 is equal to + - + 1 by C 1 - C square W square. That is V - V0 square + W square equal to1 by C square. So, this looks like circle in coordinate, in XY, in VW coordinate but if I look at on the surface, so geodesics will be, geodesics are image of, image of image under Sigma tilde of parts of circle because this is a circle in WV coordinate.

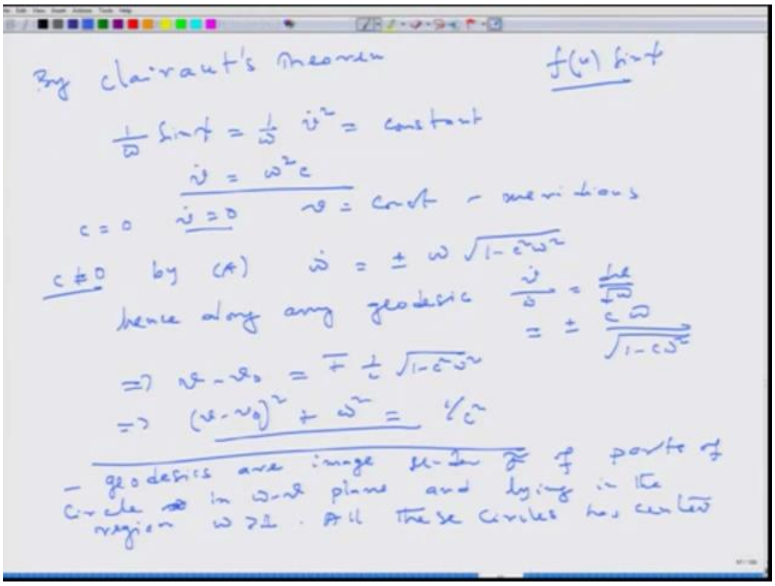
Parts of circle in WV plane and lying in the region W greater than 1 why, because U is less than 0, W is E power - U, remember each time,

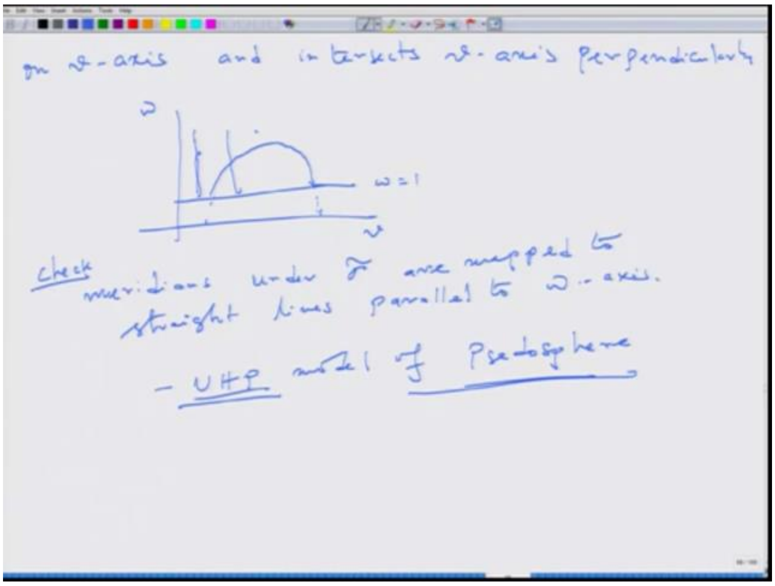
(Refer Slide Time: 16:35)



this was the reparameterisation, U is less than equal to 0, so W, actually less than 0 you can take, and W is greater than 1. Okay.

(Refer Slide Time: 16:57)





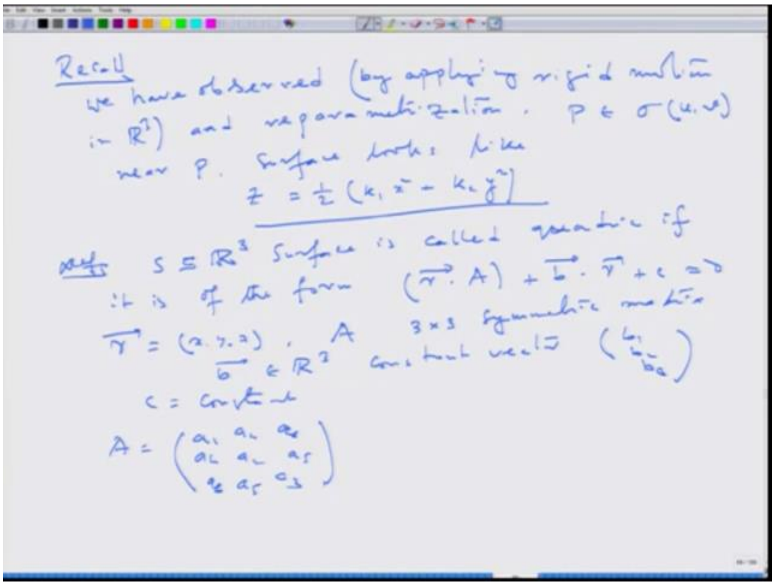
All the circles, so all the circles have Centre on V axis, V0 0 is the Centre for different V0 and intersects V axis perpendicularly. So, if I draw this in WV plane, this is W equal to 1, this is W axis, V axis, the geodesics will look like, this part is not there, part of this semicircle, okay. And what happens to the meridians? This you check, meridians are mapped to straight lines parallel to W axis.

So, the straight lines, they are also, they are meridians and they are also geodesic. These are the only 2 geodesics available for pseudo-sphere. This picture of pseudo-sphere is called the upper half plane model, this is called that UHP model of pseudo-sphere, actually it is connected to so-called, this is beginning of as I said hyperbolic geometry, so there is some inherent notion of projective planes. And if you look at the projective planes, then geodesic on projective planes, so pseudo-sphere in the upper half plane is precisely a projective plane, two-dimensional projective plane and pseudo-sphere on, geodesics are straight lines and semicircles.

So, if you want to continue with this geometry, topic of geometry or you start any course all differential geometry from here, you will be talking about the language of manifolds and also the language of, I mean this, whatever we have, as I said in the beginning, we have not actually uttered the word manifolds throughout this course but differential geometry, course will start with the word manifolds but do not, but you will have the confidence to go ahead because the manifolds word you see, the 2 and three-dimensional manifolds, what you will be working on, they are the curves and surfaces what we have dealt with.

So, that is the beginning of differential geometry course and end of curves and surfaces course. But we are left with one thing that I promised some time that I will do and so for the last part of the lecture, last part of this lecture series, I want to do this part of, I want to recall, I want to just complete that gap and that is about coordinate surfaces.

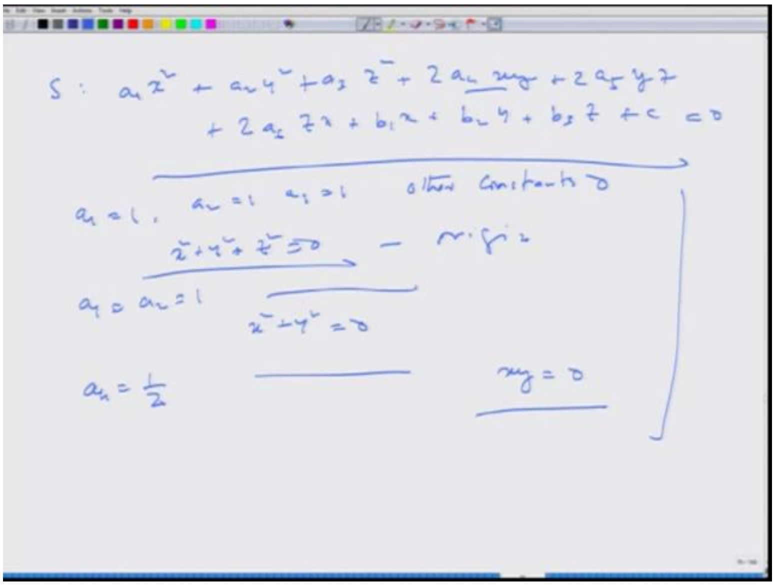
(Refer Slide Time: 21:16)



You recall, while talking about principal curvatures and this geometry, we have observed that by applying rigid motion possibly, rigid motion in R3 and reparameterisation, if I have a P on a smooth surface, then near P, the surface looks like something around Z equal to half K1 X square + K2 Y square. So, it looks like a coordinate surface, so K1 and K2 are principal curvatures.

So, here I will make a definition and then we make some observation. S in R3 surface, smooth surface, smooth regular surface, okay, we will see the smoothness and regularity condition. Surface is called Quadric if it is, okay, let us say, if it is of the form… What does it mean, if R is the vector X, Y, Z, A is 3 cross 3 symmetric matrix, B is a constant vector, B1, B2, B3 and C is some constant. So, if you write A to B, I write in this form A1, A2, A3, then A4 here, A4 here, A5 here, A5 here, A6 here, A6 here.

(Refer Slide Time: 24:37)

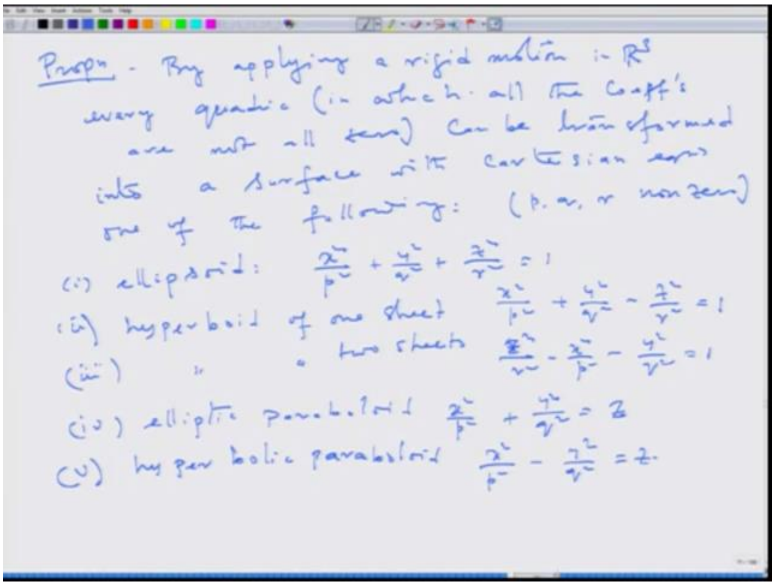


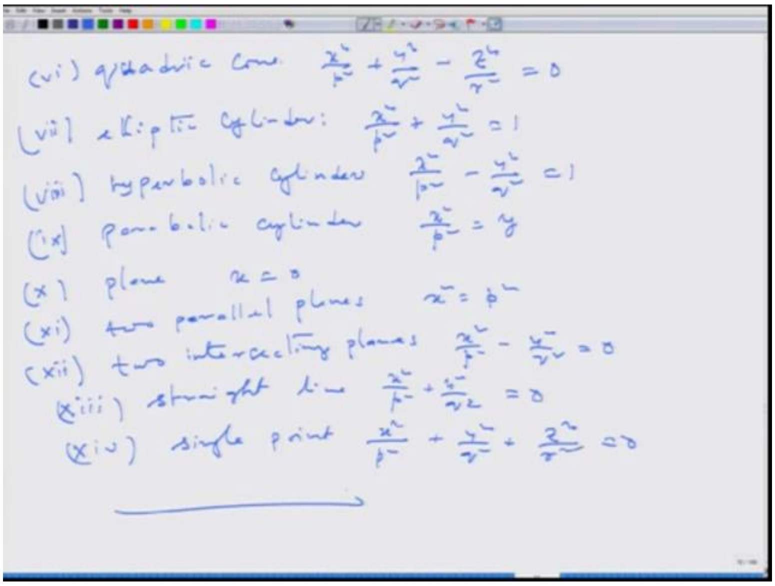
If I write it in this way, then S becomes similar, just A X square, A2 Y square, A3 Z square, 2 A4 XY, +2 A5 YZ +2 A6 ZX + B1 X + B2 Y + B3 Z + C3 equal to 0, this is a quadratic equation.

So, if you do this fellow with this matrix, you get this one. See, this is not necessarily always a surface, why? For example, if you take A1 equal to1 A2 equal to1 A3 equal to1, the constants 0, then you get X square + Y square + Z square equal to 0. This is not a surface, this is just a origin. Similarly should take just A1 equal to A2 equal to 1, the constants 0, you will get X square + Y square equal to 0. This is not a surface in R3, right, this is just a pair of straight lines.

Similarly if you take this say A4, only A4 equal to 1, A4 equal to half and other constants 0, then you will get XY equal to 0. So, this is again not a circle, it is union of 2 intersecting straight lines.

(Refer Slide Time: 26:43)



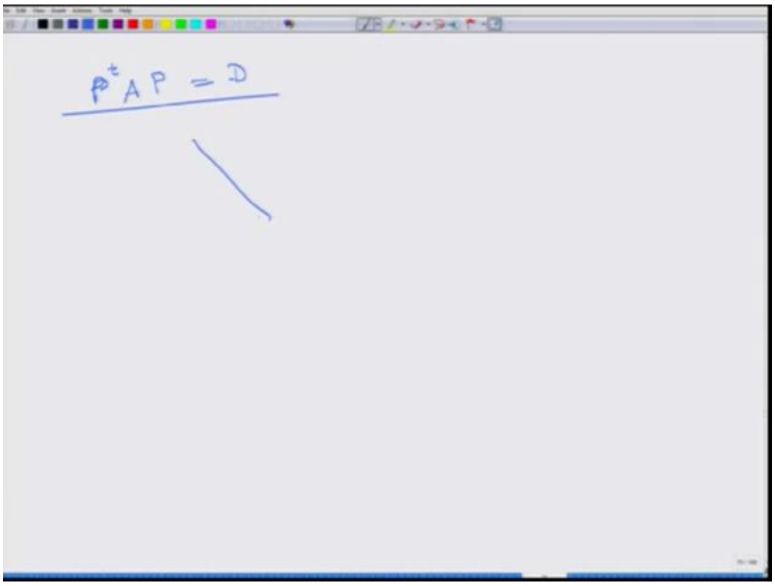


So, this any quadric may not be a surface always, so I need condition on A1, A2, A3, you A1, A2, AIs and BIs, but what is true is this proposition and that I want to write the entire proposition. We will not prove it, and proof is, but I will integrate, I will tell you how to prove it. Proof is very easy but it is very, does total classification of all quadric surfaces, all quadric.

It says by applying a rigid motion in R3, every quadric in which all the coefficients are not all 0, okay, all the coefficients should not be 0, can be transformed into, into a surface with Cartesian equation one of the following. 1 ellipsoid, this is of the form PQR throughout are nonzero constants, okay. Either ellipsoid or hyperboloid of one sheet, this is X square P square + Y square Q square - Z square R square equal to 1. Hyperboloid of 2 sheets, that is X, that is let us write it this way Z square R square - X square P square - Y square Q square equal to 1 or elliptic paraboloid, this is simply X square by P square, Y square by Q square equal to Z.

Hyperbolic paraboloid, this is simply X square - P square, Y square P square Y square P square equal to Z, okay. Say quadric cone which is simply X square P square + Y square by Q square - Z square by R square equal to 0. Elliptic cylinder, which is simply… Hyperbolic cylinder… Parabolic cylinder… Maybe just a plane X equal to 0 or Y equal to 0 or Z equal to 0 of 2 rigid motions. 2 parallel planes… 2 intersecting planes… Straight line, in this case, this is no longer a surface and finally single point… These are the all 14, only 14 exclusive possibility for a quadric surface of 2 rigid motions.

(Refer Slide Time: 33:43)



Now, how do you prove it? Proof is that I have taken this A, symmetric matrix, right. So, that means this A was a symmetric matrix 3 cross 3 symmetric matrix, what do we know about symmetric matrix, that if A is a symmetric matrix, 3 cross 3 symmetric matrix, then by applying, I can find orthogonal matrix P since the P transpose P is diagonal. So, that is, applying, orthogonal matrix to coordinate system means it is a rigid motion. It is a rotation, maybe you have to follow by translation, okay, 1st you go to the rotation, so you get a D, diagonal matrix.

Now if I have a diagonal matrix, then if you look at these equations, this will become, no, not this one, this one, this will become easier to handle now by translation depending on the entries of the diagonal and B1, B2 and B3 and C, you will get one of these equations, 1 to 14. So, the proof is very easy, you can try it yourself and for instance, okay… One may continue with many examples, I will write down one example in the assignment 4. So, that is the end of this course, thank you, I hope you get some benefit out of it. Thank you.