**Curves And Surfaces.**
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**Module-IV.**
**Surfaces-3: Curvature And Geodesics.**
**Lecture-21.**
**Classification of Quadratic Surface.**

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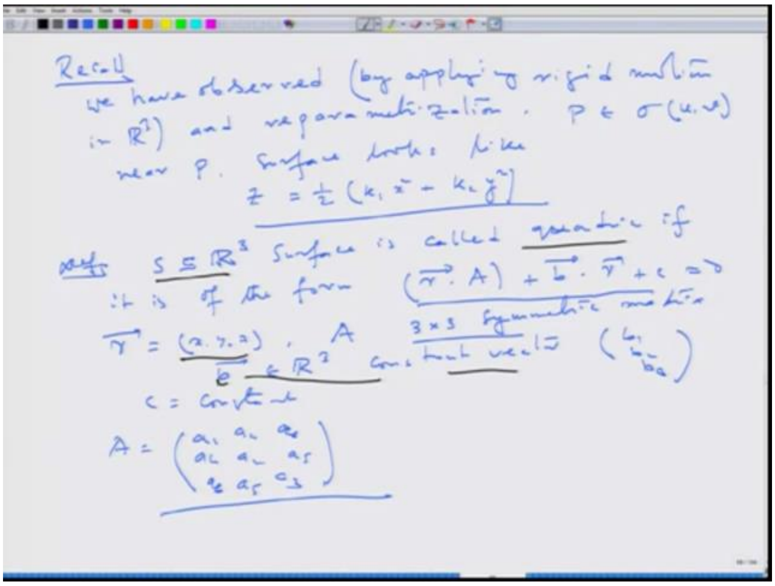


Okay, today we will 1st try to see a proof of this proposition which I wrote down last time. Recall the proposition that we are concerned about quadric surface, quadric surface is defined as S in R3 is called quadric if it is of the form R dot A plus B dot R + some constant C where R is a vector X, Y, Z, A is 3 cross 3 symmetric matrix and B is a fixed vector, constant vector in R3 and C is a constant in R. And A we have taken let me write it again.

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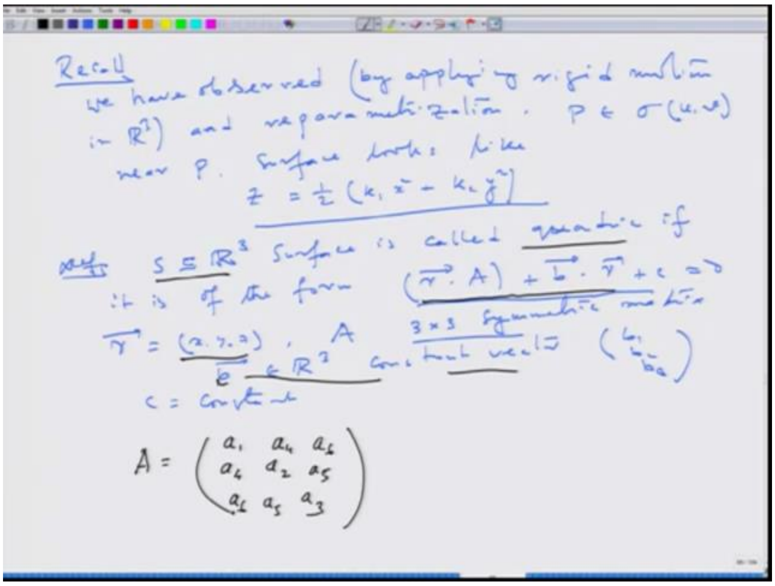


We will take A just for our calculations, we will take A in this form, so 3 cross 3 symmetric matrix, so I will put the diagonal entries as A1, A2, A3 and off diagonal entries A4 here, if I put A4 here, then I have to put A4 here, I put A5 here, I have to put letter A5 here, A6, A6. That makes a symmetric matrix, right.

So, this is just for convenience I write it in this way, because in the proof we will be manipulating with these numbers. So, such a thing is a quadric surface. And if I expand this in terms of this matrix A R and B and C, then I get equation like this.

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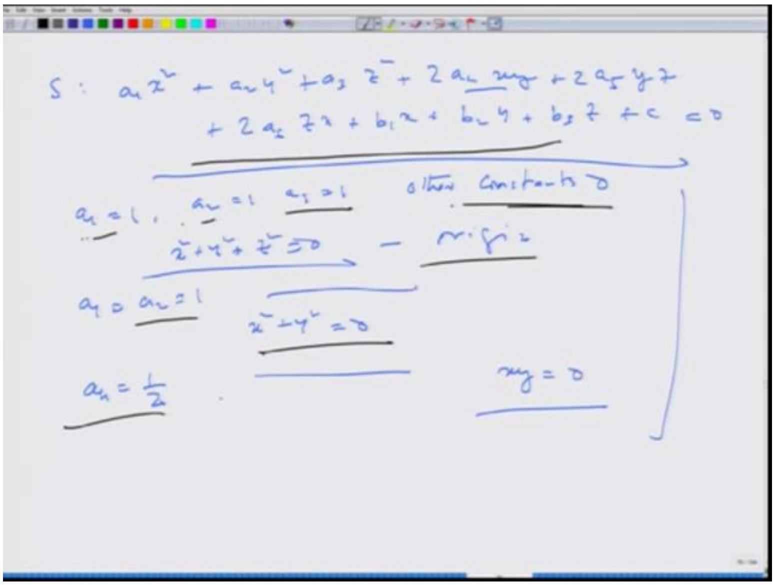

And we have noted that it may not be always a surface because in particular cases when A1 equal to 0 equal to 0, XT equal to 0, the constants are 0, sorry, A1, A2, A3 are 1 and constants are 0, this becomes X square plus Y square plus Z square equal to 0, this becomes origin. Similarly can be a circle, it can be pair of straight line.

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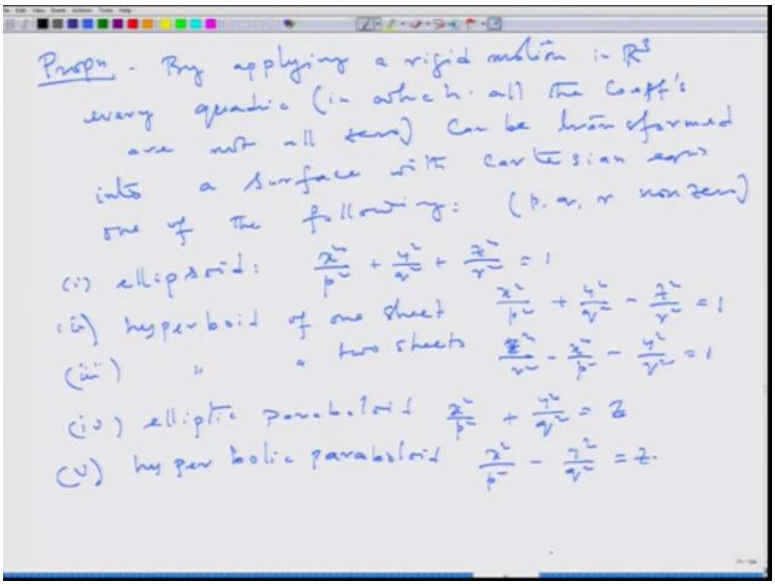


So, accordingly we wrote down this proposition. Just read it again, that by applying a rigid motion in R3, every quadric surface in which all the coefficients are not all 0, so I am not looking at the equation Z equal to 0.

Can we transform into a surface with Cartesian equation? One of the following, where P Q R nonzero constants. 1st of all it can be an ellipsoid, 2nd, so possibilities there are 10 possibilities, right. I note down somewhere.

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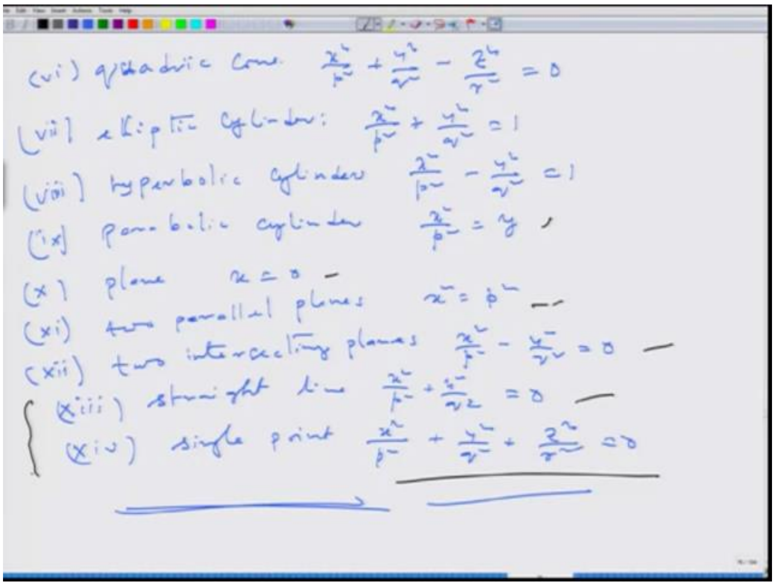


Sorry, 14 possibilities are there. And I listed down all the possibilities, ellipsoid, hyperboloid of one sheet, hyperboloid of 2 sheets, elliptic paraboloid and equations are written accordingly, hyperbolic paraboloid, this is 5th one. 6th is quadric one, a quadric cone, 7th is elliptic cylinder, hyperbolic cylinder, parabolic cylinder, this one, it can be a plane, 2 parallel planes, 2 intersecting planes, straight line, a pair of straight line, the suggested straight line, X square plus Y square, so this is a straight line or a single point.

In this case, in last 2 cases they are not surfaces in R3. So, how does one prove this? As I said this all depends on this since I have a symmetric matrix A, let me write down A again.

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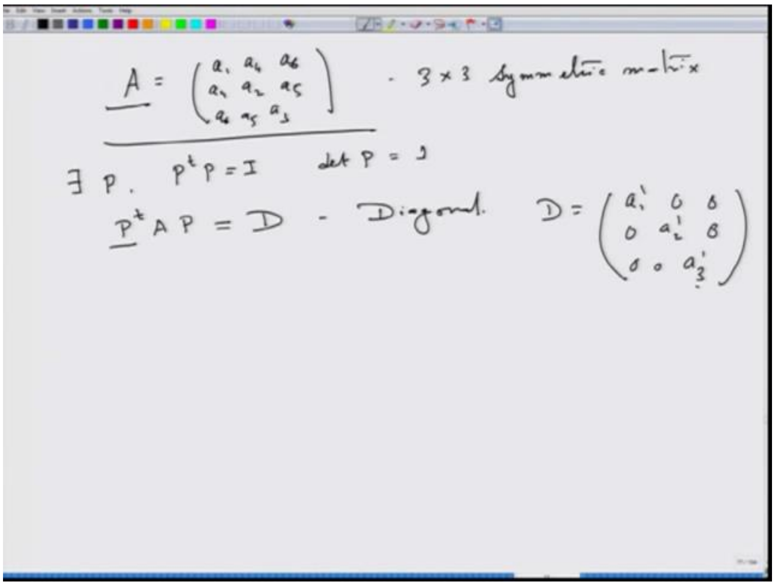

A1, A2, A3, A4 here, so A4 here, A5 here, so A5 here, A6 here, so A6 here. Now, all you know if I have a symmetric matrix 3 cross 3 symmetric matrix, then my knowledge of linear algebra tells me that there exists a orthogonal matrix actually, there exists a matrix P such that P transpose P is identity, also determinant P is one and P transpose AP is a diagonal matrix. Right. So, let this diagonal matrix D be A1 prime, A2 prime, A3 prime, rest entries are 0.

So, any 3 cross 3 symmetric matrix with diagonal nonzero with real entries. I mean, these eigenvalues are there, these are eigenvalues, right, A1 prime, A2 prime, A3 prime R eigenvalues of A and corresponding columns of P transpose are the Eigen vectors of A. So, P transpose or rows of P, they are Eigen vectors of A.

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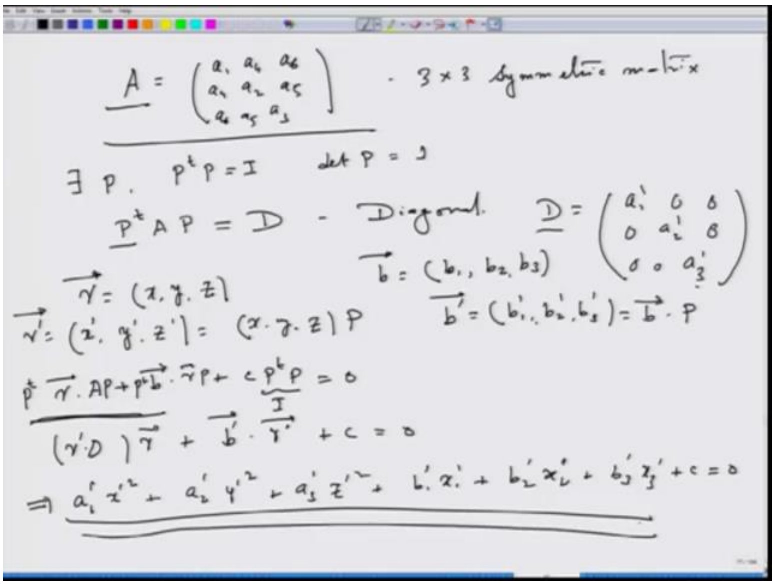


So, what we do, we will make a transformation that our R was X, Y, Z, what I do I put X prime, Y prime, Z prime correspondingly as X, Y, Z, this row vector P, this is a row vector again. Similarly our B was, we have taken B1, B2, B3, so I take, so let us put it R Prime, similarly I put B prime, B1 prime, B2 prime, B3 prime is equal here I put the vector sign also.

So, what will happen to the original equation? What was the original equation? R dot A plus B dot R Plus C equal to 0, right. So, if I multiply P transpose on this size and P, so I put a P transpose P and P transpose, so C does not matter, so this identity, C remains C equals P transpose P is identity. C is a constant but this fellow is identity, this is 0. Now, what will happen according to these equations? This will become now R Prime D, R Prime plus B Prime R Prime plus C and now D is this matrix, so this becomes A1 prime X prime square plus A2 prime Y prime square A3 prime Z prime square plus B1 prime X1 prime B2 prime X2 prime plus B3 prime Z3 prime, sorry, X3 prime plus C equal to 0.

So, this equation I have reduced into this and for reduction what I have done, I have applied some rigid motion in R3 because multiplying with a orthogonal matrix, P is orthogonal, P transpose is orthogonal, multiplying in orthogonal matrix will give you a rotation around 0. So, that is a rigid motion of R3. So, by a rigid motion, that is a rotation around origin, I have reduced the original equation of the quadric surface to this one.

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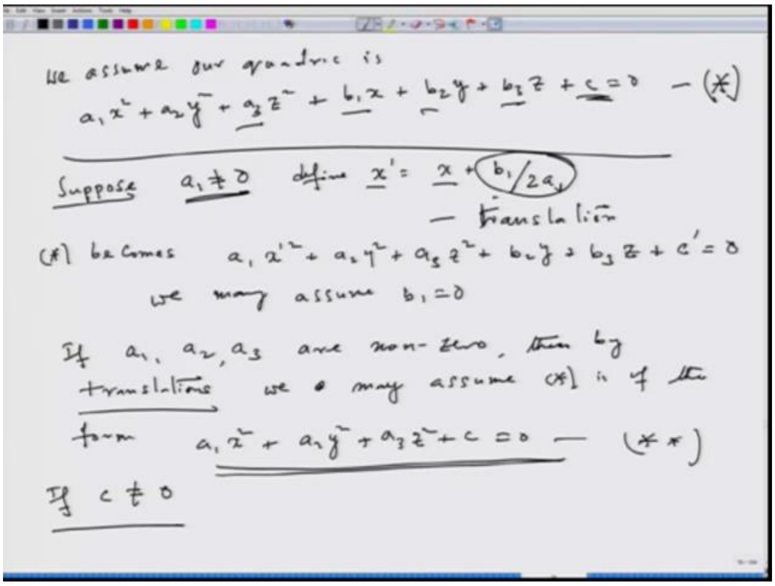


So, let me now get rid of this prime and we will assume that our surface is, our quadric is A1 X square plus A2 Y square A3 Z square plus B1 X, just for avoiding this writing prime everytime, B2 Y plus B3 Z plus C equal to 0 so, I have got the nothing, I have just written this equation instead of X1 prime, instead of prime I have just get rid of prime and assumed that our equation is giving this. So, let us call this a star.

Now, observe, how do I reduce all the cases. A1 is not equal to 0, then the idea is I will get rid of B1. How? Define X prime equal to X plus B1 by 2 A1 which is a translation. Right. Then I can write this equation, star becomes, you put X X equal to X prime minus this, so X square there, so what will happen, star becomes A1 X prime square plus A2 Y square, these are not touched Z3 square, sorry Z square plus B2 Y, B3 Y, B3 Z is not touched, this constant will change because they are here, something here, some C prime equal to 0.

Try that this will… Just put instead of X you put X prime minus B1 - 2 A1. So, this is B1 X term will go out. Correct. Because 2 X1 B1 divided by 2 A1, coming, A1 we cancel that and this negative, that will cancel B1 X. So, we can assume B1 is 0. If A1 equal to not equal to 0, we may assume B1 is 0, B1 is not there, I can get rid of it by a translation. Similarly for, if A2 is not equal to 0, I can get rid of B2, A3 is not equal to 0, I can get rid of B3, so and so if A1, A2, A3 are all nonzero, then by tree translation, I need exactly tree translation, we may assume star is of the form A1 X square A2 Y square plus A3 Z square plus some constant equal to 0 on interval star. Right.

So, if C not equal to 0 what happens? Let us go back, so this is my after rotation and translation, my surface becomes this, C not equal to 0.

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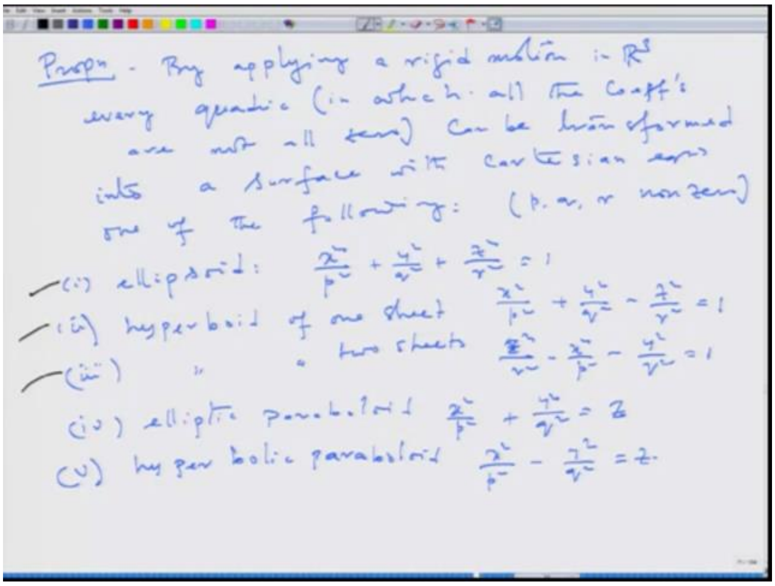


Then you see depending on the sign I can normalise, so depending on the sign of A, A1, A2, A3, we will get case 1, case 2, case 3. I will divide by C, minus C, I will get one here, one on the right inside and depending on this case, so what I mean here

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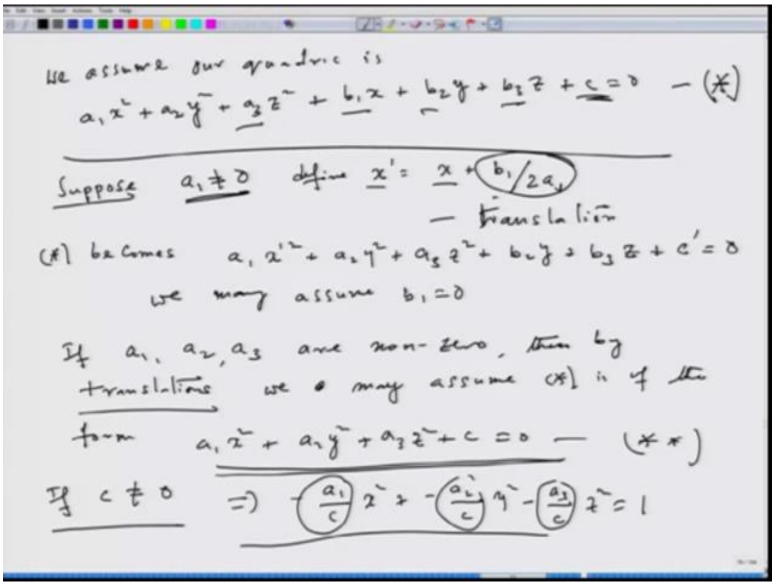


That now, take this C right-hand side, so this is actually minus A1 by C1, sorry C X square plus minus A2 C Y square A3 by C Z square equal to is equal to 1. So, this is my, these are my one by P square, one by Q square, one by R square. So, depending on sign of A1, A2, A3 I will get either ellipsoid, hyperboloid of one sheet or hyperboloid of 2 sheets. Correct. What happens when C equal to 0? You can see from now.

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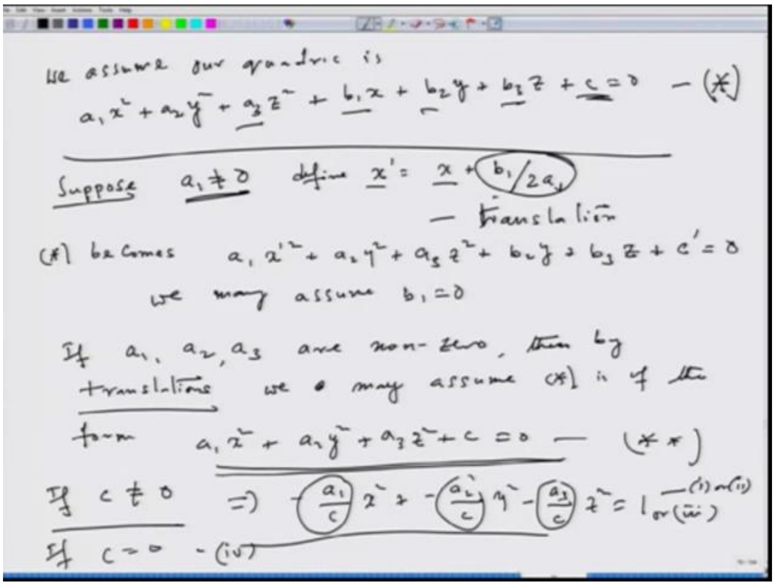


If C equal to 0, so C not equal to 0 I get case 1 or 2 or 3. If C equal to 0, then we will get 4 elliptic paraboloid, C equal to 0, sorry, not 4, 6, right, yah this one.

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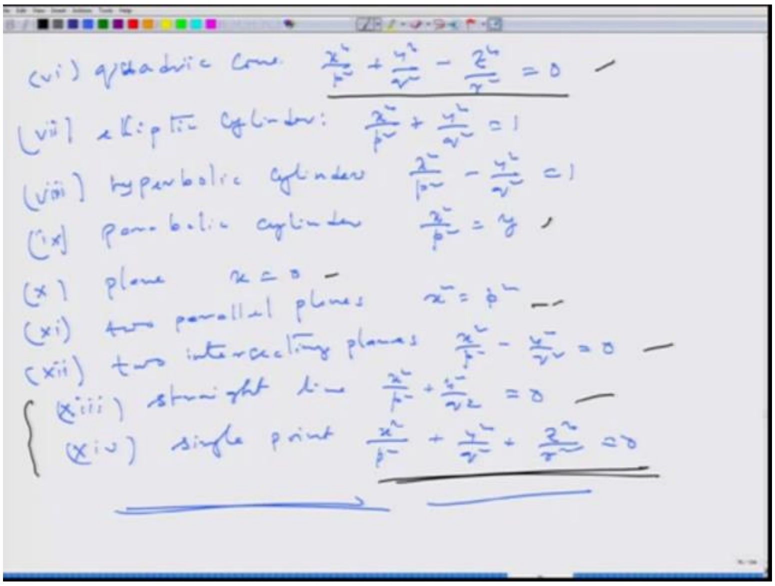


Either 6 or there is another one. That is or this one, 14. C equal to 0, so this is 6 or 14. A quadric cone or a single point. Okay, let us go ahead. So, we covered what are the cases, 1, 2, 3, 6 and 14. Other cases.

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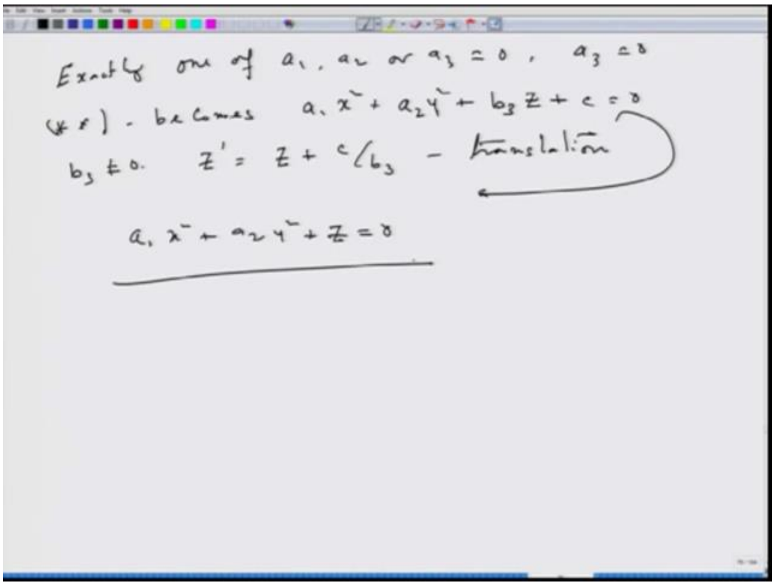


Let us say exactly one of A1, A2 or A3 equal to 0, say let us say A3 equal to 0. Then double star becomes A1 X square A2 Y square, now let B3 Z will remain plus some C equal to 0, B3 I cannot get rid of if A3 is 0. Correct. Now, B3 itself is nonzero, we make another translation, Z prime equal to Z plus C by B3. That will reduce this fellow to A1 X square plus A2 Y square plus Z equal to 0. This gives what?

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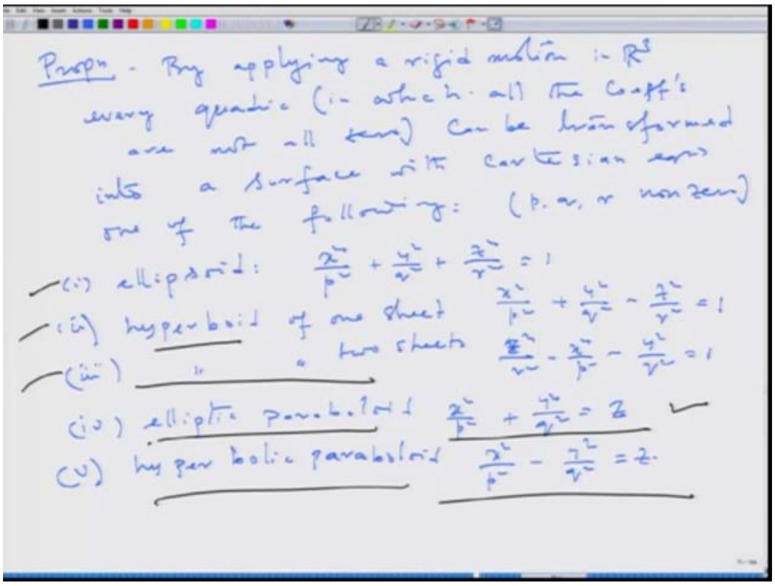


Let us go back, what does it give? A1 square, A1 X square, A2 Y square plus Z equal to 0. So, this is either this case, as elliptic paraboloid or this case, hyperbolic paraboloid, so 4 or 5 is covered. So, you get either 4 or 5.

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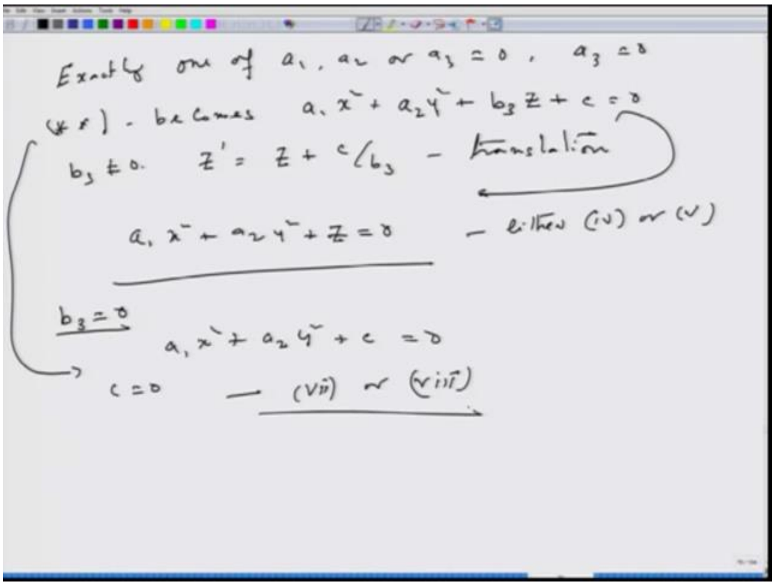


Now, B3 equal to 0. Then this equation becomes A1 X square plus A2 Y square plus C equal to 0. Now, C equal to 0, this gives, which one according to our number?

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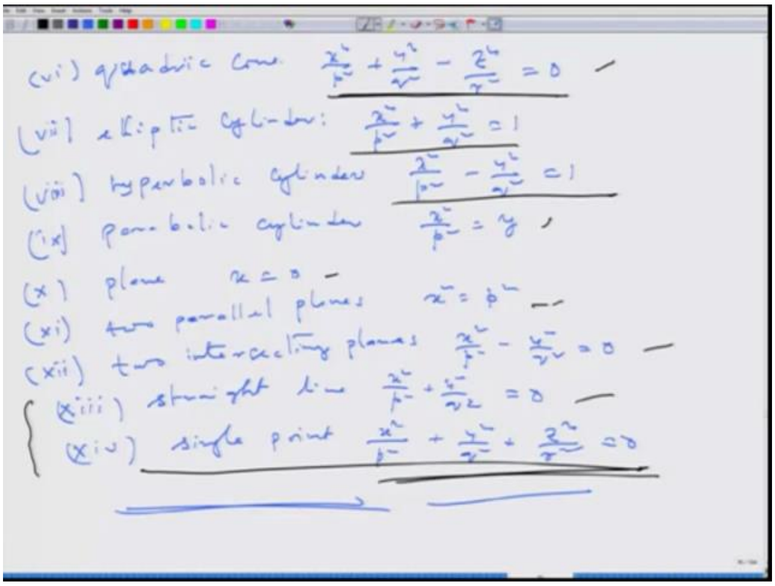


C equal to 0 C equal to 0 will give us 8, 7 or 8, right. So, you have to turn your pages everytime to go back to the equations. Now, suppose 2 organ are 0.

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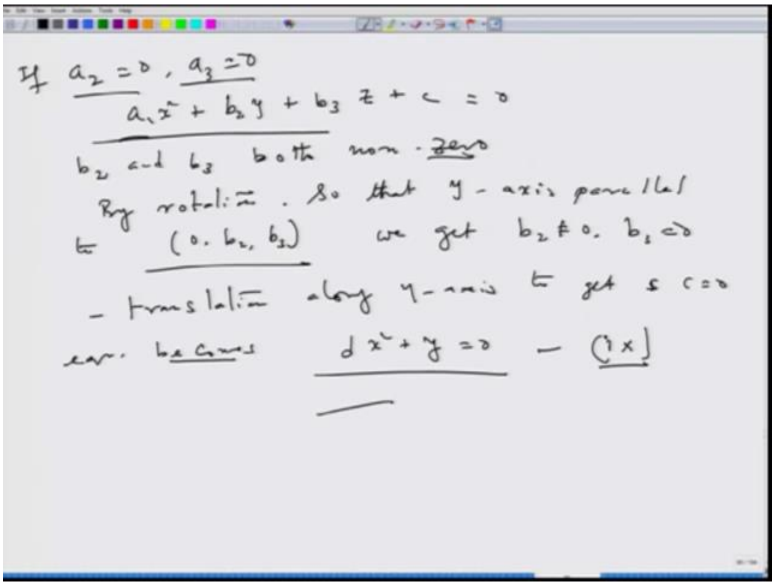


If A2 equal to 0 and A3 equal to 0, you know the trick now, then actually, the equation becomes A1 X square, B2 will remain, B3 will remain plus C equal to 0. Remember, whenever we have A1, one of the A1, A2, A3 nonzero, we can remove B1, B, corresponding Bs. So, A2 equal to 0, A3 equal to 0 I cannot remove B2 and B3, so B2 and B3 will remain. Now, if B2 and B3, both nonzero, then by rotation, by rotation so that Y axis becomes parallel to the vector 0 B2 B3, we get B2 nonzero B3 as 0. Now, apply a translation, another translation along Y axis to get C equal to 0. So, equation becomes A1 X square plus Y equal to 0, I have normalised it actually. Not, this is not same A1, I have normalised it, maybe some other A, I mean some DX square plus Y equal to 0.

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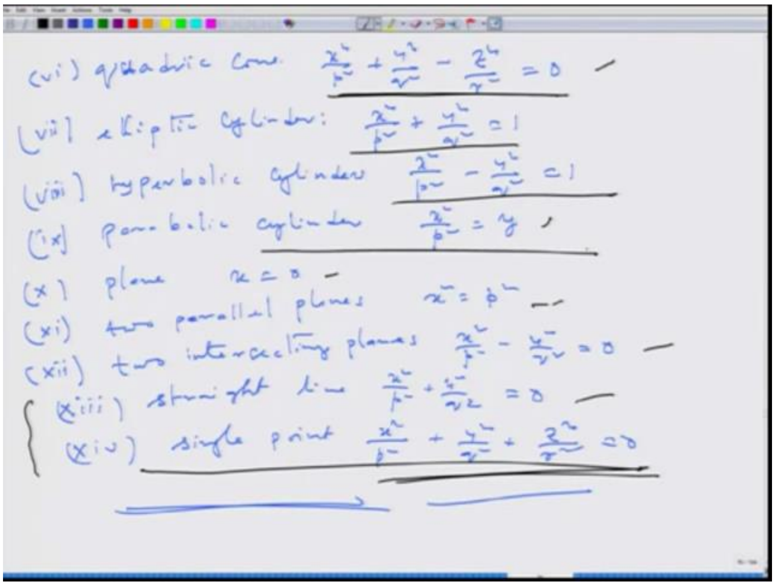

So, it gives which case, which one? Let us go back. This one, right. So, this gives case 9. What was it, 9 or 11? 9.

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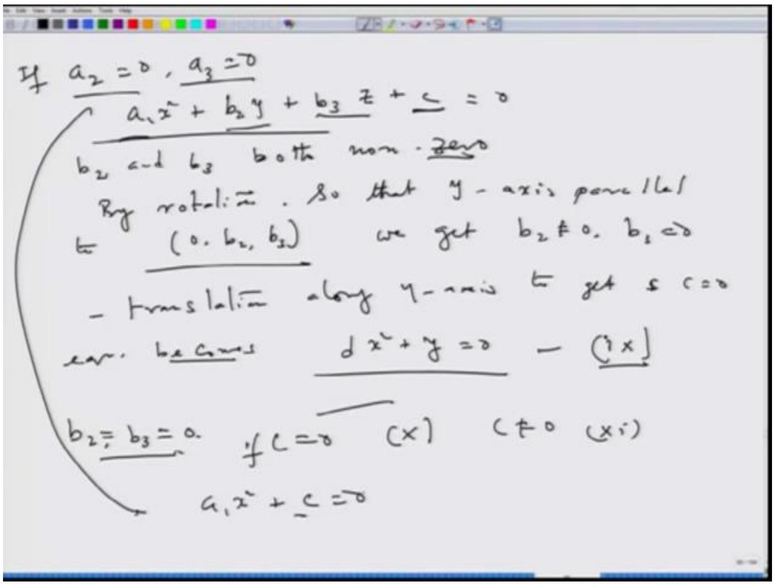

Now, if B2, B3, both 0, then in this equation B2, B3 is 0, C will be 0, A1 is nonzero, so C is 0, so I get, so, if C is 0, I will get case 10 and see not equal to 0 gives case 11. What is the equation? This equation becomes A1 X square plus C equal to 0, so C is 0, so A1 X square equal to 0.

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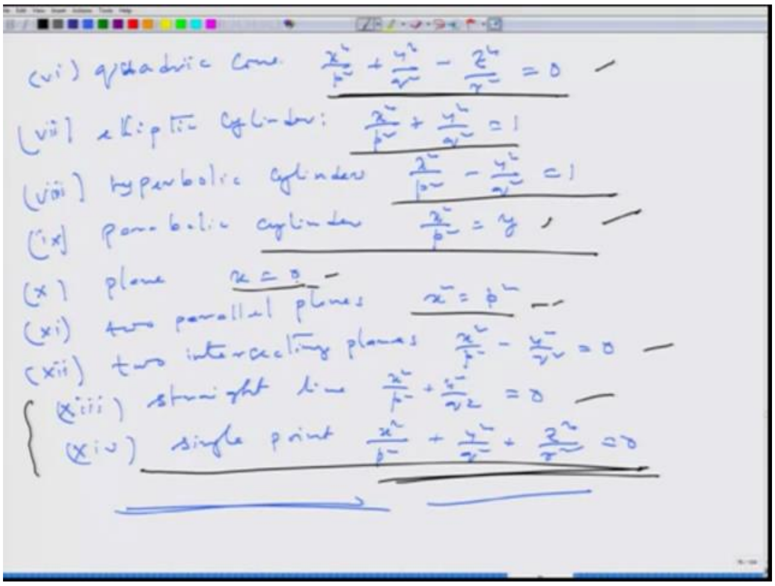


So, this is 11 and C equal to 0 just gives the plane X equal to 0. One more case left, what is that? A1 equal to A2 equal to A3 equal to 0.

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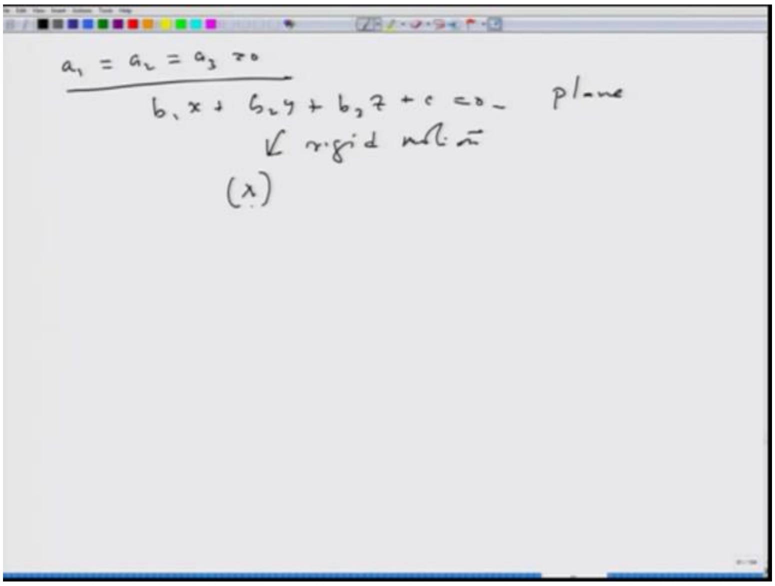


Then my equation is simply B1 X plus B2 Y plus B3 Z place equal to 0. So, this is a plane, so, after rigid motion, any plane is gives me on the form 10.

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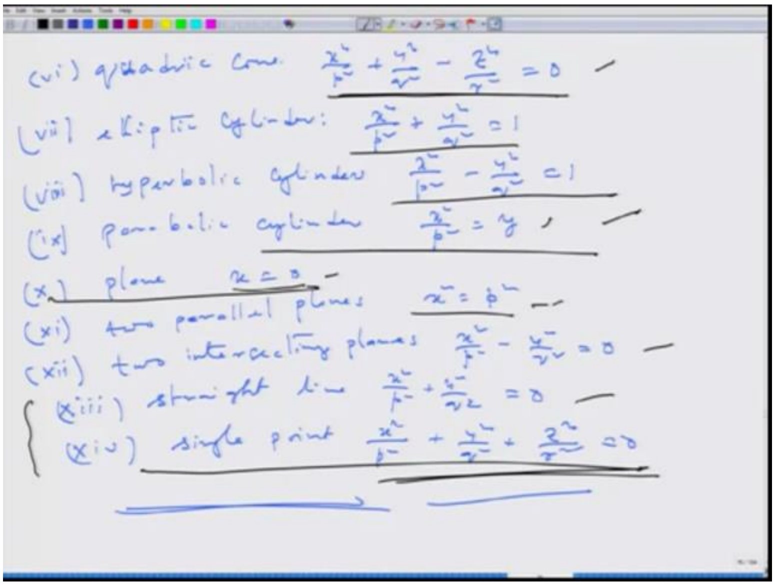


Any plane, so they can assume after rigid motion is YZ plane. This is YZ plane, right. So, all the cases are covered. So, this is the proof for so-called classification of quadric surface. Quadric surface is very important because we know every regular surface locally, we have done this thing, done this exercise after doing principle normal, so after doing principle curvatures, so depending upon principal curvatures K1 and K2, every surface locally looks like one of the quadric surface and I have only 14 possibilities of quadric surfaces. So, any surface locally in one of these 14, any regular smooth surface is locally one of these 14.

But when you patch together it may around one point it may look like a elliptic cone around one point it may look like look at and at another point it may look as Elliptic paraboloid or hyperboloid of one sheet or hyperboloid of 2 sheets but locally they will be one of them. So, that is a point, so that completes the roof of this theorem. And that is the end for this lecture. In the next lecture I will talk about, which I have not covered, something on surface area and equilateral maps.