Functional Analysis

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Module No. # 01
Lecture No. # 11
Finite Dimensional Normed Spaces and Subspaces

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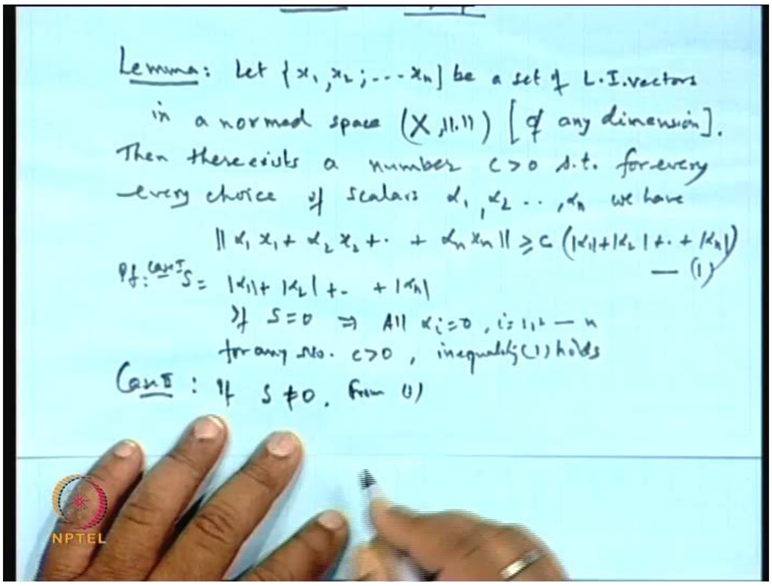

So, today, we will take up the finite dimensional normed space and subspaces. You know, we have seen so many examples of the normed space like R n, C n, then l 0, l infinity, l p and so on and so forth. Some of them are of finite dimension like R n and C n, where the others are infinite dimensional. So, we will first take up the case, when the dimension of the normed space is finite and we will see that, in what respect these finite dimensional normed space is much, is simpler than the infinite dimensional case. In fact, there are many branches, where we use the finite dimensional normed space only, just like approximation theory, spectral theory and so on. So, there we require the finite dimensional case.

Now, in order to develop the results or to useful results for a finite dimensional case, we require a lemma, which is the backbone for this total study and that lemma is as follows. Lemma says that, let x 1, x 2, x n, x 1, x 2, x n be a set of linearly independent vectors in a normed space X, of any dimension, of any dimension, dimension, then, there exists, there exists a number C, a real number C greater than 0, greater than 0, such that, for every choice, for every choice of scalars alpha 1, alpha 2, alpha n, we have norm of alpha 1 x 1 plus alpha 2 x 2 plus alpha n x n is greater than equal to C times mod alpha 1 plus mod alpha 2 plus mod alpha n, ok.

So, this lemma, what is the meaning of this lemma? What does it tells to us? x 1, x 2, x n are given to be a linearly independent vectors and what we are taking is, the length of the vector, which is the linear combination of x 1, x 2, x n. So, what this lemma says that, in case of a linearly independent vectors, one cannot find a linear combination of vectors x 1, x 2, x n, involving large number of scalars, alpha i's, but with a minimum length, minimum, small vector, because it will always be greater than equal to C times sum of these scalars. So, you cannot expect that, a large number of scalars are involved, using this linear combinations of the vectors, but the length of the vector cannot be so small; it will be greater than equal to certain number C greater than 0. So, that is the consequence of this lemma.

The proof of that lemma goes like this. Let us suppose, S is alpha 1 mod alpha 2 plus mod alpha n. Now, if S is 0, it means, all alpha i's are 0's, all alpha i's are 0. So, once all alpha i's are 0, then, obviously, for any number of C, so, for any number C greater than 0, this inequality 1, 1 holds, is it not? For any C. Because, left hand side is also 0, right hand side is also 0. So, whatever the C you put it, greater than 0, it is true. So, there is nothing to prove and or sum of these are, S is 0, means, all alphas are 0. But second case, if this is case one, if S is not equal to 0. So, if we can divide the equation 1 by S, this term by S, so, from equation 1, yes, so, if S is not equal to 0, then, let us divide the each term by S.

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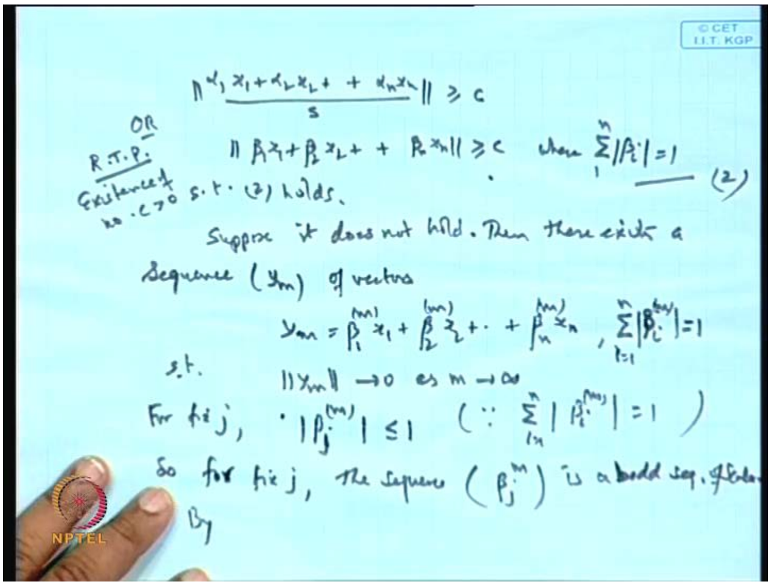

So, from 1, we get alpha 1 x 1 plus alpha 2 x 2 plus alpha n x n divided by S, norm of this is greater than equal to, say mod of C, is it not; greater than equal to C. Or, we can say like this that, norm of beta 1 x 1 beta 2 x 2 plus beta n x n is greater than equal to C, where sigma of beta i, sigma of beta i is equal to 1, 1 to n, because, what is the mod of beta i? Beta 1 is alpha 1 by S; beta 2 is alpha 2 by S and beta n is alpha n by S, ok. So, if we take the mod of beta i, i is 1 to n, this equal to 1. So, it is required to prove, required to prove, is the existence of C, which satisfies the equation 2; the required prove is existence of a number C greater than 0, such that, 2 holds. So, finally, we are landing to the proof of equation 2, ((for sense)), ok.

Now, suppose, this is not true; suppose, suppose, this is not true; suppose, it does not hold; it does not hold ((or)). It means that, there must be a some sequence of beta 1, beta 2, beta n, for which sigma of this is 1, but the length of this is not greater than equal to C or greater than equal to 0, is it not. So, it means, there exists, then, there exists a sequence, there exists a sequence, say y m, y m of vectors, where the y m is beta 1 m x 1, beta 2 m x 2, beta n m x n, x n, where the sigma of beta i m, i is 1 to n is equal to 1. So, suppose, it does not hold means, for any set of scalar, we are claiming that, this C must hold good; there will exist some C, for which the equation 2 is true. Suppose, this is not true. It means, there exist a sequence y m of vectors, where y m can be represented in this form; sigma beta i is here, such that, norm of y m, this must go to 0 as n tends, m tends to infinity, ok. So, in case of the 2 does not hold good, then, we can find a sequence, where norm of y m tends to 0. Now, for the fix j, this sequence…

Sir, (( )) m is power on beta?

m is, yes, not power; it is basically, it shows the, corresponding to y m, we are using the scalar index. If suppose, y 3, then, b 1 3, beta 2 3, beta like that. So, it is not a power; it is an index. You can just put it within bracket also, if you have some confusion, like this. So, suppose, this is bracket, ok. Now, since the sigma beta i m is 1, it means, beta i m for fix j, or fix i, this sequence beta j m, this sequence, mod of this, is less than equal to 1, for this fix j, because of this result, because sigma of this thing is 1. So, this one is less than equal to 1. So, it means, for fix j, for fix j, the sequence beta j m is a bounded sequence, is a bounded sequence, is it not. This is a bounded sequence of a scalars. And, we know by Bolzano-Weierstrass result that, every bounded sequence has a convergent subsequence.

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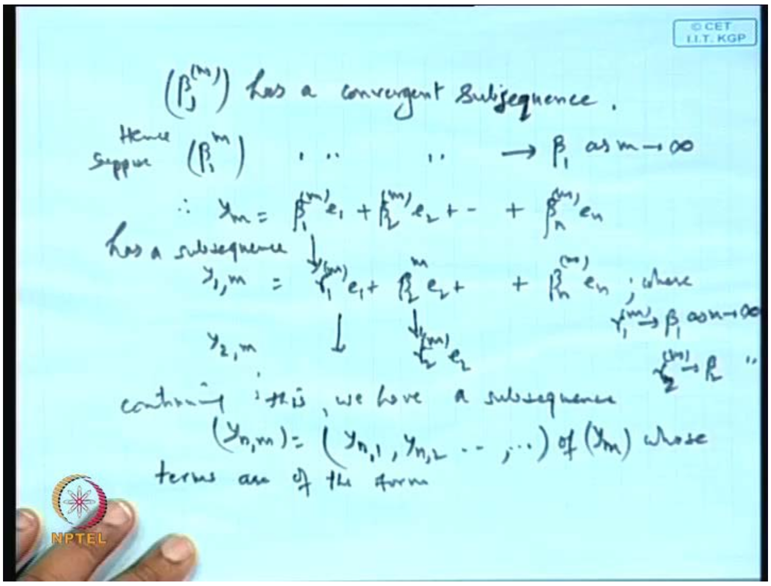

So, by Bolzano-Weierstrass theorem, yes...So, by Bolzano-Weierstrass, e r s, Weierstrass, Bolzano-Weierstrass theorem, the sequence beta j m, the sequence beta j m, has a subsequence, has a convergent subsequence, subsequence, has a convergent subsequence. So, this, for fix j, it has a converge. So, for each j, it is true. Hence, beta 1 m and suppose, beta 1 m has a convergent subsequence, which tends to beta 1, as m tends to infinity; beta 1 m has a convergent subsequence. Therefore, the y m, which is equal to beta 1 m e 1, beta 2 m e 2, beta n m e n, this sequence will have a subsequence y 1 m, which is, say, this subsequence, say, suppose, gamma 1 m e 1 and then, this is ok, beta 2 m e 2, beta n m e n, because this is a bounded sequence, beta 1 m. So, it has a convergent subsequence, say gamma 1 m, where the gamma 1 m, gamma 1 m, this goes to, gamma 1 m will tends to beta 1, as m tends to infinity, is it ok or not.

By Bolzano-Weierstrass property, Weierstrass theorem, if a sequence is a bounded sequence of real numbers, then, it has a convergent subsequence, converging to certain point. So, this y, beta 1 m has a subsequence, say gamma 1 m, which goes to beta 1, as m goes to infinity. Clear? Hence, we have a sequence y 1 m, which is a subsequence. So, y 1 m has a subsequence, say y 1 m, which has this representation. Continue this. Now, again, y 1 m, because, it has a beta 2 m. So, beta 2 m has a subsequence, say gamma 2 m, which goes to beta 2, which goes to beta 2, as n tends to infinity. So, correspondingly, we have y 2 m, is again a subsequence and continue this upto, say n step; then, we have, we have a subsequence y n m, a subsequence y n m, which is equal to say, y n 1, y n 1, y n 2, y n, like this, of y m, of, say y m, whose terms are of the form, of the form y n m, is equal to sigma i is 1 to n, gamma i m x i, where the sigma of gamma i m, i is 1 to n, is 1, with scalars gamma i, is it not.

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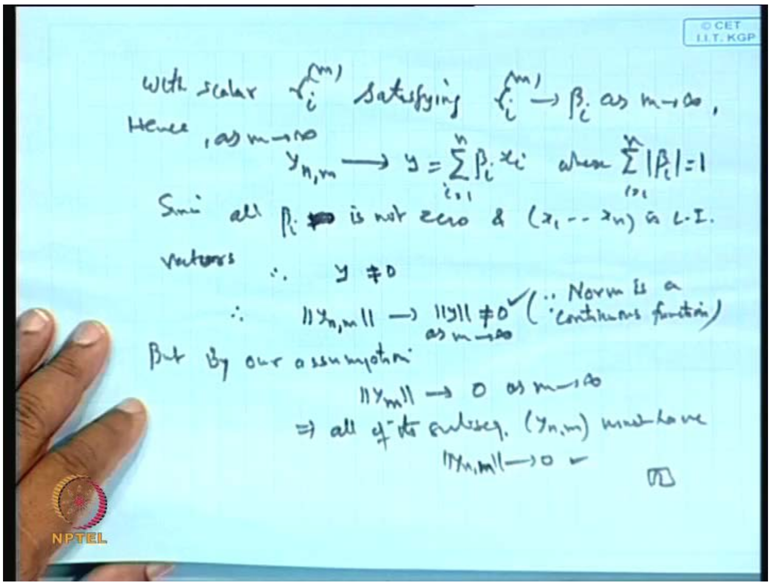

So, this sequence y n m, it will be a subsequence of y m. Here, y n m is this. Sigma of gamma i m is 1 and with scalars, with scalars gamma i m satisfying the condition that, gamma i m goes to beta i as m tends to infinity, as m tends to infinity. Hence, basically, we get that, the sequence, as m tends to infinity, the y n m tends to a point y, which is sigma beta i x i, i is 1 to n and sigma of beta i is 1; mod of beta i, i is 1 to n is 1. So, what we have assumed is that, there exists a sequence y n, which does not follow the lemmas' result. It means that, sequence y n m goes to 0, as m tends to infinity. So, where the sigma of beta i's m is 1. So, using the Bolzano, we can construct a subsequence and the subsequence is coming like this, where it goes to y, clear. Now, x 1, x 2, x n are linearly independent vectors; sigma beta i is 1. So, all the beta i cannot be 0, clear; all beta i cannot be 0. This is linearly independent set. So, it means that, linear combination of this vector y cannot be 0.

So, since, since all beta i's is not equal to 0, all beta i's is not equal to 0, or since all beta i is not equal to 0, is not 0, all beta i is not equal to 0 and x i, x 1, x 2, x n is a linearly independent, set of linearly independent vectors, therefore, the vector y is not equal to 0 vector, ok. Hence, the norm of y n m, which tends to norm of y is not equal to 0, because norm is a continuous function, continuous function; this I will show you. So, if this sequence goes to here, corresponding norm will go to norm of y. Now, since y cannot be 0 vectors, so, norm length cannot be at zero length; it will be non-zero, clear. So, norm of y n will be different from zero. But, by our assumption, assumption, the sequence y m is such that, norm of y m tends to 0, as m tends to infinity.

It means, all of its subsequence must go to 0. So, this implies, all of its subsequences, that is, y n m must go, must have this property that, norm of y m must go to 0, is it not; otherwise, it will contradiction the limit property. So, here we are getting, this goes to 0; here, we are getting, this does not go to 0, as m tends to infinity. So, this itself, shows a contradiction. And, contradiction is because, our wrong assumption that, we are able to get a sequence y n, which, which have, norm of which goes to 0, where y m is the linear combination of vectors involving large number of scalars, but the length of this, is a minimum vector. So, that assumption is wrong. It means, we cannot obtain a vector involving the large scalars and a minimum length 0. So, this proves the result, is it clear. Now, this result, lemma, has a wide application in establishing so many result, in case of finite dimensional space. One can establish beautifully, by using this lemma, this results, which are very very important in case of the finite dimension.

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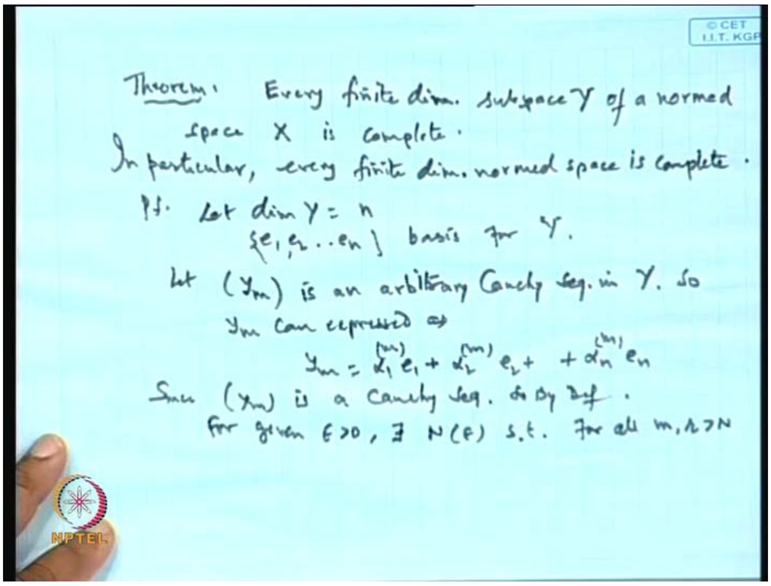

So, the first result, which we had, is in the form of theorem. Every finite dimensional, every finite dimensional subspace capital Y of a normed space, of a normed space X is complete. That is, in particular, you can say, every finite dimensional normed space is complete, is complete. So, if the space is finite dimensional, then, by this result, you need not to prove the completeness, because every finite dimensional space is complete by this result. So, R n is a finite dimensional, is a complete space; R 1 is complete; C 1 is, C is complete; R n is complete; C n is complete; any finite dimensional, if we picked up, it will be a complete normed space. So, that.

So, in order to prove the completeness, what we will prove is that, every Cauchy sequence in this, if it is convergent, then, we say the space is complete. So, let us see the proof. First, we have assumed the subspace Y is of finite dimension. So, assume, let the dimension of Y is, suppose n; and let e 1, e 2, e n, these are the basis elements, basis for Y. So, any element of Y can be expressed in terms of the linear combination of the basic element. Now, we want this Y to be complete. So, let us choose an arbitrary Cauchy sequence. So, let y n m is an arbitrary Cauchy sequence in Y. Let it be an arbitrary Cauchy sequence. Since y m is a point in Y and Y has a basis e 1, e 2, e n, so, y m can be expressed as…So, y m can be expressed as a linear combination of the elements of basis.

So, alpha 1 e 1, alpha 1 m, because it corresponding to y m, alpha 2 m e 2 plus alpha n m e n. So, for each y m, we have a unique representation for this, ok.

Now, since the y m is a Cauchy sequence, so, apply the condition of Cauchy sequence. So, for, since y m is a Cauchy sequence, so, by definition, for given epsilon greater than 0, for given epsilon greater than 0, there exists, there exists an N, depending on epsilon, such that, for all m n r, sorry, for all m r greater than N, we have norm of y m minus y r is less than epsilon for, or less than epsilon, is it not.

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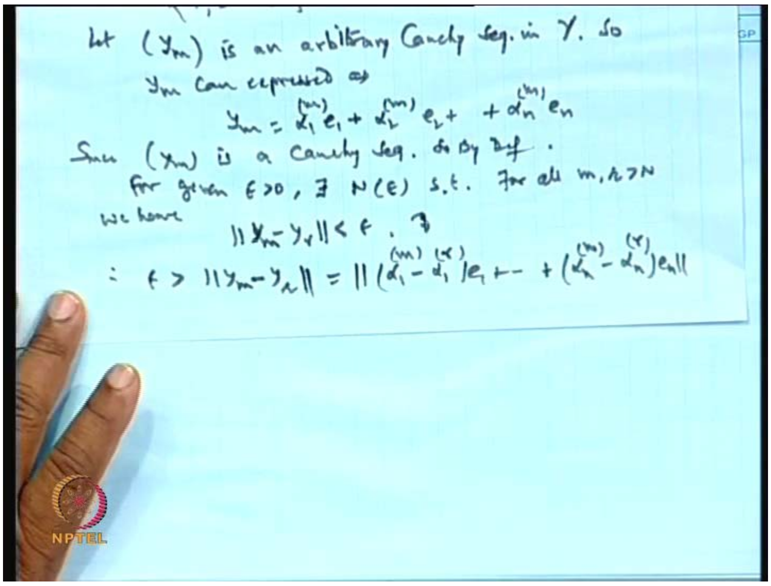

If y m is a Cauchy sequence, then, for a given epsilon, there exist an N, depending on epsilon, such that, for m and r greater than N, we have this. It means that, epsilon is greater than this, epsilon is greater than this. Now, write down this expression. So, y m corresponding to this expression; y r will correspond to alpha 1 r e 1, alpha 2 r e 2, alpha n r. So, when you subtract, you will get like this; alpha 1 m minus alpha 1 r e 1 and like this; and rest, last term will be alpha n m minus alpha n r e n, like this, is it not. This is our norm of this, ok; which is, can be written as…

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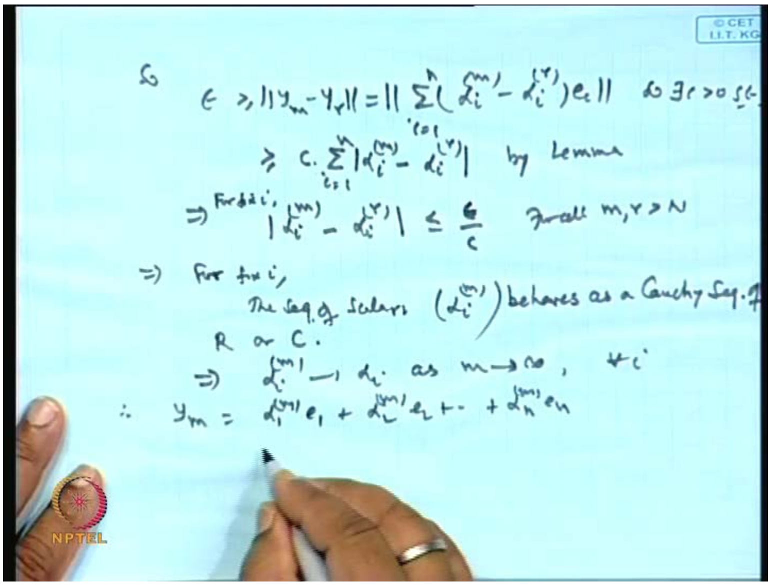

So, we get, epsilon greater than equal to norm of y m minus y r, which is equal to norm of sigma alpha i m minus alpha i r e i, i is 1 to n, and this is greater than equal to C times sigma of mod alpha i m minus alpha i r by the lemma; because e 1, e 2, e n is a linearly independent vectors and these are simply scalars. So, a linear combination, finite linear, linear combination of these vectors is there. So, we can find a C. So, there exist a C greater than 0, such that, this is true, by lemma, i is 1 to n. Therefore, from here, we can say, mod of alpha i m minus alpha i r is less than equal to epsilon by C, for all m and r, greater than N, for fixed i. So, for fixed i, we get this term, is it not; sigma is less than, here; so, for each i, you can get this one. What does it mean? It implies that, for each i, for fixed i, the sequence of scalars alpha i m behaves as a Cauchy sequence, Cauchy sequence of real or complex numbers, is it ok; because, this is the Cauchy, definition of Cauchy; difference between any two term after certain stage is less than epsilon. So, it is a Cauchy sequence, but real and complex number, it is complete. So, this Cauchy sequence must be convergent.

So, alpha i m goes to alpha i, as m tends to infinity; because this is a Cauchy sequence in a real or complex is a convergent sequence. Therefore, for each i, this is true for each i, we get. So, what we get is, now... So, the sequence y m, which is alpha 1 m e 1, alpha 2 m e 2, alpha n m e n, this will go to the element y, which is of the form alpha 1 e 1, alpha 2 e 2, alpha n e n, as m tends to infinity, clear.

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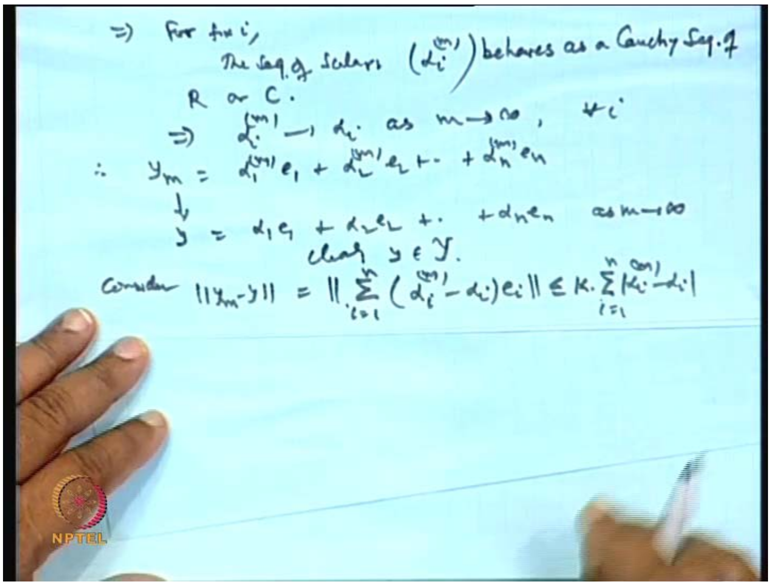

Now, y is a linear combination of this basis element e 1, e 2, e n. So, is it not a point of this space? It belongs to Y? So, obviously, this is an element of Y. Clearly, y belongs to capital Y, is it clear. Now, the question arise, whether this convergence is in the norm of y or not; because, if it is in the norm of y, then, the every sequence has a convergence, of convergence, every Cauchy sequence converges in the norm of y. So, we have to show that, this convergence is in the norm of y. So, let us take the norm of y m minus y. Now, this can be written as norm of sigma i is 1 to n, alpha i m minus alpha i e i. Now, this will be less than equal to some constant times k sigma of norm alpha i m minus alpha i, i is 1 to n. Here, what is k? k is, here, what is k? k is the maximum value of, yes, where k is the maximum value of the norm e i, when i varies from 1 to n, ok.

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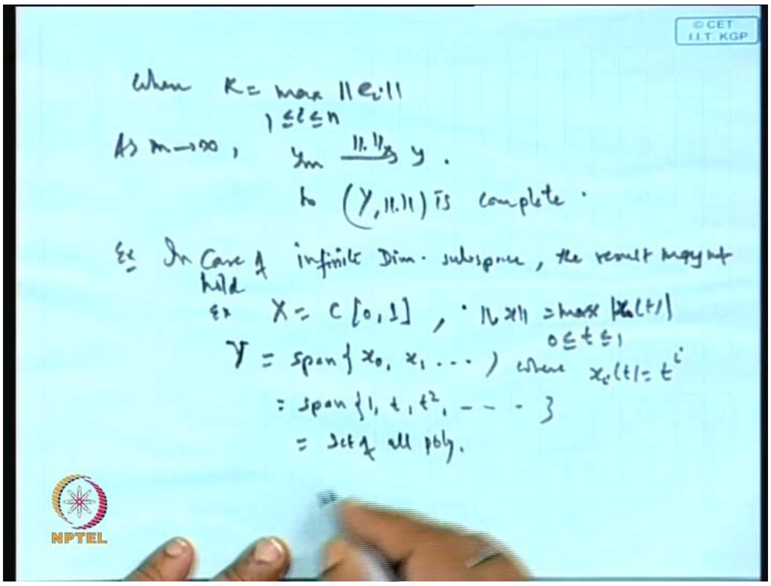

So, we are taking, where k is the maximum of norm e i, i is 1 to n. So, this is the maximum value of e i and (( )). Now, let us look this again. If we look this equation, say equation 3, as m tends to infinity, alpha i m is tending to alpha i, because of this result, is it not? Because of this A, already shown. So, this part is tending to 0, this is finite number and k is already finite. So, basically, as m tends to infinity, y m go to y, under the norm. So, as m tends to infinity, y m tends to y in the norm of y and this completes this. So, Y is complete, is it clear? Now, this result is valid for, in case of the finite dimensional; that is, if we take any finite dimensional subspace of a normed space, then, it must be complete; but if it is not finite dimensional subspace, then, it may or may not be complete. So, in case of the infinite dimensional subspace, in case of infinite dimensional subspace, the result is not true; result may not hold, may not hold, ok.

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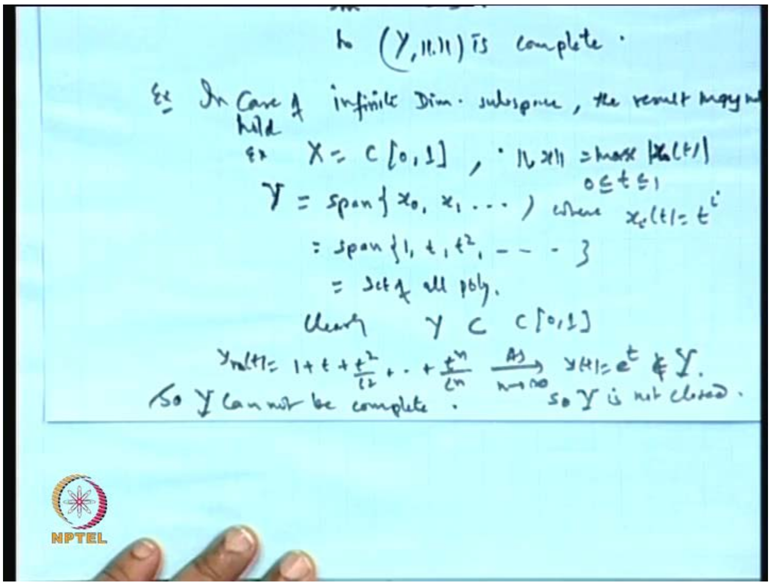

For example, if we take the space X as C 0 1, set of all continuous function defined over the closed interval 0 to 1, and this is a complete under the metric, norm of x as the maximum of mod x i, maximum of mod x i t or x t, maximum of mod x t, where the t ranges from 0 to 1; under this norm, it is a complete metric space, clear. Now, I am choosing the Y, as the span of x naught, x 1, x 2 and so on, where x i t is equal to t to the power i. So, I am taking this Y as the linear combination of the elements of 1, t, t squared and so on. So, basically, it is the span of 1, t, t square and so on, clear. So, it means Y is a polynomial; set of all polynomials, because when you are taking a linear combinations of this 1, t, t square etcetera, you have to picked up a finite number of the points only at a time; and once you take the finite number of point, the linear combination of this will give a polynomial. So, basically, it is a polynomial. So, collection of the polynomial and this polynomial is also continuous function. So, Y is clearly, the set of all polynomials, is it not? This is the set of polynomials; elements will be the polynomials in that.

Now, this Y is clearly, Y is a subset of C 0 1, is it not; because, every polynomial is a continuous function and C 0 1 is the set of all continuous function, defined over the closed interval 0 to 1. So, it is a subspace, subset or subspace of C 0 1. Now, question is, whether it is complete or not? Because, this is a infinite dimensional. The dimension of Y is not finite, because it is a infinite dimensional. So, we claim that, this will not be a complete space. Why? Suppose, I take the sequence y n, y n t as 1 plus t plus t square by 2 plus t to the power n by factorial n, ok. Suppose, I take this sequence. It is the point in

y. What is the limit point of this? As n tends to infinity, it go to the point y t, which is e to the power t, is it not; as m, n tends to infinity, it will go to…As, because, it is an expansion of e to the power t, basically; 1 plus t plus t square by factorial (( )) t n to the power etcetera, which is not a point in capital Y, is it clear.

So, it means, this limit point of this is not a point in Y. So, this Y is not closed. So, Y is not closed. So, Y is not closed. And this, already we have one result, a subspace of a metric space is complete, if and only if, it is closed. Similarly, a subspace of a normed space is Banach, if and only if, it is closed. So, if this is not closed, it means, Y cannot be a complete normed space. So, Y cannot be complete; so, Y cannot be complete, clear. So, though it is a subspace of C a b, but it is not complete, because the dimension is finite, here. So, it…Then, in case of the infinite dimensional subspace, the result is not true. But finite dimension, it is always true. Whatever the x may be, if I picked up a finite dimensional space, it must be a complete space, of any terms. So, this is one.

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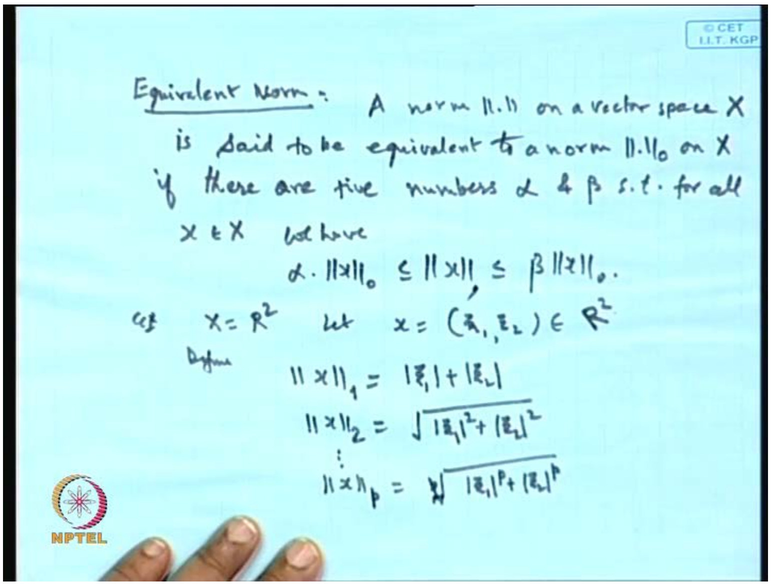

Now, another interesting application of this lemma is about the equivalent norms. That equivalent norms, we have defined already, it is my reviser; a norm on a vector space, a norm on a vector space, a norm on a vector space X is said to be, is said to be equivalent, is said to be equivalent to a norm, to a norm 0 on X, if, if there are positive, there are positive numbers alpha and beta, such that, for all x belonging to capital X, we have alpha times norm of x 0 is less than equal to norm of x, which is less than equal to beta times norm x 0, norm x 0. For example, let us take the examples here. Suppose, I take…

(( ))

Yes, alpha and beta. Yes, that is why. Means, if there are two norms are there, find out the value of x, any arbitrary x, in one norm and then, find out the value of the same x in other norm. If this condition is verified, that is, norm x will lies between this and this, with a suitable alpha and beta, then, we say both the norms are equivalent norm, ok. For example, I am taking this example, just given.

Sir, this is (( )).

No, it is, it is only norm x; no, norm x 0. This is not, it is a norm; no norm x 0. It is only…Suppose, I take this example, say. Suppose, x is equal to, say R 2 and I define and, let x is x i 1, x i 2 be an element of R 2, R 2; be in an R 2. I am defining the norm x 1 or x 1 as mod x i 1 plus mod x i 2. The norm of x, another norm, I am defining as, under root mod x i 1 square mod x i 2 square and continue this. Suppose, I define norm of x p as under root mod x i 1 p, mod x i 2 p and power 1 by p, this is power 1 by p, power 1 by p.

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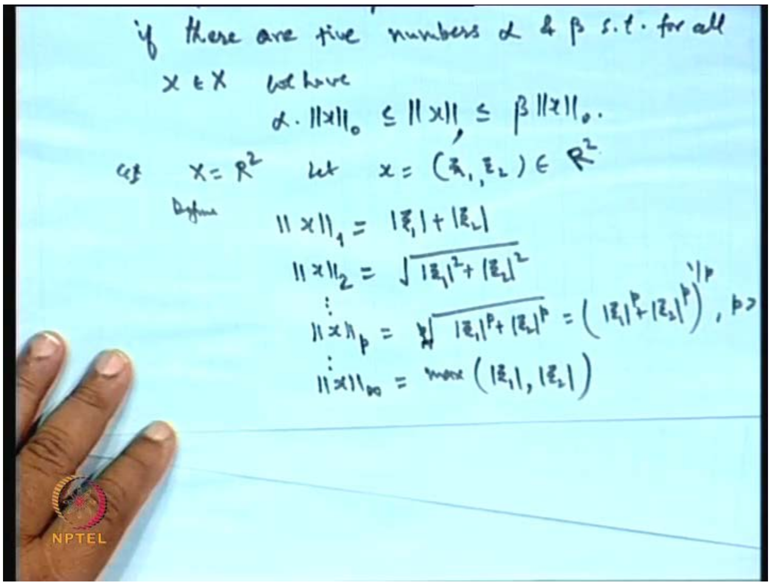

So, that is, the meaning is, mod x i 1 p plus mod x i 2 p power 1 by p, where the p is greater than 1. And, norm x infinity, say, I am defining as, maximum of mod x i 1 comma mod x i 2. Suppose, on R 2, I am defining this. You can verify that, this forms the norm.

Sir, p is greater than…

1. p is greater than 1, because we are starting 1, 2, 3 and so on. So, p must be greater than 1. For various p's, you can find out these thing. Now, if I look this, say 1 and 2 norm, x 1 and x 2, norm x 1 and norm x 2, then, what we get it from here is, norm x 2, norm x 2.

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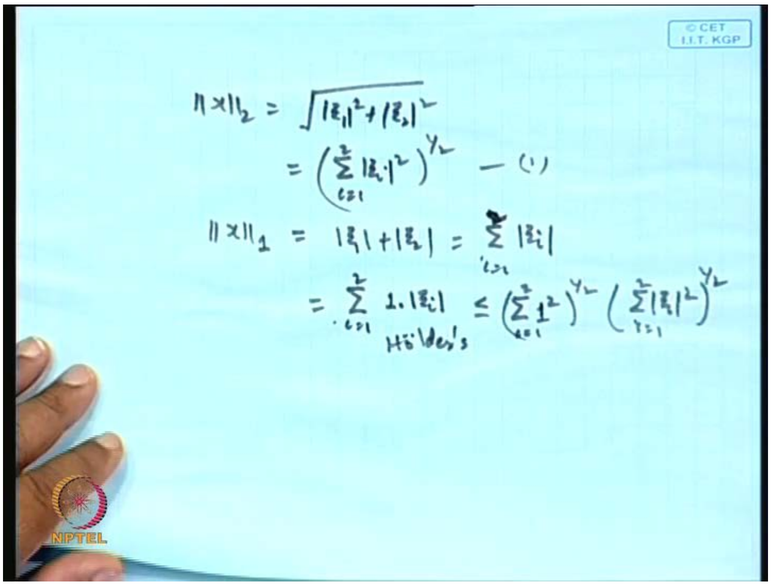

What is this is, under root mod x i 1 square mod x i 2 square, is it not. Now, this will be written as sigma x i square, i is 1 to 2, and power half, clear, power half; let it be 1, ok. What is the norm x 1? Norm x 1. The norm x 1 is mod x i 1 plus mod x i 2. Now, this is equal to sigma mod x i i, i is 1 to n, 1 to 2, 1 to 2, is it ok or not. Now, if I apply the Holder's inequality, keeping this as, this is the point; i is 1 to 2. Suppose, x i and y i are the two terms and apply the Holder's inequality; then, what is this result says, sigma of 1 square, i is 1 to 2, power half, is it not. Then, sigma of mod x i i square, i is 1 to 2, power half, is it not; by Holder's inequality or Cauchy-Schwarz inequality, you can say; for p equal to 2, it is Cauchy-Schwarz inequality. So, we are saying this one, clear.

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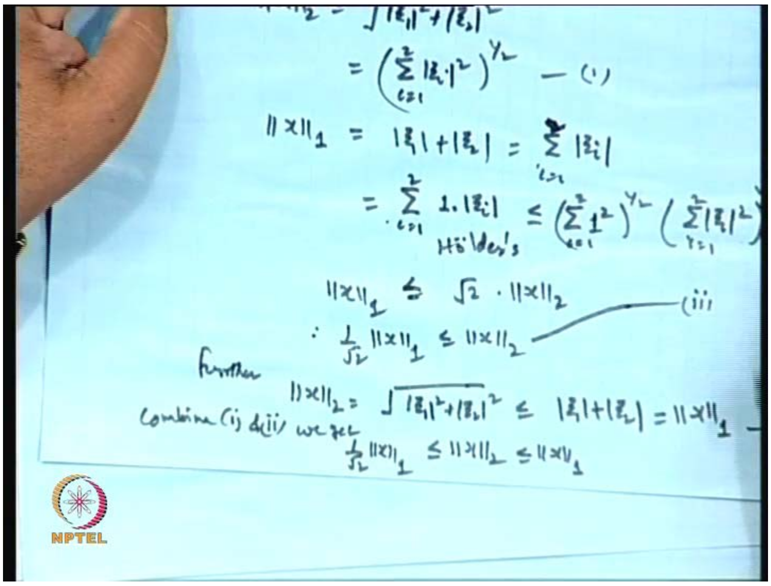

Now, what is the sigma 1 square? 1 square is 1. So, basically, 1 plus 1 is 2. So, basically, this is coming to be root 2 into, is it not a norm of x 2? So, what we conclude that, norm of x 1 is less than equal to norm of x 2, is it ok, clear. So, a constant beta can be obtained under 2. So, norm of x 1 is lying between, is this one, is it ok or not. So, we can say, 1 upon root 2, norm of x 1 is less than equal to norm of x 2. Let it be, this, second; not this; let it be, put it this form, clear. Further, norm of x 2 as is given by this, under root mod x i 1 square mod x i 2 square. Now, is it not less than equal to mod x i 1 plus mod x i 2? This is valid, because, if I square both side, square both side, then, this square will be high, more than this value. So, this is valid always, but what is this? Is it not the norm of x1? So, this is third. Combine 1 and 2, we get norm of x 2 is greater than equal to 1 by root 2 norm x 1, is less than equal to norm of x 1.

It means, here the alpha is 1 upon root 2, beta is 1. So, we can find out the scalar alpha and beta, so that, these two norms can be put it into this (( )) can satisfy this. So, both these norms are equivalent norm, clear. Similarly, we can go for any other, any norm. So, all the norms, which we have defined in this fashion, they are all equivalent norm. Now, what is the advantage of the equivalent norm, ok.

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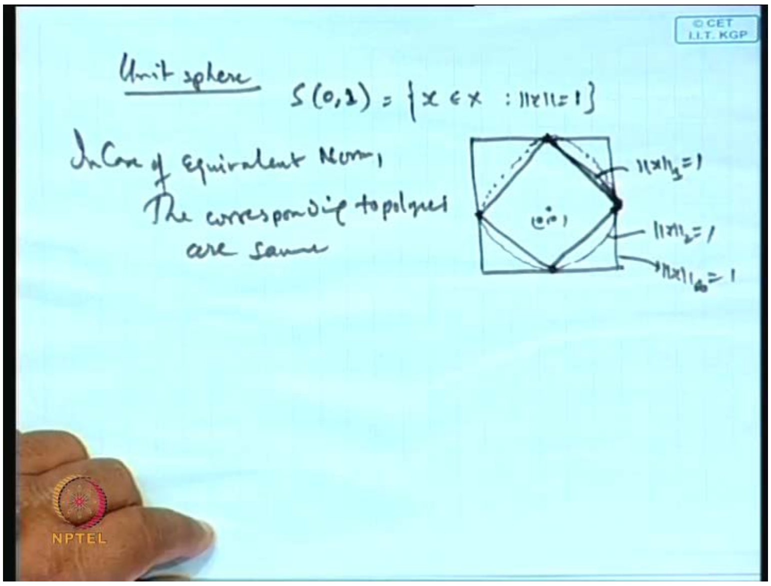

If I define the, here, that sphere, say, if I take this problem, like, yes, the sphere, unit sphere, unit sphere as 0, 1, with a center 0 and radius 1, is the set of…

Sir, that is (( )) number, they are defined (( ))…

Metrics also, yes, is same to same

(( )) such an example which can be (( )).

Which can be metric?

Yes, metric (( )).

Metrics.

Function number of a metric.

Oh, matrix also we can do it. Yes, we can, ok.

We defined the (( )).

Equivalent norms or the matrix of the R n matrix, they will all…Whatever the important matrix are there, whatever the norm, I will give it example, where the norms, we can define in many ways and this becomes a equivalent topology.

Sir, we can define the ((condition)) of the matrix based on this norms.

Yes.

One of these norms…

Yes.

So, can you discuss this point?

Yes we can.

An example…

Example, only finite case; infinite dimensional, we cannot say anything. Only finite case, it is true; that we can…Let us see this one. x belongs to capital X, such that, norm of x equal to say, 1. This is the unit sphere in a normed space X. Now, if we look this previous results, then, here we can say, this one, that, suppose, we have this. This is the center here, here. So, basically, the unit sphere, yes, yes, this is the unit sphere, in case of norm x 1 equal to 1; because the, it is a mod x 1 plus this is the center 0, 0. So, and then, slowly it goes like this, this, like this and something like this. So, it goes to the norm x 2.

And this one, basically, finally, norm of x infinity is 1. Now, when these norms are equivalent, the corresponding topologies are the same. So, in case of the equivalent norm, the corresponding topologies are same. They give the same topology. Topology means collection of the sets, that open sets, that is, I have, did not define the topology earlier; we will take the topology, we will teach you later on, when we go for the some other, further. But this, you can simply understand that, we are getting the open sets. So, whenever you pick up any open set in one norm, correspondingly, you can get another open set in the, suitable open set in another norm, so that, one can take a sequence which converge in one open set, it will converge into the other open set of the other norm; like then, vice versa. So, this way we can prove. Thanks.