Functional Analysis

Prof. P. D. Srivastava

Department of Mathematics

Indian Institute of Technology, Kharagpur

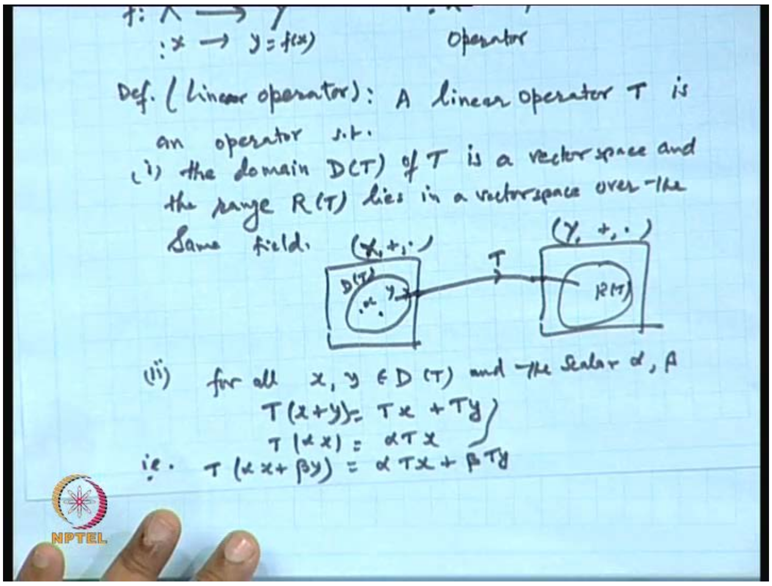
Module No. # 01

Lecture No. # 13

Linear Operators – definition and Examples

So, in the last lecture, we have discussed the concept of the compact sets. Today, we will take up the concept of linear operators over a vector space and then, we will slowly introduce the linear operator concepts in a normed space, which brings the norm addition and scalar multiplication together.

(Refer Slide Time: 00:47)

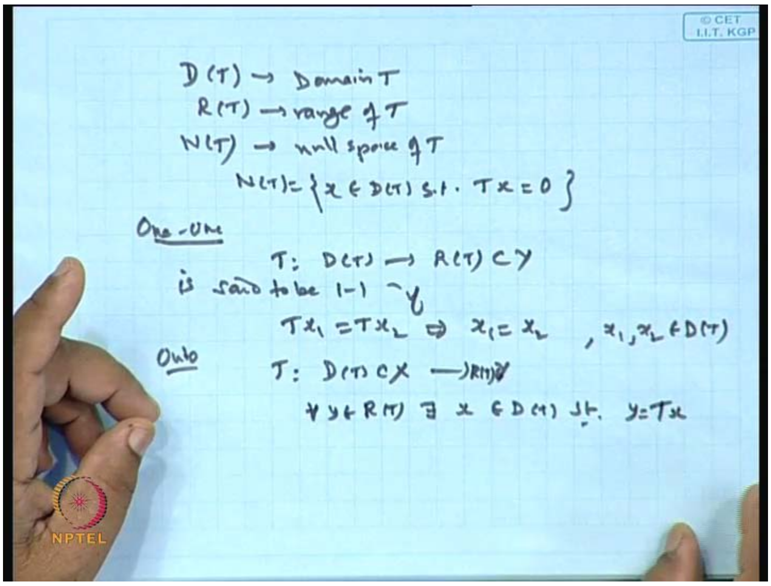


So, we know that, if X is a arbitrary set and Y is another arbitrary set, then, a mapping f from X to Y is a rule, which assign each element of x to an unique elements of y in capital Y, as y equal to f x. ((audio not available: 01:06 to 01:19)). Y vector space is, instead of choosing simply, a set, that is, X and Y both are the algebraic structures with suitable addition and scalar multiplication, then, the corresponding f, we denote it by T, and we call it, is an operator. So, operators are basically the extensions of the functions, where the domain as well as the range, they becomes, they are the part of the vector spaces.

We are interested in that particular operator, which gives a relation between addition and scalar multiplication and certain property of the linearity is followed. So, we gave the word linear operator, which is very interesting, linear operator and useful. We define the linear operator as a linear operator T, a linear operator T is an operator, such that, one, the domain, which we denoted by D T of T is a vector space, vector space and the range, and the range R T lies in a vector space, over the same field, over the same field. That we have, X and Y are the two vector spaces suppose, with addition and scalar multiplication. Here also, Y which, say, same addition, I am using the same notation; well, basically, it will be different; then, our operator T from the domain of T to range Y, this operator is a linear operator, when the domain of D T is a vector space and the range of this T lies in a vector space Y, over the same field. We, the, whatever the field of X and Y may be, the same field must be written.

Then, second property is that, for all x and y belongs to T, for x y belongs to D T, that is, we pick up that any two point x and y and the scalars, and the scalars alpha and beta, the condition T of x plus y equal to T x plus T y, T of alpha x equal to alpha of T x; that is, if we combine these two, then, we get T of alpha x plus beta y is equal to alpha of T x plus beta of T y. So, if this conditions hold, then, the operator T is said to be a linear operator. So, basically, the operator T, which preserve the operation addition as well as a scalar multiplication is said to be a linear operator, where the domain T is always a vector space and range should lies in a vector space. And, in fact, we will prove that, for in case of a linear operator, the range of T will also be a vector space; that is all.

(Refer Slide Time: 05:55)

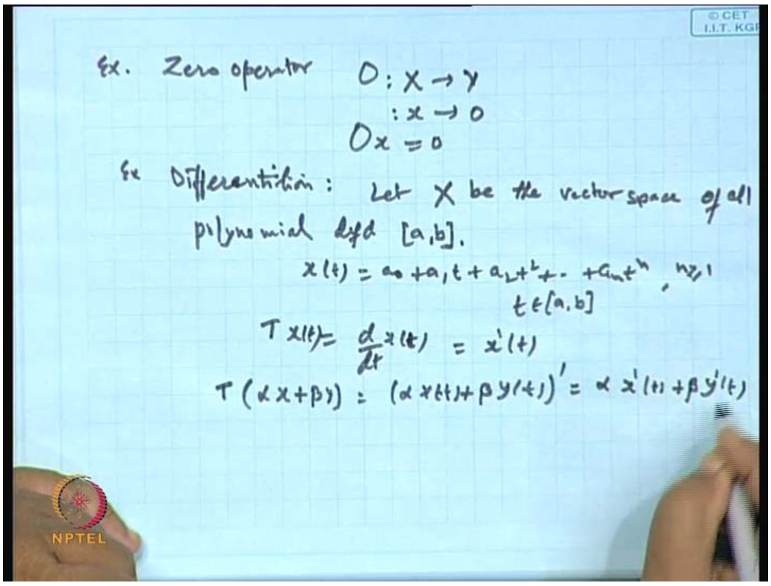


So, this is the general definition we are giving. Here, we will use the same notations as we are using as, (( )). D T we will use for domain of T; R T, we will use for the range of T and N T, which is a new notation, we use for the null space of T. The null space of T, we mean that, set of all points x belong to the domain of T, such that, the T of x becomes 0. This is the null space and so, sometimes, we also call it as a kernel of T; but, since kernel, we will use for the functional, functional, so, we will use the word null space of T; nullity, ok. So, here, we have taken. Then, h, we have in case of the mapping, the one-one mapping, one-one, onto mapping, many-one mapping and like this; so, similar concept, we have the one-oneness. An operator T, from the domain D T to the range T, which is a subset of Y, is said to be one-one, if it sends the image of the different points to different numbers; or, if T x 1 equal to T x 2, this should imply x 1 equal to x 2, where the x 1 and x 2 are the point of D T, ok.

So, when each having a unique image, each point is having the N, suppose, the two images are equal, this is only possible when the points are coincident; then, we say, the mapping is one-one. And onto, whenever the T is a mapping from onto, we say that, T is mapping from D T, which is a subset of X to Y and is said to be onto mapping, if for a range, range of T, R T, which is subset of Y, then, it said to be an onto mapping, if for every y belongs to the range T, there exist a point x, belongs to the D T, such that, the image point y comes out to be as a form of y equal to T x; that is, corresponding to each point in the range, if we are getting a point x in the domain, whose image falls on Y, then, we say T is a onto mapping. So, one-one, onto mapping; whenever is the mapping is one-one and onto, the inverse exist. So, we will talk about the inverse only case, when T is one-one and onto.

Obviously, when T is a mapping from domain to D T to R T, it will be onto mapping, because all the points, images are lying here; no point, or range is only R T, we are restricting. So, that is why, it will be a onto, there. Now, as a particular examples, we will see, that is, there is many examples, which we come across of the functions, of the operator T, which are linear operators. The one is, a simple one is, identity operator. This is an operator from, denoted by say, I x, from a vector space X to X, such that, the image of each element under I becomes the x itself; this is true for every x belongs to capital X and such an operator we call as an identity operator; and, it is easy to verify that, this is a linear operator. Because, if we take the I x alpha x plus beta y, then, obviously, the image comes out to be alpha x plus beta y and that can be written as alpha of x ((comma)), within bracket x, plus beta of I x, within bracket y. So, it is a linear operator. A very simple and (( )) example.

(Refer Slide Time: 10:02)



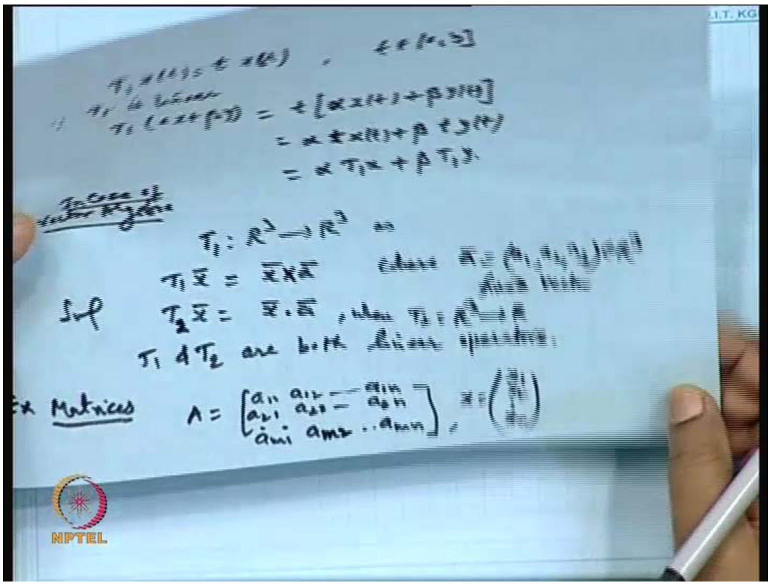
Another operator, which is also linear, is the zero operator, zero operator. This operator is also a linear operator. An operator 0 from x to, say, in fact, that range will be the 0 set only, so, any set X to Y, such that, image of x comes out to be a singleton set 0; that is, the zero operator, which maps each element to a 0. This is an operator and this is a small value 0. And, its operator comes out to be a linear operator. Then, differentiation is also an operator, differential operator or differentiation, (( )) d. Let X be a, X be the vector space of all polynomials, all polynomials defined on the interval, say a, b; so, x be a polynomial x t, which is of the form, say, a 0 plus a 1 t, a 2 t square and so on, say a n into t to the power n, where n could be greater than or equal to 1 and t belongs to a, b.

So, it is a polynomial. Each polynomial is continuous function and well, differentiable function. So, one can find out the operator T of x as the derivative of x t; T of x t, you can write this also, T of x t is d. Now, this is well defined, exist; because, the polynomials are differentiable function. We can take the set X to be the set of continuous function, but there is no guarantee that, every continuous function is differentiable; that is why, we did not take the X to be vector space of continuous function. We, for the safer side, let us take the domain of this as a space of all polynomials, defined on a, b; it forms a vector space of degree less than or equal to n.

So, this will be a vector space and correspondingly, the operator T is defined on this. So, we get the polynomial x prime, the value x prime t, which is again a polynomial, of course. And, this will be a linear polynomial operator, because T of alpha x plus beta y, this is equivalent to alpha x t plus beta y t prime, prime denotes the derivatives of this. So, when you take the differentiation, the constant will come out and we get this beta into y prime t. Therefore, this can be written as alpha times of T x t plus beta times of T y t. So, T becomes linear. This is, differentiation is used widely. So, in fact, the linear operator has a wide scope; is a lot of applications are there, in case of, for the linear operators.

Another operator, which is also well known, is the integration operator. If we take space X as C a b, set of all continuous functions, defined, continuous functions defined over the closed interval, say a, b. Then, and let us put that T of x t as the integral of this, a to b, x t tau, this tau. Then, this will be an operator, a linear operator…First thing, it is well defined thing, because, every continuous function defined over closed interval is integrable. And, in fact, we get the area bounded by this curve y equal to x t and to (( )) is equal to a x equal to b and where x is a continuous curve. So, this is a well defined thing and then, the operator T, which you are getting T of x t is equal to a to b x tau t tau, x t; better take it x t, because this is a operate function of t. So, we get this as a linear operator of this.

(Refer Slide Time: 15:02)

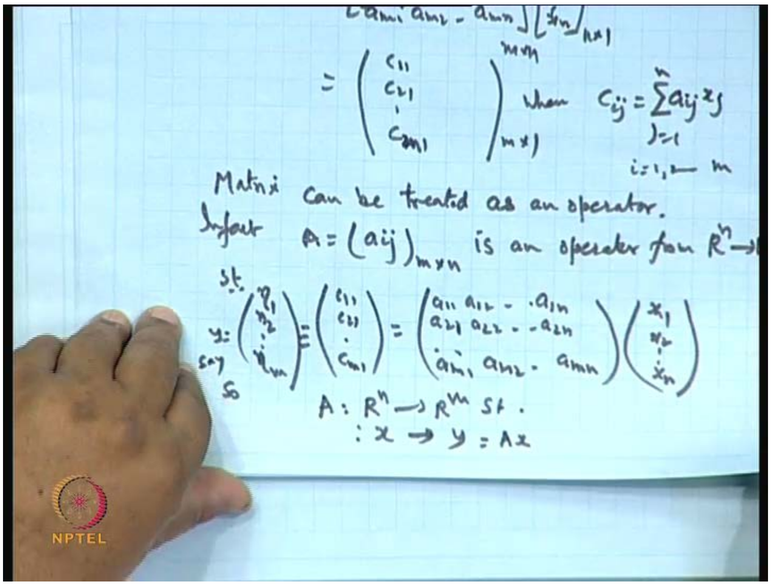


Another operator, one can define on this C a b, the operator, let us define by T 1 x t as t times of x t. Now, this will also be a linear operator, because t is a point in between a b. So, because, if we take T 1 is linear, because T 1 of alpha x plus beta y, this will be the same as, t of alpha x t plus beta y t. And, this can be further expressed as alpha times t of x t plus beta times t of y t and that is (( )) as alpha T 1 of x plus beta T 1 of y. So, T 1 becomes linear operator.

So, this also…Another interesting examples of the vectors, element vector algebra. In case of the elementary vector algebra, in case of the elementary algebra, elementary vector algebra, we see the cross product and dot product. If we define the T 1 from R 3 to R 3 as T 1 x bar is equal to x bar dot a bar, where a bar is a vector a 1, a 2, a 3, belongs to R 3 is a fixed vector; fixed vector, sorry, cross, this is cross; because, you want the vector, ok. So, if we define T 1 as from R 3 to R 3, as x bar cross a bar, where a bar is a fixed vector, then, this cross product of the two vector is again a vector quantity and it will be a point of R 3; and, we see this is a linear operator. Similarly, if we define T 1 x bar as x bar dot a bar, T 2 x bar as x bar dot a bar, where T 2 is a mapping from R 3 to R, then, this scalar product also gives you a linear vector. So, T 1 and T 2, linear operators. T 1 and T 2 are both linear operator, operators. And, this can be easily viewed. So, we do not go for verification.

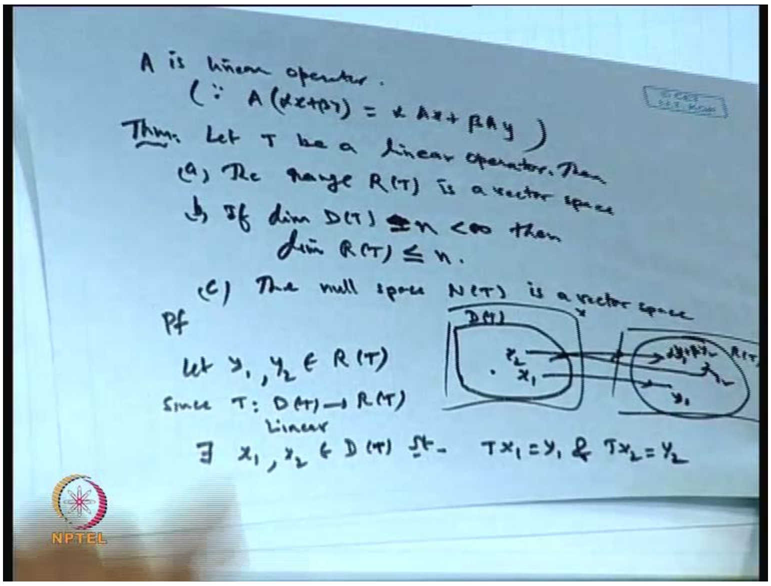
The interesting example is our multiplication, matrix multiplication. So, matrices, we know, if A is a matrix having the coordinates, say a 1 1, a 1 2, a 1 n, a 2 1, a 2 2, a 2 n, a m 1, a m 2, a m n, and x is a vector, as x 1, x 2, x n, x 1, x 2, x n, then, if you multiply a into x, what you get?

(Refer Slide Time: 18:54)



This is a matrix a 1 1, a 1 2, a 1 n, a 2 1, a 2 2, a 2 n, a m 1, a m 2, a m n; when it is multiplied by x, x 1, x 2, x n…We know this is a matrix of order m cross n. Here, this is a matrix of order n cross 1. So, product will be a matrix of order m cross 1. There will be m rows, n columns and 1 column. So, we get c 1 1, c 2 1 and c 2 n, c m 1, m rows and 1 column, where the c i j is the sigma a i j x j, j varies from 1 to n, is it not? i is varying from 1 to m. So, this is the dot, this is the product of the two matrices. Therefore, matrix can be treated as, can be treated as an operator, as an operator; and, in fact, in fact, a matrix a of order m cross n is an operator from R n to R m, from R n to R m, such that, such that, the matrix c 1 1 or you can write that c 2 1, c m 1, or better equivalent to, this is a matrix of a 1 1, a 1 2, a 1 n, a 2 1, a 2 2, a 2 n, a m 1, a m 2, a m n into x 1, x 2, x n; this will be... So, if we put it this thing in a suitable form, then, we can say, the product of a x is equal to y. So, we can say, or let us take it this as eta 1, eta 2, eta m, ok; suppose, this is y, say y, say y. So, we can say that, A is a, matrix A is an operator from R n to R m, such that, it carries the x to y, where the y is equal to A of x and y is this.

(Refer Slide Time: 22:17)



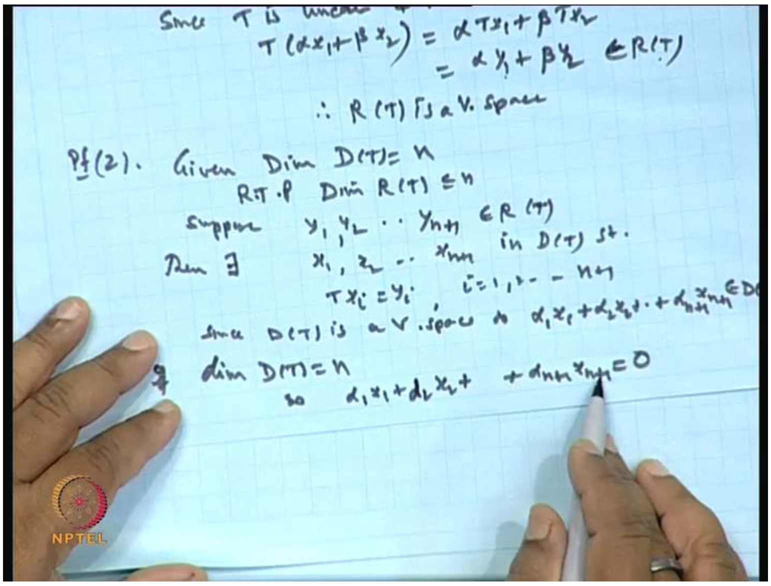
Now, this operator is a linear operator. Matrix A is linear operator, which can easily be viewed, seen. So, it nothing to prove, because if we take A alpha x plus beta y, then, just a ordinary multiplication; one can verify that, this comes out to be this part. A is linear. So, what we seeing, there are many examples of the linear operators, which are important and very useful in practice. They, in general, these linear operator satisfies certain properties and those properties are given in the form of theorem. The first theorem is that, let T be a linear operator. Then, the range set, the range, that is, R T of the operator T is a vector space. Because in the definition, if you remember the, when you define the linear operator, we have not considered R T to be vector space. What we have thought that, D T is a vector space and R T lies in a vector space. But basically, if T is a linear operator, then, R T comes out to be a vector space.

Then, second is, if the dimension of D T is finite, say n; if the dimension of D T is, say, suppose, equal to n, which is finite, then, the dimension of the range set, then, the dimension of the range set T, cannot exceed by n. So, if the dimension of the D T is finite, the dimension of the range set will be finite and in fact, it will not be higher than the dimension of T. Dimension we know, because D T is vector space and every vector space has a basis, unless, it is zero vector space. Then, the set which is linearly independent and spans the whole space and the number of elements is called the dimension. So, number of elements in the basis is called the dimension of the vector space.

So, and third property is the null space, the null space N T is a vector space. Let us see the proof of this. Proof is simple, but…So, what we want is, range of T is a vector space. So, suppose, this is our D T and here is range R T; this is say, Y and this one is suppose, X. And, T is a mapping from this to this, D T to Y. The range, T is a linear operator. So, it will transfer the x plus y to T x plus T y; T of alpha x is equal to alpha of T x; we want the range of R T is a vector space. So, if I prove this is vector space, means, pick up any two arbitrary point of this, if the linear combination of this, that is, alpha x plus alpha y 1 plus beta y 2 also belongs to the class R T, then, the R T will be a vector space. So, let y 1 and y 2, they are two arbitrary points of the range set. Since T is an operator from domain to range, so, since T is an operator from D T to R T, which is a linear operator, it is also given to be a linear operator, so, if we take any point y 1 in R T, the correspondingly, there will be point x 1 available in D T, whose image will falls on y 1. So, this will be there.

Similarly, there is another point x 2, whose image will fall on y 2, like this. So, there exists x 1 and x 2, belongs to the domain D T, such that, T of x 1 is equal to y 1 and T of x 2 is y 2, ok. Now, we want the linear combination of this. Now, since x 1 and x 2 are available in D T and D T is already a vector space, D T is a vector space, which is by definition of the linear operator, so, any linear combination of this alpha x 1 beta x 2 must be a point in D T.

(Refer Slide Time: 27:30)

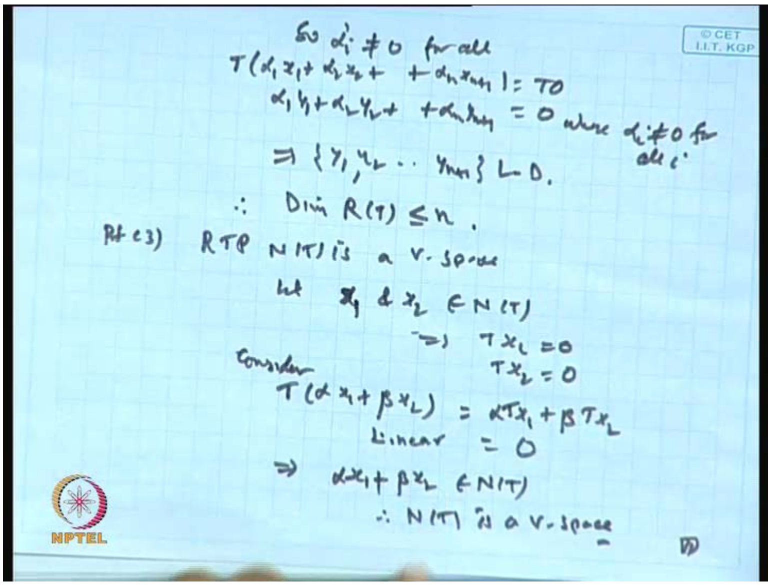


So, since D T is a vector space, therefore, alpha x 1 plus beta x 2 will be point on D T, ok. So, we are getting this point, alpha x 1 plus beta x 2. Now, since T is linear, so, T of alpha x 1 plus beta of x 2, this will be equal to what, alpha of T x 1 plus beta of T x 2. And, that will be equal to alpha y 1 plus beta y 2; but T is a mapping and T is a mapping from domain to range set. So, if alpha x 1 plus beta x 2 belongs to D T, then, this has to belongs to R T. Therefore, if we, what we have seen, if y 1, y 2 are the point in R T, then, the linear combination will also be point of R T. Therefore, R T is a vector space, ok.

Now, second comes proof of two. Given that, domain of D T, dimension of D T is n, which is finite, of course. Dimension means, it has a basis elements, linearly independent elements are available here and that is, number is n. We are interested to show the dimension of the range set, dimension of the range set cannot exceed by n. Suppose, this is not true. Suppose, we have the points y 1, y 2, y n plus 1, be the points available in R T, suppose, ok. Then, there, correspondingly, there exists, the points x 1, x 2, x n plus 1 in domain D T, such that, T of x i equal to y i, i is 1 to n plus 1, because T is a mapping from domain to R T. Since D T is a vector space, so, alpha 1 x 1 plus alpha 2 x 2 plus alpha n plus 1 x n plus 1 must be a point of D T, must be a point of D T.

But x 1, x 2, x n, these are…Now, once we take y 1, y 2, y n are these. Now, if the dimension of this D T is, say n, then, it cannot exceed, or it cannot have a range, cannot have a set, which is linearly independent and spans D, intersect D T, but linearly independent. So, the maximum, the largest linearly independent sets available for D T, has only n elements. So, since the dimension of D T is n, therefore, if we take this alpha 1 x 1 plus alpha 2 x 2 plus alpha n plus 1 x n plus 1 is suppose, zero; suppose, this is 0, it means that, x 1 x 2 x n plus 1, they are the elements of this and the image of this, T of alpha 1 x 1 plus alpha 2 x 2 plus alpha n x n plus 1 will be T of 0, that is 0. So, we get alpha 1, because T is linear, alpha 2 y 2 plus alpha n y n plus 1 is 0, ok. So, if we take x 1, x 2, x n (( )) satisfy this condition, then, all this x 1, x 2, x n will be linearly dependent set. From here, it implies that, x 1, x 2, x n plus 1 is linearly dependent. Why, because the dimension of this n, so, it cannot have more than n element. And, if the linear combination of this is 0, then, this is only possible, it is not possible that, all alphas will be 0; otherwise, the dimension will exceed by this, ok.

(Refer Slide Time: 32:39)



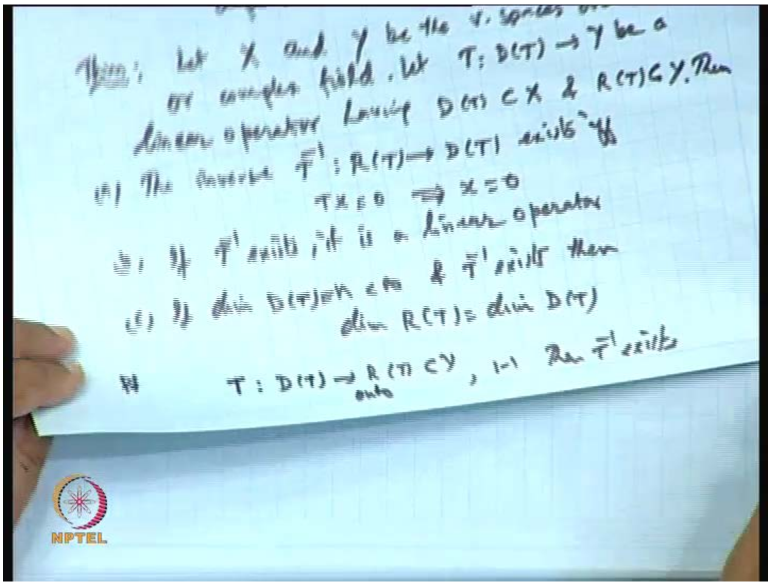
So, if this linearly dependent set and here, all alphas cannot be 0. So, all alphas are not 0. So, alpha is not 0 for all. It means, some alpha will be there. So, here, all alphas are not 0. This is, which we are, alpha is, is not equal to 0, for all i. So, this imply that, y 1, y 2, y n plus 1 is a linearly dependent set. It means, if I start with any set having n plus l elements in the Y, in the range set, then, we start with any n plus 1 element in the range set, then, we cannot get a set, which is linearly independent it will come out to be the linearly dependent set. Therefore, the dimension of this range set will be at the most n or may be less than equal to n and that proves the result, is it clear?.

Then, proof third part. Third part is the, was that, if we have, yes, this was the third part. The null space N T is a vector space. I think, it is simple to show the, required to prove, null space T is a vector space. So, pick up the two arbitrary points and if the linear combination belongs to this, then, it will be a…So, let y 1 and y t, or x 1, sorry, x 1 and x 2 be that point of the null space T. It means, T of x 1 is equal to 0, T of x 2 is equal to 0, by definition. Consider, T of alpha x 1 beta x 2. Since T is linear, so, it will be alpha x T x 1 plus beta T x 2 and that comes out to be 0. The T x 1 is 0, T x 2 is 0. So, this shows that, alpha of x 1 beta of x 2 will be a point of N T, null space. Therefore, N T is a vector space.

So, this completes the whole results. Now, if we look this results once again, what conclusion can be drawn? The linear operator, which is a mapping from D T to R T, D T is already taken to be the vector space and the range set R T, which we have proved to be vector, already proved, it is a vector space, fine. And, the dimension of the range set cannot exceed by the dimension of the domain and third is, null space is a vector space. Now, we look the second part. This shows that, if we have a linearly independent set in the domain, then, it can be transferred to a linearly independent set in the range set, because it cannot be a linearly independent. Because here, we have started, proof if you see that, x 1, x 2, x n are such, where this is, x 1, x 2, are such, where this is 0; where not, not all alpha is are 0; not alpha is 0.

So, it is a linearly dependent sets. And, what is that image? Image comes out to be the alpha 1 y 2 etcetera is 0, where all alpha are not 0; it means, it is also linearly dependent. So, the linear operator is such, which preserve the linearly dependence criteria or linearly dependence property; that is, we can say from here that, the linear operators, or conclusion… Linear operators preserve, preserves, linear preserve, sorry, preserve linear dependence.

(Refer Slide Time: 36:45)

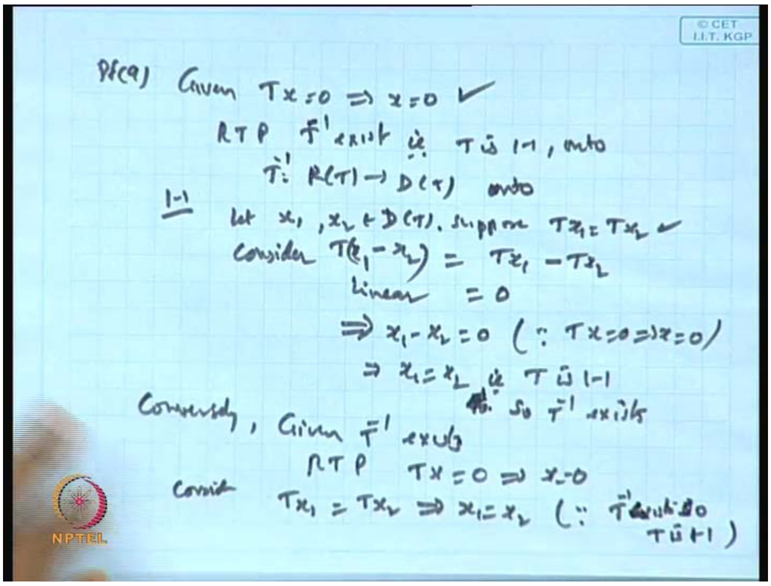


We cannot say, what we think about the linearly independence. If a set is linearly independence, then, the may be linear independent, may not be linearly independent. But, in case of the linearly dependence, this is granted. If a collection or set of the point in linearly dependent, the image under the linear operator must be linearly dependent. So, that is really interesting; an important thing, which linear operator preserves. Then, we have some results on the inverse operator. The inverse operator, the results are: let X and Y be, X and Y be the vector space over, be the vector spaces over the real or complex field, or complex field. Let X, let T is a mapping from D T to Y be a linear operator, be a linear operator, whose domain is D T lies in X, having the domain D T, which is subset of X and range set R T, which lies in Y.

Then, what this result says that, the, the inverse operator T inverse, then, the inverse, T inverse, which is a mapping from range set to the domain exist, if and only if, T x equal to 0 implies, x equal to 0. And second is, if T inverse exist, then, it is a linear operator; means, if T is a linear operator, the inverse exist, it will also be a linear operator. And, c is the, if dimension of R, if dimension of D T, which is n, is finite, then, and T inverse exists, then, both will have the same dimension. Then, dimension of R T will be the same as dimension of D T, ok.

So, proof is, how to define this T inverse? Let us see, first. We have, we have told that, a T is a operator from, T is an operator from D T to range set R T, which is subset of Y. Then, we say it is an all onto operator. Now, if this operator is also one-one, then T inverse exist. So, only the one-one, onto operator, we can talk about the T inverse or the Y naught. So, first thing is, here, already given that, T is an operator from D T to Y. So, range set lies in it. It must be operator from, means, if it is, we want it to be onto, then, it should be operator from D T to R T, ok. So, that is why, in case of the inverse operator, the first one, if you say, it is, domain is R T and range is D T. So, what the first result says that, if the inverse operator exist, then T x equal to 0 will imply x is 0.

(Refer Slide Time: 41:23)

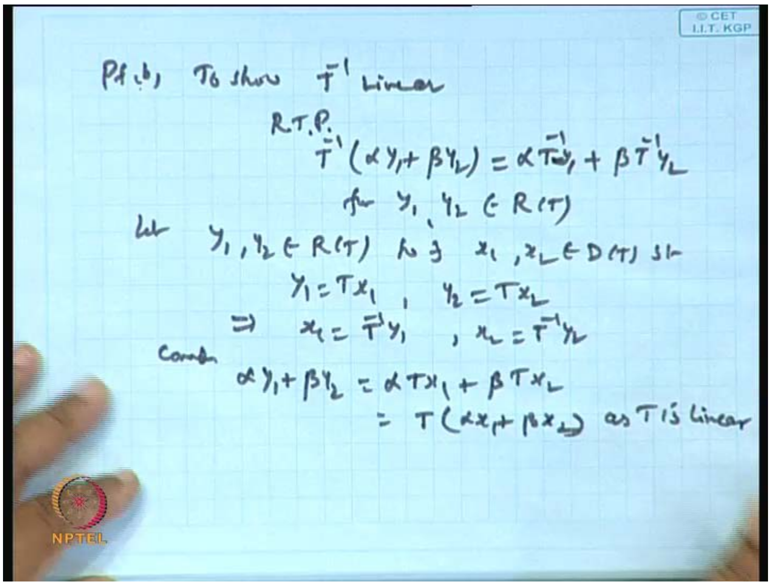


And, similarly, if T x equal to 0 implies x is 0, then, T inverse will exist. So, let us see the proof for the…The first part, proof is follows like this. Suppose, given, T x equal to 0 implies x is 0. What is required to show, T inverse exist, is it not. It means, that is, if I prove T is one-one and onto, then, T inverse will exist. Already, T is given to be operator, or T inverse is given to be an operator from R T to D T. So, it is onto operator. Nothing to show here. One-oneness we wanted. So, to show the one-oneness, what we do is, let x 1 and x 2 are the two points belonging to the domain D T, and suppose, T x 1 equal to T of, suppose, or T x 1 equal to T x 2; suppose, T x 1 is equal to T x 2, ok. To show the one-oneness, if I show x 1 is equal to x 2, then it is ok. So, consider, now, T x 1 minus x 2. Since T is a linear operator, so, it will give the value T x 1 minus T x 2; but it is given, T x 1 equal to T x 2, so, it is 0. So, T x 1 minus x 0 is 0 will implies, x 1 minus x 2 is 0, because of this given condition; because this condition is given; given is T x equal to 0, implies x is 0. So, this shows x 1 is equal to x 2; that is T is one-one.

T is already onto; T is already onto. So, T inverse exist. So, T inverse exist, clear, from this. So, conversely, if T inverse exist, what will it do then? Conversely, given T inverse exist, so, what is required to prove? Required to show is, T x equal to 0 must imply x is 0, ok. So, let us take, consider T x 1 equal to T x 2, clear. Now, x 2 is equal…Now, this will imply x 1 is equal to x 2; why, because T is, T inverse exist. So, T is one-one; T is one-one; because T inverse exist, therefore, T will be one-one. So, here, put x 2 equal to

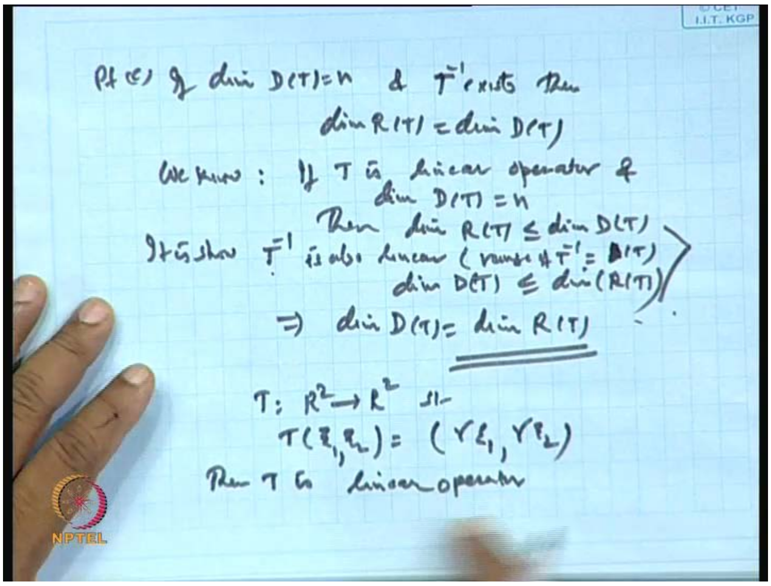
0 and x 1, you replace by x. We get immediately, T x equal to 0 will imply x is…And, that completes the proof (( )). So, this part is also clear.

(Refer Slide Time: 44:53)



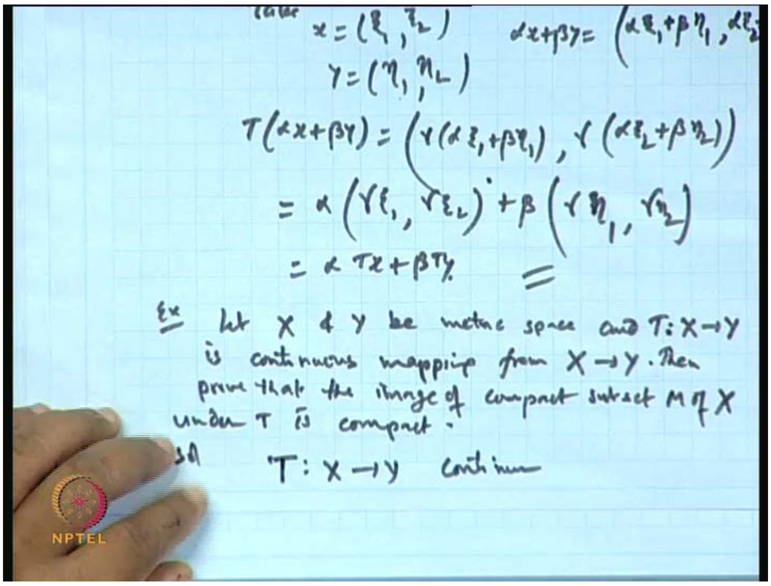
Then, T inverse is a linear operator. Proof for the second is, to show the T inverse is linear. What we are required to show is, let us pick up the two points y 1 and y 2. And, if I prove, T inverse alpha y 1 plus beta y 2 is alpha times T inverse y 1 plus beta times T inverse y 2, then, our proof is complete. So, for y 1 and y 2 belongs to the range set. So, let us start. Let y 1, y 2 belongs to the range set. So, there exist x 1 and x 2 in the domain, such that, y 1 is equal to T x 1, y 2 is equal to T x 2. Therefore, we get from here is, x 1 is T inverse y 1, x 2 is T inverse of y 2. Now, consider this. Alpha of y 1 plus beta y 2; alpha, y 1 is given to be T x 1, y 2 is given to be T x 2; T is linear. So, we can write this as this form, as T is linear, ok. So, once it is there, then, T inverse exist. Since T inverse exist, so, we can write T inverse of alpha y 1 plus beta y 2 equal to alpha x 1 beta x 2; and, that is equal to alpha T inverse y 1 beta T inverse y 2, because x 1, x 2 are these. Therefore, T inverse is linear. I hope it is clear. So, this way, this came.

(Refer Slide Time: 46:57)



The third one is, if the dimension of D T, dimension of D T is n, dimension of D T is n, and inverse exist, and T inverse exist, then, then, both will be same; dimension of range set is the dimension of domain. Now, this result follows from the first two. We know, if T is a linear operator and the dimension of domain is n, then, dimension of the range set cannot exceed by n, by the dimension of the domain; that is, n. We have already shown, it is shown that, T inverse is also linear. So, dimension of this, for the T inverse, what is the range set? The range set becomes domain of T and domain of that is, becomes the range of T. In case of the T inverse, what is the range? Where the range set, range of T inverse is, is the same as R T, sorry, D T and domain, and domain is the R T. So, replacing this, we are getting here. So, combine these two, we get, dimension of the domain is the same as dimension of the range set. So, this proves the result, ok. So, just using the previous knowledge, we can go this far.

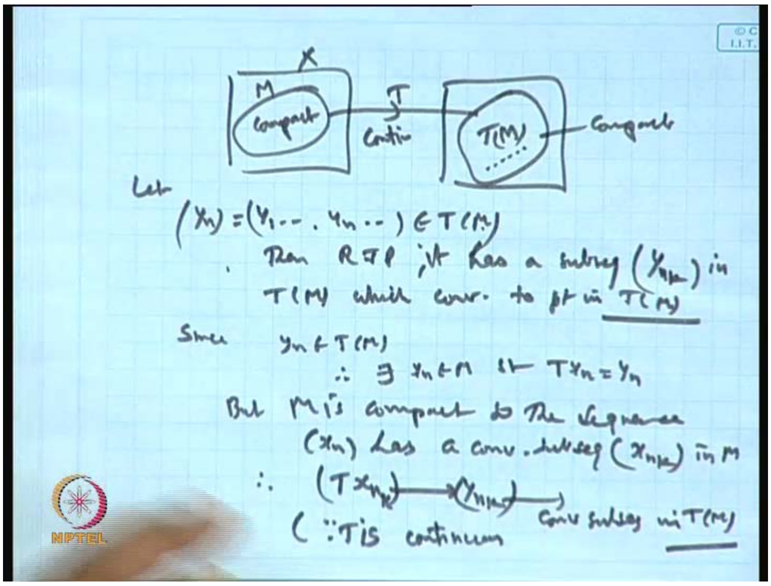
(Refer Slide Time: 49:52)



Now, let us see few examples, where it is…Just a simple, few examples of this. Suppose, we have an operator T from R 2 to R 2 and such that, T of x i 1, x i 2 is suppose, gamma x i 1, gamma x i 2. Then, this T is a linear operator. One can easily check it. Take two points, because if we take, take x as x i 1, x i 2; y as eta 1, eta 2 and alpha x plus beta y will be what, alpha x i 1 plus beta eta 1, alpha and comma alpha x plus beta. So, alpha x i 2 beta eta 2, this is the point. So, T of alpha x plus beta y, this is defined as T of x i 1, x i 2 is defined as what? It is a new point in R 2, where the x i 1 coordinate is accelerated or decreased by gamma and then, same multiple with the second coordinate also, x i 2. So, we are taking gamma times of this, comma gamma times of this. So, this is gamma times of eta 1, comma gamma times of eta 2. Now, can it not be written like this, alpha gamma time x i 1 gamma time x i 2 plus beta gamma times eta 1 comma gamma times eta 2? You can just, simple multiplication and addition. So, this will be alpha of T x beta of T y. So, this shows the linearity.

Now, yesterday, we have taken few, that compact set. So, the relation between the compact set and continuous functions we will give it in the form of this exercise. The exercise says let X and Y be the metric space, be metric spaces and T is an operator, is a continuous mapping from X to Y, is a continuous mapping from X to Y. Then, prove that, the image of compact subset M of X, compact subset M of X under T, is compact. Now, T is a mapping from X to Y and this is given to be a continuous mapping. We know, the continuous mapping cannot always transfer the open set to open set; but here is a compact set; M, we are choosing a compact set in X. What we are interested is image of a M under T should also be a compact.

(Refer Slide Time: 53:19)



So, if we take M to be a compact set instead of taking a open set, and T is a mapping from this X to Y, then, image of this T M under this compact set, under the continuous operator T, will be a compact; this we wanted to show. It means, if we take any arbitrary sequence y 1, y 2, y n and so on, if it is an sequence in M, T M, then, what is required to prove, it has a subsequence, say y n k, in M, T M, which is convergent; which converges to a point in T M, is it not. Then only, there is a compact. Every sequence has a convergence subsequence, then, a set will be a compact set.

So, let us take, y n be a sequence, be a, let y n be a sequence in T M. We wanted to show a subsequence lies in y n, which is convergent, ok. So, since y n belongs to T M, therefore, there exist x n in M, such that, T of x n will be y n, because of this. But M is compact. So, the sequence x n has a convergent subsequence x n k in M, because M is compact; so, every sequence has a convergent subsequence. Therefore, the image of this, therefore, T of x n, x n k, which is transferred to y n k, this is a convergent subsequence in T M; because T is a mapping from M to T; T M is image of M. So, if a sequence x n has a convergent subsequence, limit point belongs to the M. Then, is, the corresponding images will be converged to the same point, because T is continuous, because T is continuous. So, it will transfer the limiting point to the limiting points; that is by definition. Therefore, this sequence y n, will have a subsequence which converges in M. Hence, T M is compact; T M is compact.

Thank you. So, this. Thanks.