Functional Analysis

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Module No. # 01
Lecture No. # 14
Bounded Linear-Operators in a Normed Space

((Last)) lecture we have discussed the linear operators on a vector spaces, operators on vector spaces.

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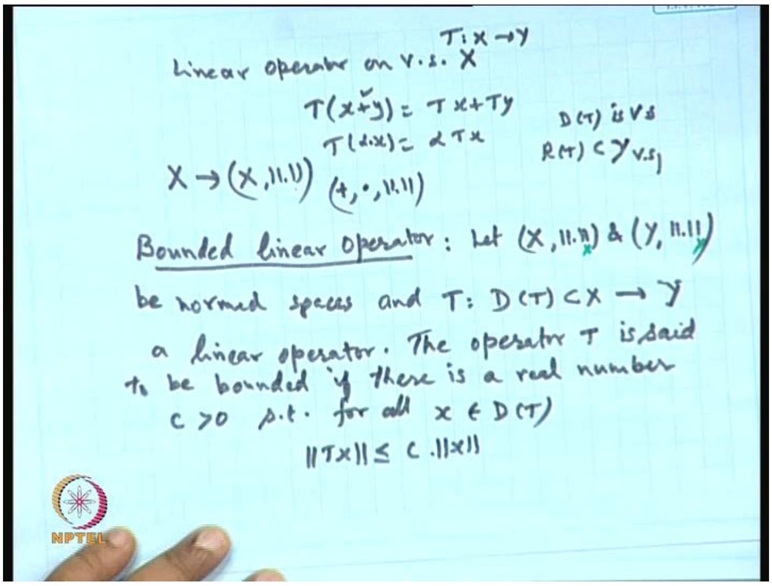

So, this operator, we have defined as T of X plus Y is equal to T X plus T Y and T of alpha X equal to alpha of T X, where the domain of T is a vector space and range of T lies in a vector space, lies in Y, which is a vector space. So, that way we have defined the linear operator on a vector space, say X T in operator, from X to Y. ok. Now, if we look the definition here, we are having the two operation, addition and scalar multiplication. So, this operator T has a connection between a scalar multiplication and addition and addition of the two vectors. But we do not have any things about the normed. So, if I replace the vector space X by a normed space or a Banach space, then, this norm has no function over here so far. So, we wanted to extend this concept of the linear operator, over a Banach space or over a normed space, which includes another operation, which we call it as a norm.

So, a operator which includes all these three operation, its definitions, will be more interesting and important, comparative to this one. And, this leads to the concept of bounded linear operator. The bounded linear operator is defined as follows. Let X and Y be, X and Y be normed spaces; normed spaces, here, I am taking the same norm and can use the another notation, as this as norm X and this is corresponding to the norm Y. So, normally, we do not write it, because it is a understanding that, whenever the elements of Y is there, that corresponding norm of Y is used. So, let X and Y be normed spaces and T is a mapping from domain of T, which is of course, a subset of X to Y, a linear operator, a linear operator. Then, we say the operator T is said to be, is said to be a bounded linear operator, said to be bounded, a linear operator said to be bounded, if there is a real number C, of course, C will be positive, such that, for all X belonging to domain of T, the norm of T X is less than equal to C times norm of X.

So, we have defined the bounded linear operator in this form, means, a linear operator is said to be bounded, if this extra condition is also satisfied. So, basically, if we look, an operator T from domain D T to Y is a bounded linear operator means that, this should satisfy this condition alpha X plus beta Y equal to alpha of T X plus beta of T Y; that is T is linear; T is linear and apart from this, the norm of T X should be less than equal to C times norm of X. Here, this is the norm on Y and this is the norm on X. So, this gives the relation between the norms and the operator also. So, this operator, an operator which satisfy these conditions, we call it as a bounded linear operator. Now, the question arise, what should be the minimum value of C here, so that, this condition holds. This can be obtained as a, since norm of T X, Y norm of X is less than equal to C, this is true for all X belonging to domain of T D T.

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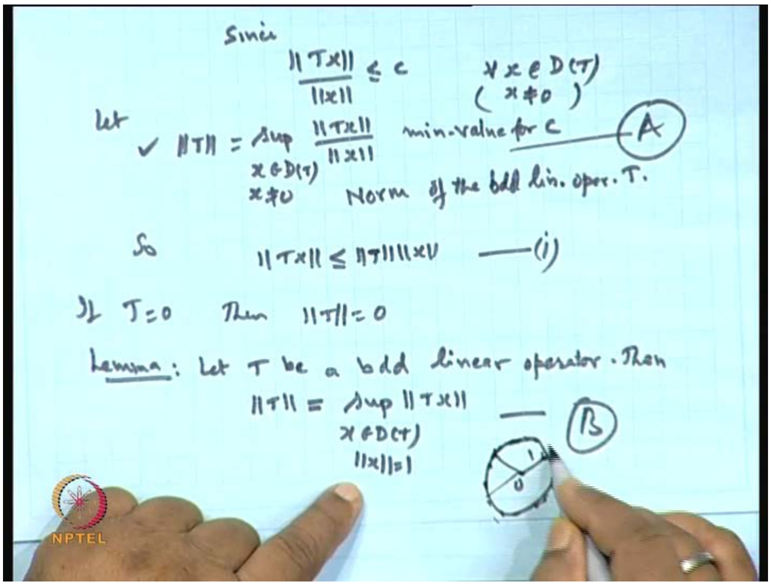

So, take the supremum of this norm of T X over norm of X, and X belongs to the domain of T. If this supremum, that will be the minimum value of C, which will be satisfy by this. So, this we denoted by norm of T. So, let, norm of T is this, which is the minimum value, minimum value for C. So, if I take the real is, then, this minimum value of C, we call it as a norm of the operator, of the bounded linear operator T. Clear? And, if you use this thing, again same, so, we get the relation, the norm of T X replaced by the minimum value norm of T into norm of X and this is a interesting equation; equation, which will be, which will be used very frequently. So, we get from here that, this norm of this, is like this.

Now, if T is 0, because 0 is also a bounded operator; so, if T is 0, here, one more thing which I write, that for all X which are different from 0, where, because, if X is 0, we cannot divide by this. So, for all X, which are non-zero, the minimum value of C will be defined like this. So, X is not equal to 0, or this will be T; is, if T is 0, then, the norm of T is considered to be the 0. So, we do not take it always, in order to complete the definition, we take it norm T; when T is 0, it is equal to 0. Then, we have certain lemma, which will give you, the another definition of the norm T. The definition, there is a alternate way of defining the norm and it is defined as, let T be a bounded; T be a bounded linear operator, as defined, bounded linear operator. Then, the norm of T can also be defined as supremum of norm T X, where X belongs to the domain of T and norm of X is equal to 1; norm of X is equal to 1. This, look the definition this, here, the definition of the norm which is assigned by A and the definition of norm, which is given by B.

This suggests that, norm of an operator T is basically, obtained by choosing the supremum value at all the point X, which are on the unit circle, centered at 0 and radius one. So, if we take any point here, whose length is 1, distance from this is 1; the supremum is taken over all such X, that will be the norm. So, you need not go for the supremum for the entire domain D T; simply choose the point, which has a length 1; those vector which has a length 1. Let us see the proof of this.

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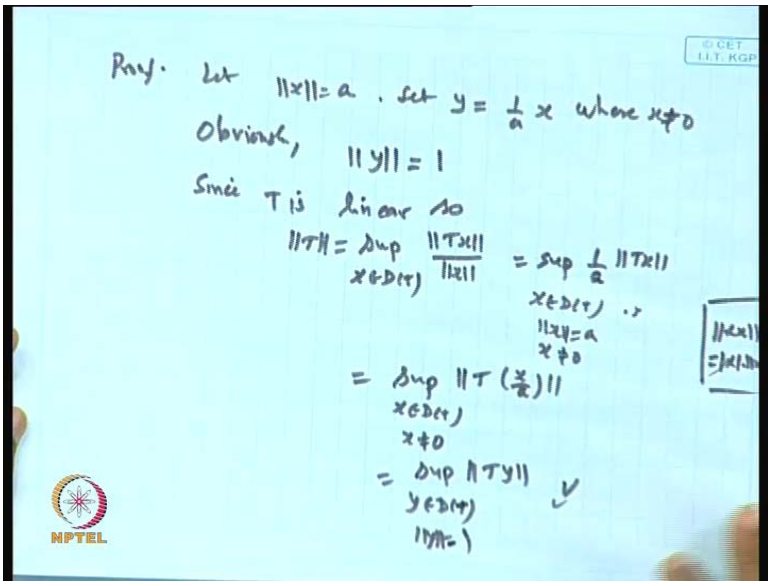

Proof of this lemma. This is very simple way. Let us suppose, norm of x is equal to a and let us set Y equal to 1 by a into x, where x is not equal to 0. Obviously, from here, norm of Y will be 1, because norm of Y equal to norm of x by a, is equal to this. So, norm of Y is 1. And since T is linear, so, we get T of norm of T, which is defined as supremum of norm of T x over norm x, when the x belongs to the domain of D T. So, but norm x is 1; so, it is equal to supremum of norm 1 by a norm of T x where x belongs to the domain D T and norm of x is equal to a. Then, this will be x is not equal to 0. Of course, x is not equal to 0; then, this will be equal to, supremum is taken over x norm of T x by a, where the x belongs to D T and x is not equal to 0; because it is a norm, the property of the norm, if you remember, the norm of alpha x is equal to mod alpha; norm of alpha x is equal to mod alpha into norm x. So, using this property, we are able to write, this 1 by a inside and we get this one; but x by a is Y. So, basically, this is the supremum of norm T y, where the Y belongs to the domain of D T and norm of Y is equal to 1; and this proves the results for it.

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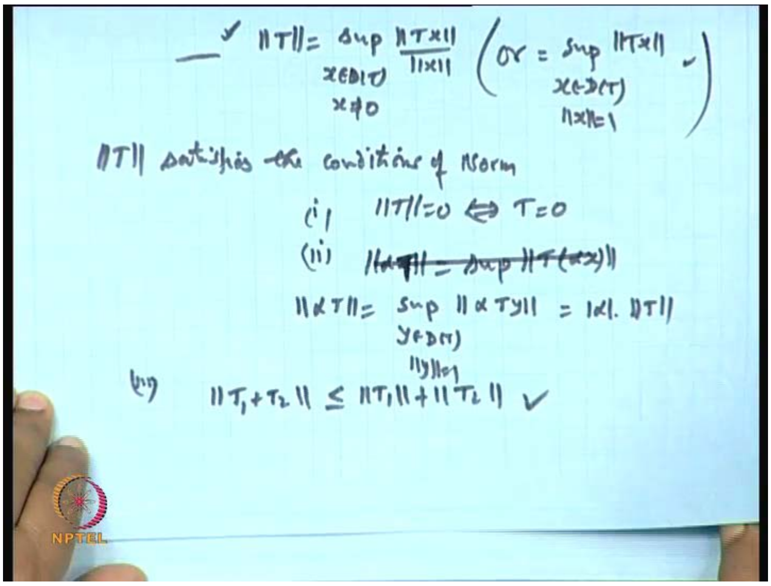

So, a norm of a bounded linear operator, one can defined in this way, as the supremum norm of T X over norm X, X belongs to the domain of D T and X is not equal to 0; or equivalently, we can also say, equal to supremum norm of T X over all X belongs to D T where the norm X is equal to 1. Clear? So, either this way or this. Now, the question, whether this really satisfy the condition of norms. So, if we look that conditions, this norm satisfy, norm of T satisfies the conditions of norm; why, because the first condition is, obviously, true; norm of T equal to 0, if and only if, T equal to 0. Because if it is 0, then, according to this, this supremum has to be 0 for all X belongs to this, but if norm of T X is 0, or norm of T Y is 0 for all Y, where Y is, norm Y is 1, then, it is only possible whenn 10 T is 0.

So, vice versa. So, norm T equal to 0 implies T is 0 and similarly, if T is 0, this entire part will be 0 and we get norm T is 0. Second condition, which we can see, the norm of alpha X, where alpha is this. So, supremum norm of T alpha, this is alpha T, sorry alpha T. So, this is equal to alpha into T X. So, alpha X; we are taking alpha T or you can write this, sorry, let me see; norm of alpha T by definition, it is the supremum of norm alpha T Y; I am taking the second definition; Y is D T; norm Y is equal to 1; but alpha can be taken outside and we get mod alpha norm of T. Similarly, the third condition, one can prove that, norm of T 1 plus T 2 is less than equal to norm of T 1 plus norm of T 2; because the supremum of the sum, is less than equal to sum of the supremum. So, this can be verified. Therefore, all the three conditions are satisfy and this gives a norm and equivalently, this gives the norm for it. So, this one. Now, there are certain examples; is the examples for the bounded linear functional as well as unbounded linear functional.

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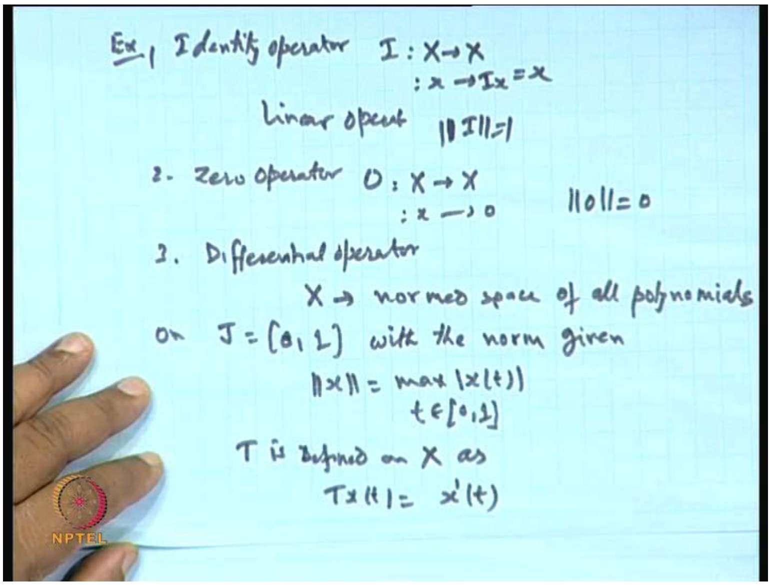

So, first, example of the bounded linear functional. The identity operator, as we have seen, the identity operator I is a mapping from X to X, on a normed space X, such that, on a normed space , which carriers the image X to I X, that is equal to X. And, obviously, it is a linear operator and bounded also; because the bound of this, we can see, it is 1; norm of this is 1. It is easily verified. Then, 0 operator; this is also a bounded linear operator; a 0 from X to X, which carries the image X to 0, and it has a operator, bounded with a bound 0. The third operator, which is a differential operator; we see that, this operator we have seen, it is already a linear operator; that is, if we take X to be the normed space of all polynomials, all polynomials, on the close interval 0, 1, on the close interval 0, 1, with the norms given as norm of X equal to maximum of mod X T and T belonging to the interval 0, 1. Now, differential operator T is defined on, T is defined on this X as T X t, is the derivative of X with respect to t; T x t is the derivative of X with respect to t, where the prime denotes the differentiation of X with respect to t.

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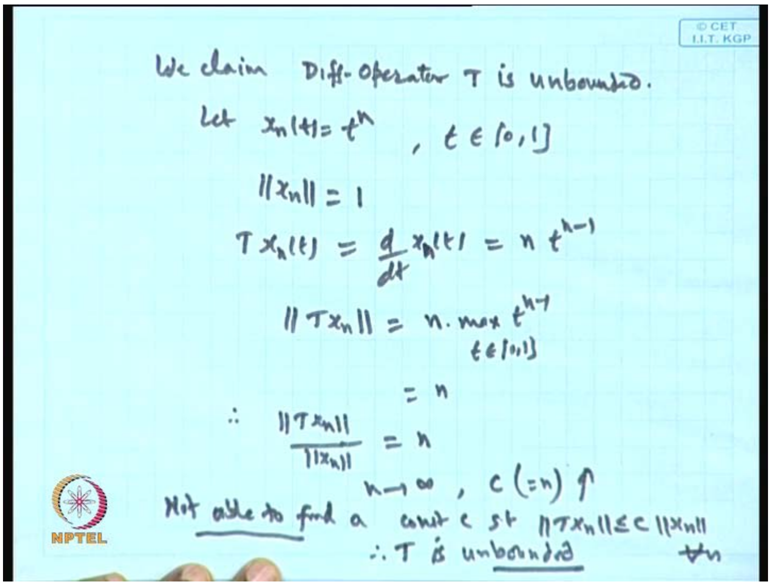

Now, we can see quickly that, T is linear; this we have already shown earlier; T is linear. Now, to show whether T is bounded or unbounded, let us check. We claim that, this operator T, differential operator T, is unbounded operator; is not bounded. It means, we are unable to get a constant C, such that, norm of T X is less than equal to C times norm of X. Let us see, for example, suppose, I take a sequence of the polynomials, a functions X n t as T to the power n, where the t belongs to the interval 0 to 1; this is a polynomial. So, it belongs to the class and find out the norm of X n; by definition the norm of X n will be equal to 1, because the maximum value is taking. So, norm of this is 1, fine.

When we operate this T of X n t, the operation will give derivative of X n t, with respect to T, and that becomes, n t to the power n minus 1. So, the norm of T X n, because it is also a polynomial, so, norm is defined as the maximum of this. So, it will be the, n into the maximum value of this, T to the power n minus 1, when T belongs to 0, 1, and that is nothing, but n, ok. So, what we get it here is that, norm of T X n divided by norm X n, that will come out to be the n. Now, as n increases, the bound C, this is equivalent to C, the C which is equal to basically n, increases. It means we are not able to get, not able to find a constant C, such that, norm of T x n is less than equal to C times norm X n, for all n; is not possible; we are not able to get. So, this will show, this shows that, T is unbounded; is not a bounded operator. So, what we see here, this is a very interesting example. Why in the interest, because the differential operator is a very frequently used operator and entire real analysis and concept is based on this continuous and differentiation. So, this operator, which is an unbounded linear operator suggest that, the theory of the unbounded linear operator plays a vital role in the development of the analysis or functional analysis or any branch of mathematics. We have a lot of application of an unbounded operators environment, ok.

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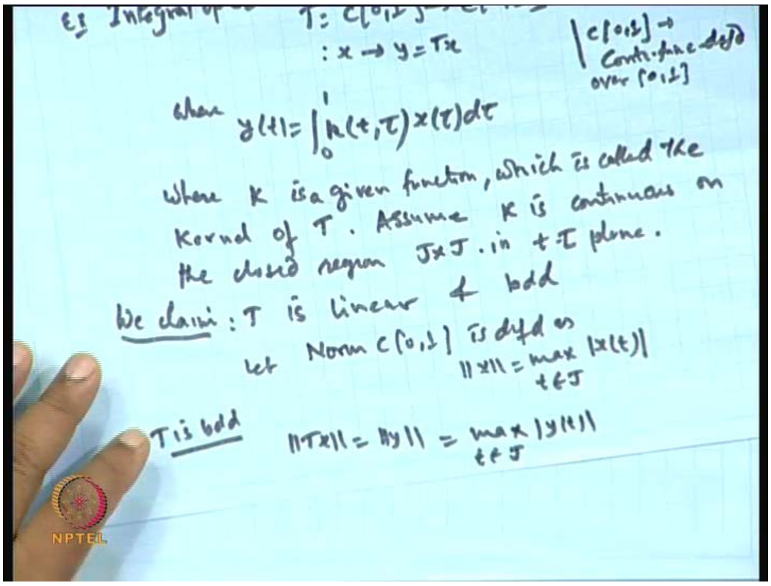

So, we have also come across about the integral operator. What is that integral tha, suppose, T is an operator from C 0, 1 to C 0, 1, where C 0, 1 is the set of all continuous functions, continuous functions, defined over the, defined over the close interval 0, 1; set of all continuous functions defined over the close interval with 0 1. So, if we take a point X here a, the corresponding value Y, is coming to be in the form of T X, which also continuous, where the Y, we have defined as integral 0 to 1, K of T tau X tau D tau, where K is known as the kernel; is a given function, is a given function, which is called, is called the kernel of the operator T, kernel of T. And, we assume that, K is continuous on this closed square, on the closed region or a square, J cross J, that is on 0, 1 cross 0, 1, 0 1 cross 0 1 in the T tau plane, T tau plane. So, this definition, this way, when we define the operator T, which maps the X to Y, where Y is defined in this way, is known as integral operator. Now, we claim that, this operator T is linear and bounded.

Linearity follows immediately, just by replace X equal to alpha X plus beta Y. So, here it will change the alpha X tau plus beta X tau, and then, it can be break up as a sum of the two integral. So, linearity of the operator T is guaranteed. To show the boundedness, let us see, the function is continuous function and the norm on C 0, 1, let the norm on C 0 is defined as, or is considered as the norm of X is maximum of mod X t D t, mod X T D T, where not X is, sorry, and T belongs to j. Because there are two ways in defining; one is this way; another is the integral form. We are choosing the norm on this form. Now, once you take this, then, T is bounded, obtained from here. What is the norm of T X? Norm of T X means, it is equivalent to the norm of Y, but norm of Y is the maximum of Y, means, this part; this is, is it not. So, maximum of mod Y T, where the T belongs to J.

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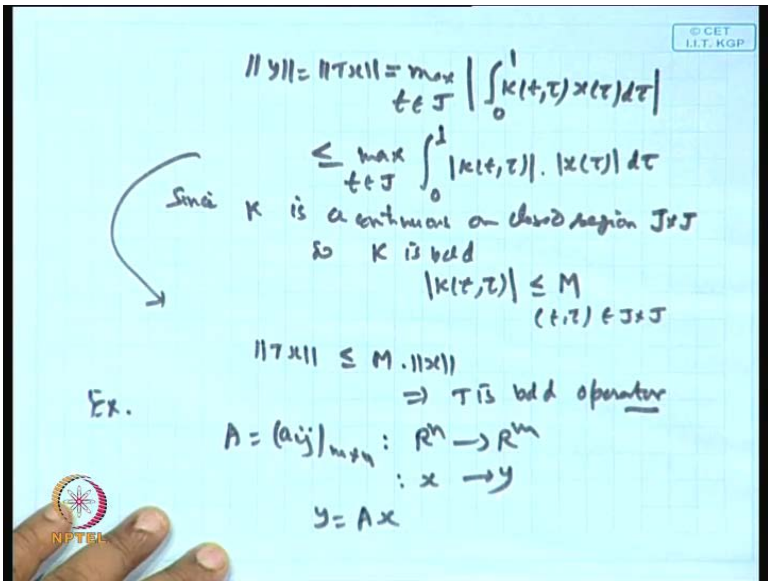

But maximum of Y T, this is equal to… So, norm of T X, norm of Y, which is norm of T X, is equal to the maximum of t, belongs to J, modulus of integral 0 to 1, K t tau X tau D tau; modulus use the modulus. So, it will be less than equal to maximum of t belongs to j, integral 0 to 1; take the modulus inside; mod of K t tau into mod of X tau D tau. Now, K is a continuous function. Since K is a continuous function by assumption, on the closed region, on closed region J cross, on a closed region J cross J, is it not. On this closed region, the function is continuous. So, it is bounded function. Every continuous function in a closed region is bounded. So, K is bounded; K is bounded means, we can find a bound for a number, say m, which is positive and mod also, that mod of K t tau is less than equal to m for all t tau, for all t tau belongs to the region J cross J. So, this is true, bounded.

Hence, from here we can say, the norm of T X is less than equal to, this will be m and what will be this maximum of this norm of X T, is the norm of X; that is all. So, we get this one. Hence, this shows that, T is bounded operator, ok. So, integral operator comes out to be a bounded linear operator. That is all. Now, yesterday, we have also taken one example of the matrices, and matrices we have to consider; in fact, it comes out to be an operator that, if a is a matrix of order m cross n, then, it behaves as an operator from R n to R m, R n to R m, such that, the image of X goes to Y, where the Y is equal to a X; and is defined as, if we choose X to be X i 1, X i 2, X i n, so, if we take X to be X i 1, X i 2, X i n, of order n cross 1, Y equal to eta 1, eta 2, eta m, of order say m cross 1, then, a X Y is equal to a X will give, Y equal to A X will give the eta i, sigma a n K X i k, K equal to 1 to n and that will give the eta a n K, so, eta n a i K.

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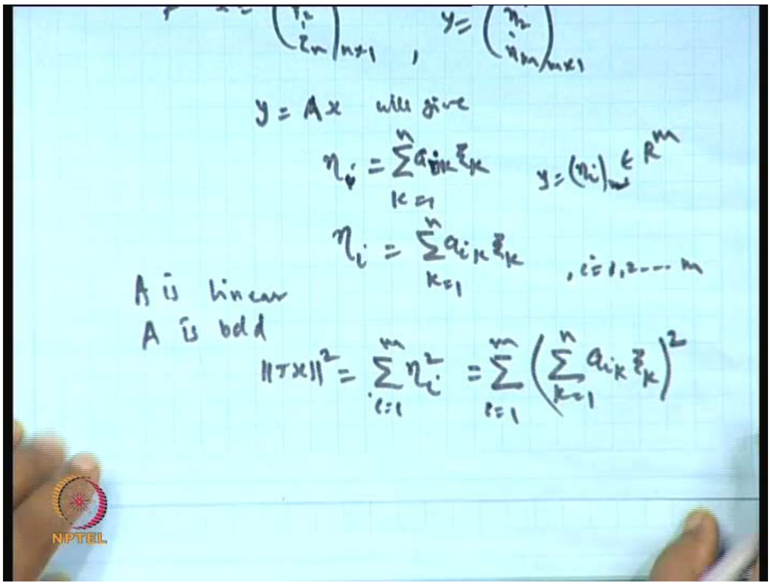

Here, I am taking i, sorry this is i. So, it will be written, eta I, as sigma a i K X i K, K is 1 to n, like this, where the i will varies from 1 to m, 1 to m. So, this will be our corresponding map. Now, let us see the matrix A. We have seen matrix A is linear. Now, we also claim, A is bounded. It is not only linear, it is bounded; why? What is the norm of T X? Norm of T X is square. This is equal to sigma eta i square, i is 1 to m; because this Y is equal to eta i and eta i is varying from 1 to m, and it belongs to… So, Y is equal to eta i of order, eta i belongs to R m, ok. So, the norm will be defined in this fashion, raised to the power half. So, we are getting norm. But eta i is A. So, we are getting sigma, i is equal to 1 to m, then, sigma K equal to 1 to n, a i K X i K square X i K, k square, that is all.

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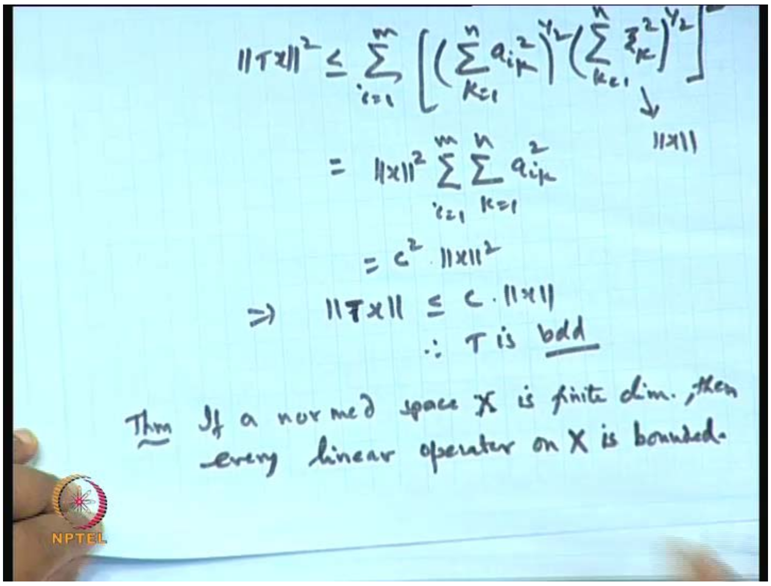

Now, apply the Cauchy Schwarz's inequality. So, by Cauchy Schwarz's inequality, Cauchy Schwarz's inequality, we get that, this norm of T X square is less than equal to sigma, i equal to 1 to m, as usual, then, sigma K equal to 1 to n, a i K square power half, sigma K equal to 1 to n X i K square power half, that is all. Power half and then, whole square, because this is square. So, we have applied this Cauchy Schwarz's inequality over the product of these two things. So, sigma of, if these are a i K and X i K, they are nonnegative numbers, then, we are getting this sigma a i K, this whole square into power half, sigma X i K square power half and then, square will be there.

Now, this shows from here that, this is equal to… Now, this part is what; this is basically, is equal to norm of X. So, we are getting, this is norm of X square into double summation i equal to 1 to m, K equal to 1 to n, a i K square, a i K square and square this. Now, this is nothing, but simply a constant. So, let it be replaced by another constant, say C squared into norm of X square. Therefore, we get from here is, the norm of T X is less than equal to C times norm of X . So, T is bounded. So, every matrix of order m cross n represents a bounded linear operator from R n to R m, that is all. And, this will finite case, it is very simple, because we are dealing basically, the matrices. Now, there is another advantage of the finite dimensional normed space. There is so many concepts are valid for a finite dimensional case. Just like in, we have seen earlier, if the two norms are equivalent, then, over a finite dimensional case, the topology generated by these two norm will be the same. But this may not be true, in case of the infinite dimensional case.

So, here also, there are many results, which are valid for a finite dimensional case, in general, but may not be true, in a infinite dimensional case. So, one of them, results we are telling is that, if a normed space is a finite dimensional, then, every linear operator must be bounded; but result is, if a normed space, if a normed space is finite dimensional, normed space X is finite dimensional normed space, then, every linear operator on X is bounded. Let us see the proof of this. In case of the finite dimensional normed space, any linear operator is a bounded operator.

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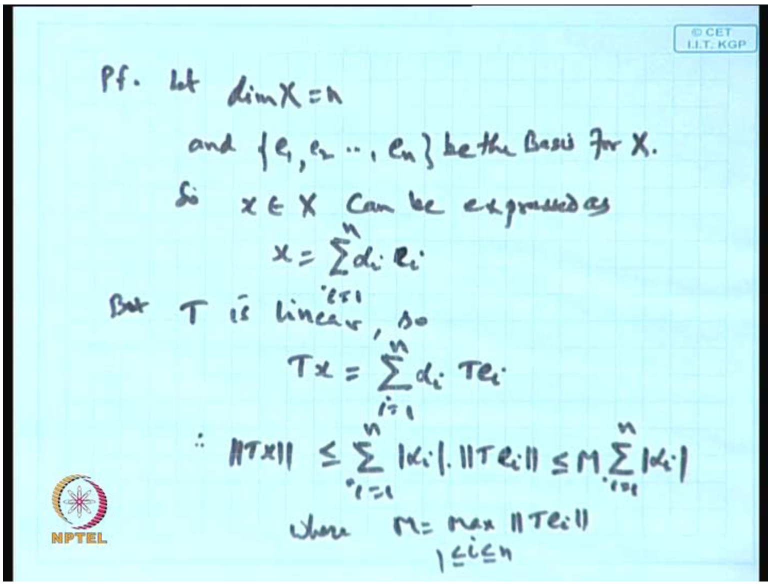

The proof of it. Suppose, we take X to be a finite dimension. So, let the dimension of X be n and e 1, e 2, e n, be the basis element, basis for X . So, any element x belonging to capital X, can be expressed as sigma alpha i x i alpha e I, because e 1, e 2, e n are basis element, i equal to 1 to n. But T is linear. T is linear, so, the image of X under T become sigma i equal to 1 to n, alpha i T of e i, T of e i. Therefore, norm of T X is less than equal to sigma i equal to 1 to n, mod of alpha i into norm of T i, T e i. T e 1, T e 2, T e n, these are the finite in number and norm we are taking. So, this is length of the vector T e 1, T e 2, T e n; replace this by a maximum value. So, let it be m into sigma i equal to 1 to n, mod of alpha i, where m denotes the maximum value of norm T e i, where i e varying from 1 to n. So, we are getting this. Let it be equation 1.

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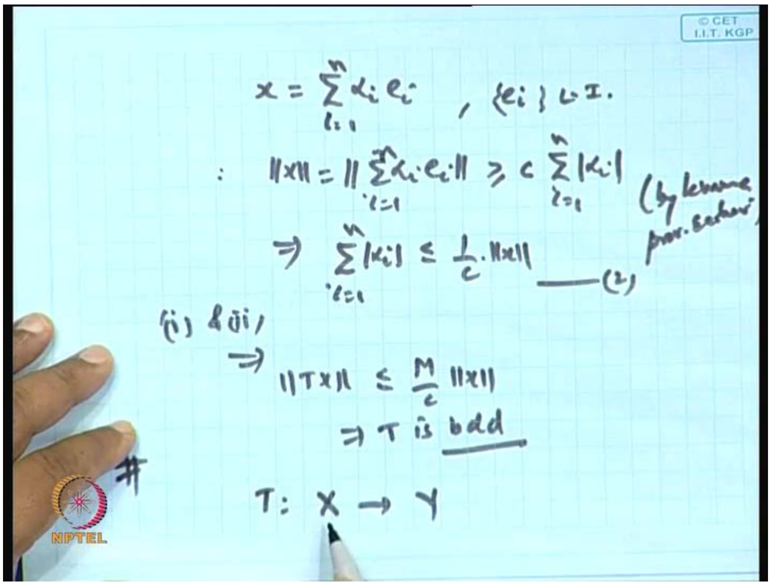

Now, since X is, can be, is expressed as i equal to 1 to n, alpha i e i, where e i, e 2, e n these are the linearly independent set of vectors; because these are the elements of the basis. So, for a linearly independent vector, we have seen that, one result that, if e 1, e 2, e n are linearly independent vectors, then, one cannot expect a vector involving the large number of scalars, but a minimum length; that is one can find a constant C or epsilon, such that, norm of this is greater than equal to epsilon times sigma of this. So, we get from here is, the norm of X which is equal to norm of sigma, alpha i e i, i is 1 to n and again apply this, so, there exist a constant C, such that, this will, condition holds. By, there, by the earlier lemma, proved earlier; this is the lemma proved earlier.

So, from here, we get sigma mod alpha i, i is 1 to n, is less than equal to 1 by C into norm X. Let it be 2. Now, if we combine first and 2, 1 is norm of T X is less than equal to this; 2 is sigma alpha is less than… So, 1 and 2 gives that, norm of T X is less than equal to m by C into norm X; that is, there is a constant, some constant this, which is… So, this implies that, T is bounded. So, every linear operator in a finite dimensional case, is a bounded linear operator. Now, since the linear operators, they are basically a mapping; they are a mapping. Only difference is that, when we operate from vector space to a vector space, then, this is called an operator. So, whether vector space replaced by the norm or a Banach space, again this is a mapping. So, in case of the mapping, we have a concept of continuity. So, similar concept, we can also define over a bounded linear operator, over a linear operator or in general, an operator, when the operator will be considered to be a continuous operator, at a certain point or over the entire domain T.

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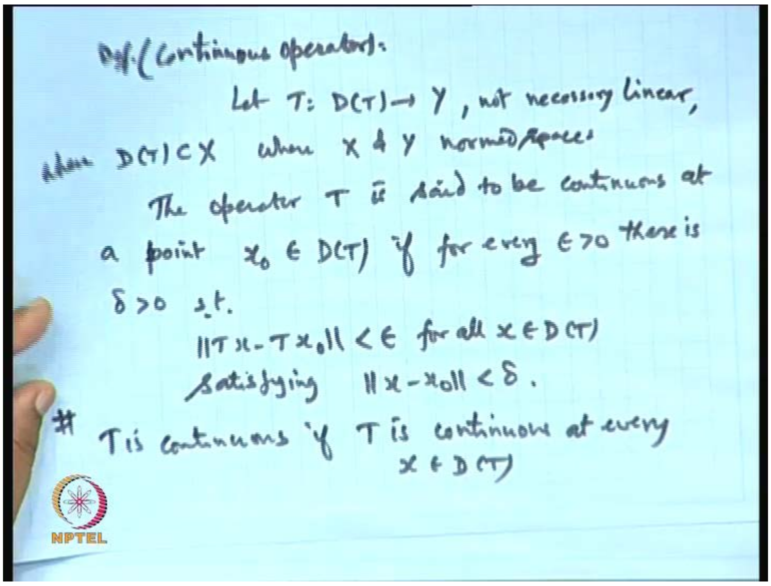

So, which define the continuity of an operator. continuity of an operator. Continuous operator. continuous operator. So, let us suppose, T is an operator, from D T to Y, not necessary to be linear, not necessary linear, any arbitrary operator, that is all, where the D T is contained in X, X is a vector space and Y is also a vector space or norm, where X and Y are chosen to be, say, normed space spaces. We define the operator T, is said to be continuous at a point, at a point X naught, belonging to the domain D T, if for every epsilon greater than 0, there is a delta greater than 0, such that, the norm of T X minus T X naught is less than epsilon, for all X belonging to the domain of T, satisfying the condition, norm of X minus X naught is less than delta.

So, it is definition is parallel to our definition of the continuous function. The only thing, the mapping is, f is replaced by operator T. So, an operator which, for which this condition is satisfied, that for any epsilon, one can identify delta, such that, difference between the images remains less than epsilon, provided the differences between the points is less than delta. Now, if X naught is an arbitrary point. If this is true for all trial, this points, all the points of D T, then, we say T is continuous over the J. So, we say T is continuous, if T is continuous at every X, belonging to domain D T . So, clear? So, where.

Now, if an operator is a continuous operator and operator is also linear operator, operator is bounded, these three concept we have introduced. What is the relation between these three concepts? The very interesting result is, we can now say that, in case of the linear operator, the continuity and boundedness comes out to be the same, ok. Because, in general, in a function, a function is continuous, it will be bounded over a; if a function is continuous over a closed interval, it will be a bounded function. But if the function is bounded, then, it need not be a continuous function, ok. But here, the continuity and boundedness will be identical, when T is a linear. And, another point, which I alsom I want to make it clear, the concept of the boundedness of a linear operator and the concept of the boundedness of a function is different. In case of the function or mapping,we say a function is bounded, when the corresponding range set is bounded. But here, we do not talk about this thing; what we say, a operator is bounded, when this satisfy the certain condition. The condition is the norm of T X is less than equal to C times norm of X; that is the image set, the image is norm of T X . So, length of this vector divided by the norm of this original vector, if we take the supremum of this, this must be some number C, greater than 0; supremum must exist for it. Then, we say, it is a bounded operator ok.

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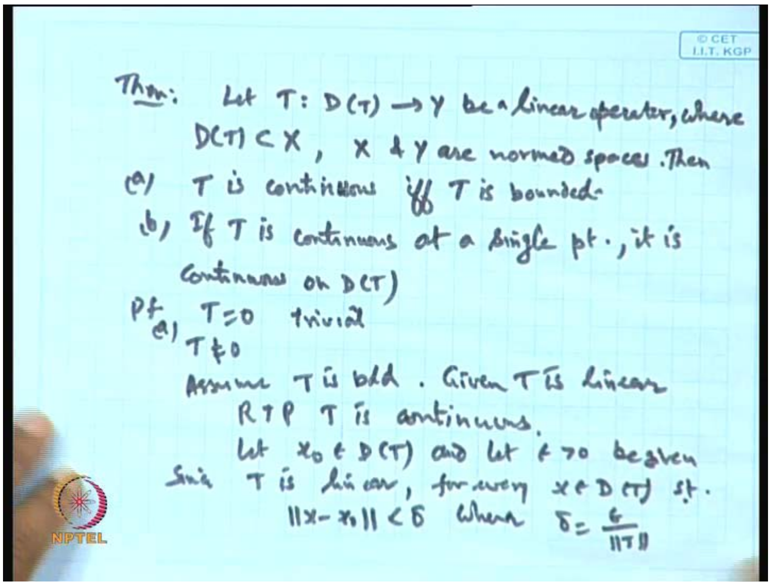

So, we have now, a relation between this; and the relation is in the form of result, we say theorem. Let T be a operator from D T to Y, be a linear operator, where D T lies in X, X and Y both are normed spaces, normed space. Then, one, T is continuous, if and only if, T is bounded. Second one is, if T is continuous at a single point, then, it is continuous, it is continuous throughout the domain T; it is continuous on D T, throughout the domain. So, in case of the linear operator, this is also, second result in testing, that to test the continuity of the entire domain, just simplifying the continuity at a single point; because the T is linear, it will automatically spread the continuity over the entire domain D T. The proof.

Suppose, T is 0; case 1. Then, nothing is proved; everything is very obvious; because in case of the 0 operator, which is a linear operator, continuity and Boundedness will be the same. So, it will nothing do. So, it is obvious. T is a statement is obvious or trivial also. So, let T is not equal to 0, ok. Now, assume… So, we wanted to proof first thing, a. So, assume, T is bounded. We wanted to show T is continuous; T is bounded; given T is linear, this is known already, linear. To show T is continuous, T is continuous. So, suppose X naught be a point in the domain D T. And, let epsilon greater than 0 be given. So, for the continuity means, we have to find a delta, such that, norm of T X minus T X naught less than epsilon, only when X minus X naught less than delta, ok. So, since T is linear, linear, so, for every X, every X, for every X, belonging to the domain D T, such that, norm of X minus X naught less than delta, where delta, I am choosing to be epsilon over norm T.

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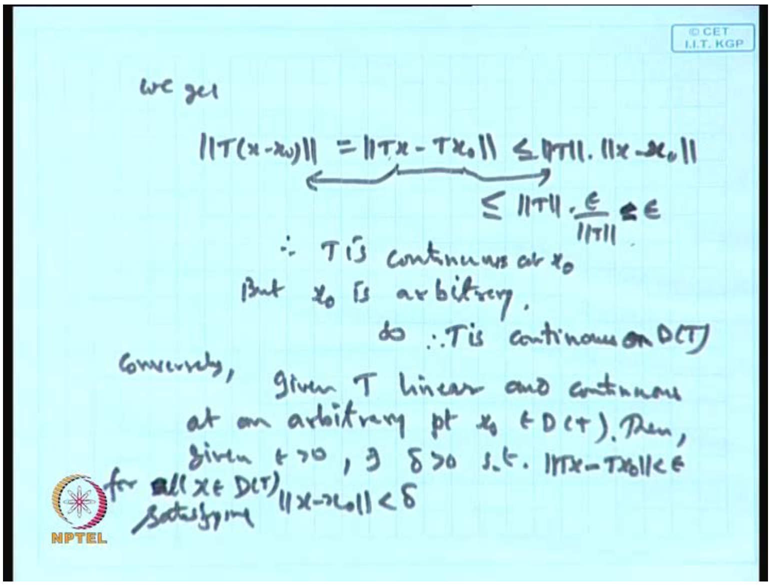

We obtained, we get norm of T X minus X naught, this is equal to norm of T X minus T X naught, which is less than equal to norm of T into X minus X naught, and this is less than equal to norm of T and X minus X naught is less than epsilon over norm T, so, it is less than epsilon; it is basically equal, ok. It means that, if I choose an arbitrary point X naught and epsilon greater than 0, then, because T is linear, this condition is, can be written. T of X minus X naught can be written as T of X minus X naught and because T is linear, by this and bounded, is also, because of the boundedness, we can write from here to here. Because norm of T X is less than equal to norm T into norm of X . So, from here, we can write this thing. And again, epsilon is chosen. So, we can find a delta in terms of epsilon, which is equal to epsilon by norm T. Substitute it, we get this thing. Therefore, what we conclude is, the T is continuous at X naught.

But X naught is an arbitrary point. So, we can choose any point. Therefore, T is continuous over the throughout, continuous on D T, throughout the domain D T. Let us see the converse part. Conversely, what is given now that, we wanted the T to be bounded. So, T is given to be continuous. Given T linear and continuous, continuous on, at an arbitrary point X naught belonging to D T, ok. So, by definition, then, for given epsilon greater than 0, there will exist a delta, which is depends on epsilon greater than, such that, norm of X minus X naught is less than delta; norm of X minus, such that, norm of T X minus T X naught is less than epsilon, for all X, for all X belonging to D T, satisfying this condition. Norm of X minus X naught less than delta; this is by definition, ok.

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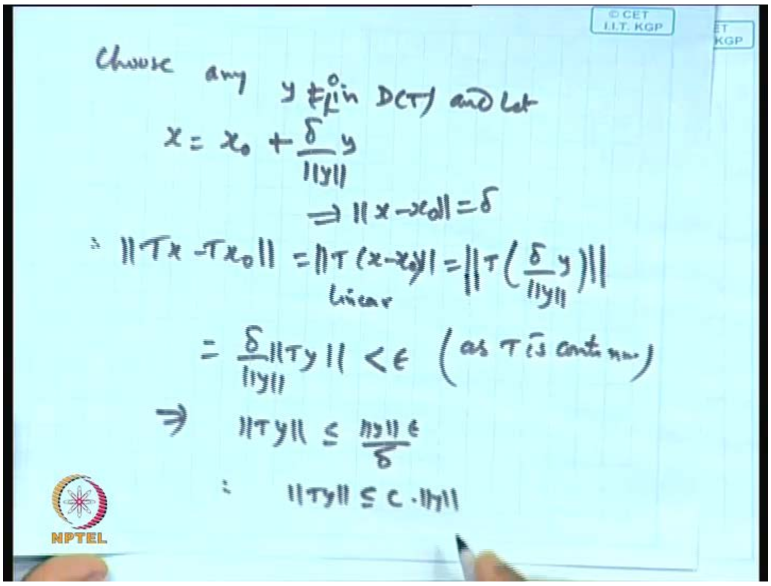

So, now, choose the point y, choose any y, different from 0 in D T. And, let x is suppose, x naught plus delta over norm y, into y, since y is not equal to 0, this is 0; y is not equal to 0, so, we can choose like this. Now, from here, obviously, x minus x naught norm y is delta. This. So, by definition of continuity, therefore, T x minus T x naught, this is equal to norm of T x minus x naught, because the T is linear, ok. Then, this will be equal to norm of T x minus x naught. You can write it, delta over norm y into y; and this will be equal to… Now, since T is, T is given to be a, this continuity is satisfying; T is continuous. So, this has to be less than epsilon, for whenever the norm of this thing is there. So, we can say, this part is less than epsilon.

So, we are taking this delta over norm y out, into T of y. Now, this is to be less than epsilon as T is continuous. So, from here, we get norm of T y is less than equal to norm of y, norm of y over delta into epsilon and that is shows that, this will be equal to, that is norm of T y is less than equal to constant times norm y; because this is nothing, but simply a constant. Therefore, T is bounded, ok. So, this shows that T is bounded. So, this proves the result. The second part is very obvious. Second part, what it says is that, if it is T is continuous at a single point, it is continuous on this. So, if suppose, T is continuous at the single point, then, according to this, it must be bounded; bounded means throughout the bounded. So, it is continuous again at any arbitrary point, and this proves. So, second part follows by the… Thank you. Thanks.