Functional Analysiss

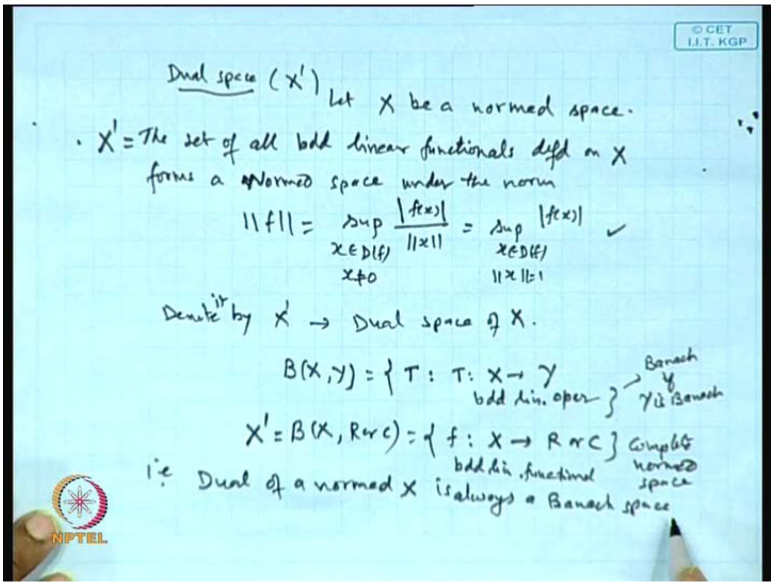
Prof. P. D. Srivastava
Department of Mathematics
Indian Institute of Technology, Kharagpur

Module No. # 01

Lecture No. # 18

Dual Spaces with Examples

(Refer Slide Time: 00:22)

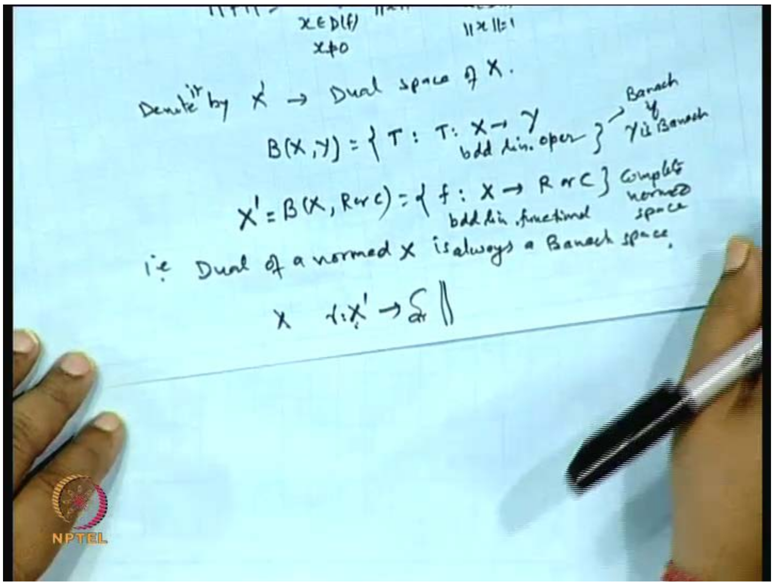


In last lecture, we are discussing the dual spaces. We will denote this by X dash. So, let X be a normed space and the set of all bounded linear functionals defined on X forms a vector; it forms a normed space, under the norm as supremum mod of f x over norm x, when the x belongs to the domain of f and x is not equal to 0. And, this is as same as, as supremum mod f x, when x belongs to the domain of f and norm of x is equal to 1. So, in fact, the operation, addition and scalar multiplication will be the same as we have discussed in the operator and this forms a normed space.

This set of all bounded linear functional defined on X, we denote it by X dash and is called the dual space of X. So, this is denoted by X dash, denote it by X dash and is called the dual space of X, ok. So, basically, this is a dual space X dash; it is a particular case of the B X Y. What is the B X Y? B X Y, if you remember, it is the set of those operator T, where T is a mapping from X to Y, is a bounded linear operator, defined from one normed space to another normed space; and this forms a norm under the same norm T, as defined by norm of mod T x over, norm T x over norm x, etcetera.

Now, Y is a normed space. In place of this Y, if I write either R or C, then, this is, is the linear functional, a bounded linear functional defined from X to R or C. So, B X R, we are denoting this by X dash and we have seen already that, this class is a Banach space, if Y is Banach, is it not. That we have already proved, the set of all bounded linear functional from one normed space to another is a Banach space, if Y is a Banach space. So, here, in place of Y, we are choosing R or C. So, but, R and C, they are complete space; so, obviously, X dash will be a complete normed space; that is, a dual of a normed space is always a Banach space. So, that is, the dual of a normed space X is always a Banach space, clear. And this follows from… So, that is the one properties.

(Refer Slide Time: 00:22)

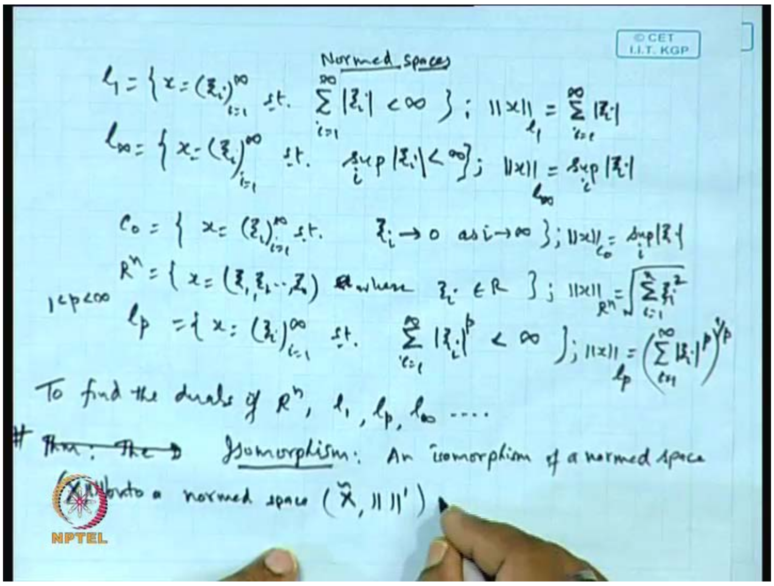


Now, in order to find out or investigate the duals of various spaces, what we do is, we use the concept of the isomorphism; that is, if any space X is given and we are interest in finding the dual of this, then, we will try to establish a relation gamma or some C from X dash to that space, say S, which is one-one onto and preserve the norms; then, we say, this two spaces are basically, identical spaces; they differ only, so far, as the nature of the points are concerned, but for other point of view, they are the same. Metric property remains the same, because the norms are same. Therefore, this two spaces are the carbon copy of the same.

So, when we say, the X dash is isomorphic to S, it means the dual of X is nothing, but S. So, that is our concern. So, in order to investigate the dual of a space, we first identify the set and a mapping, which can be a isomorphic mapping from that X to that space choosing. And, once it is established, then we say, the dual of that space, is the particular S.

For example, here, say, if I choose X equal to l 1, then, we will show the dual of l 1 is nothing, but l infinity; that is, if we are, we will be able to get a mapping from the dual of l 1 to l infinity, which is one-one onto, bijective and the norms are preserved, clear. So, this l 1 dual and l infinity, both are isomorphic spaces. So, l 1 dual is nothing, but the l infinity. Similarly, we go for the other. So, main idea is, establishing this dual, investigation of the dual requires the concept of the isomorphism in normed space.

(Refer Slide Time: 06:52)



Sir, what is l 1 and …

l 1 means, l 1 is the space, set of those sequences x, infinite sequences, such that, sigma of mod x i i, i is 1 to infinity, is finite. This is the l 1 space; l infinity space is the set of those sequences, infinite sequences x i i, such that, supremum of mod x i i, over i, is finite. While C naught is the set of those sequences x i i, i is 1 to infinity, such that, x i i, this sequence tends to 0, as i tends to infinity or convergence sequence converging to 0.

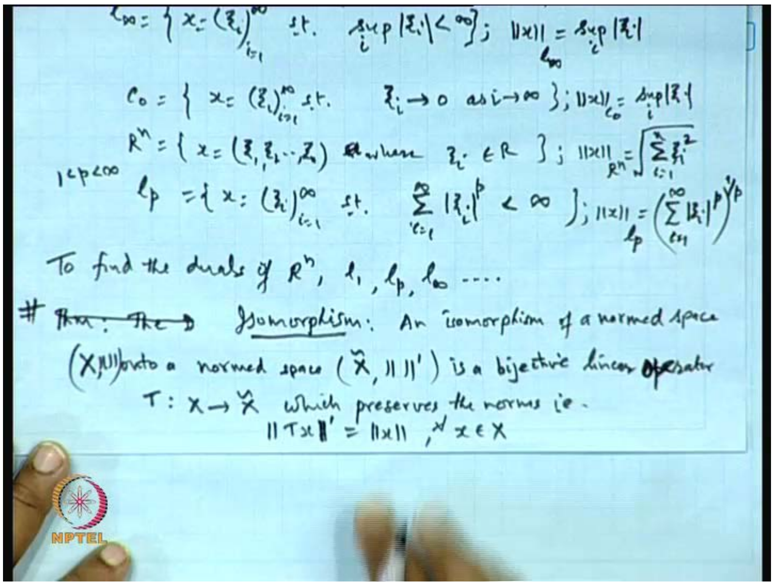
Similarly, C is the sequence, class of those sequence, which converges to limit point l, dual and R n is the set of those n points, which are the n-tuples, such that, n tuples, set of all n tuples, where xis are real; set of all interval means, point in the n dimensional plane; this is R n. C n is the point in the n dimensional plane of c n; points are complex numbers like this. So, here, R n, C n and l p, 1 is less than p less than infinity, l p is the set of those sequences, infinite sequences, i is 1 to infinity, such that, sigma of mod x i i power p, i is 1 to infinity, is finite, ok.

Now, these spaces, we have already discussed that, these are the normed space; the corresponding normed spaces, these are all normed spaces and corresponding norms are defined as norm of x l1 is nothing, but sigma of mod x i i, i is 1 to infinity, this is the norm x l 1. Here, the norm of this x l infinity, this is the norm, defined as the supremum of mod x i i over i. In fact, we can also write this thing, set of all sequence, which are bounded; that is the same as the supremum is finite, ok. And here, the norm of this thing is defined in terms of S, if it is a norm of infinity. So, norm of C naught is the same as norm of infinity, because C naught is a subclass of l infinity; and then, R n, here, the norm is define as sigma x i i whole square, i is 1 to n and under root.

x 1 is x i 1 square plus x i 2 square plus x i n square and raised to the power half. And here, the norm of x l p is defined as sigma mod x i i power p, i is 1 to infinity, raised to the power 1 by p, raised to the power 1 by p. So, that way, we can say that, this l 1 is a norm expression that, this norm will be (( )). We are interested in finding the duals of, to find the duals of, duals of R n, l 1, l p, l infinity, etcetera, etcetera, ok. Dual of this, dual of l 1, R n. In fact, this is a general practice, whenever we define any space, a collection of the point, together with certain operation, addition, multiplication and the norm, as soon as we get the normed space, then, we try to find what is the, its dual.

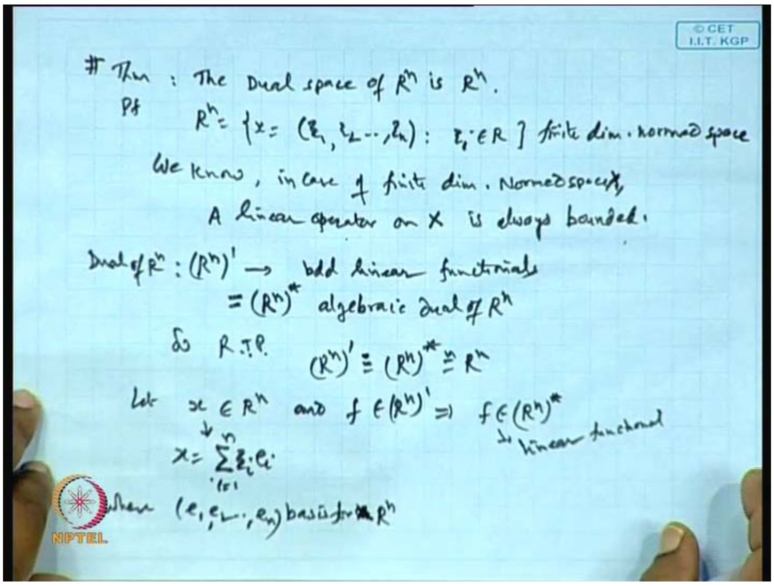
So, combination of the space with the dual is very much important, because with the help of the dual, one can investigate so many other properties of x; that we will come to know, when we go for the Reflexibility and separability and so many things can be related with the help of this duals form. So, we go further. Today, we will discuss the duals of these spaces, clear. So, first is, the result or theorem - the dual of…Now, before starting the dual, let us see that, little bit idea about this isomorphism, is it not. We have already define the isomorphism, but let us say, again, an isomorphism of a normed space X, of a normed space X, onto a normed space X delta, suppose, I take, this is the norm and here is the norm defined as this dash; an isomorphism of a normed space X onto an another normed space Y, into a norm space Y, is a bijective, is a bijective linear mapping or operator, because is operator, bijective linear operator T from X to X delta, which preserves, which preserves, the norms, which preserves the norms; that is, the norm of T x, under this norm, is the same as norm of x, when the x belongs to capital X, for all x, belongs to this.

(Refer Slide Time: 06:52)



So, an isomorphism is a one-one onto mapping, a linear mapping, which preserve the norms, clear. And, there the T X is the norm of X star and X under the norm of x, so that, this two spaces are said to be isomorphism, ok.

(Refer Slide Time: 14:57)

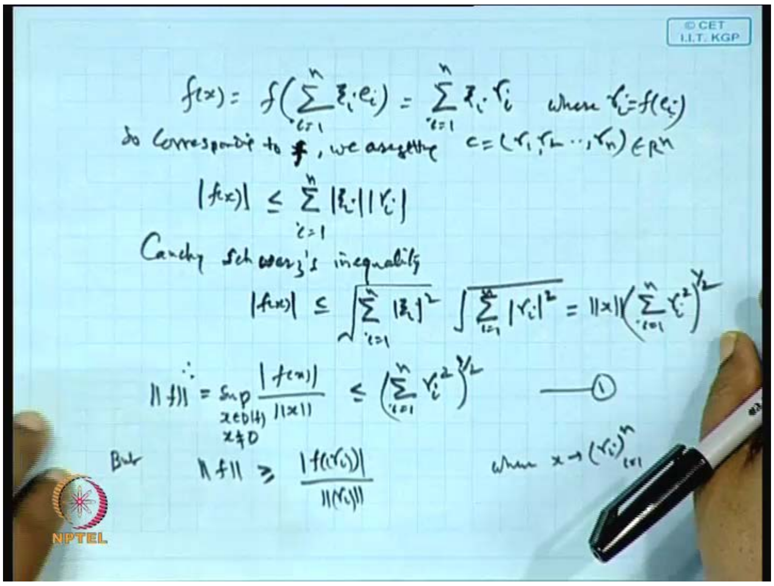


Now, this concept will be used in investigating the dual of this. So, first result is, the dual space of R n is R n; the dual space of R n is R n. Now, what is R n? R n is a set of those points x, which are n-tuples of real numbers. So, this is a finite dimensional normed space or vector space, finite dimensional normed space and we know, in case of the finite dimensional normed space, any linear operator is bounded. We know, in case of finite dimensional normed space, a linear operator on X, finite dimensional normed space X, a linear operator on X is always bounded, is always bounded, ok.

So, if we take the dual of this R n, this is the dual of R n, dual of R n; that is a R n dash. What is this? This is nothing, but a bounded linear functional, bounded linear functional; but, in case of the finite space, the linear operator is always bounded. So, will it not be the same as R n star, star, that is the algebraic dual of R n; algebraic dual means, set of all linear functional define on R n; it forms a vector space; it forms the normed space and we say, it is a algebraic dual of this, is it not. So, in case of the finite dimensional space, the dual space, the set of bounded linear of functional, will give the same set as the set of linear functional. Therefore, in order to prove that, R n dual, we will simply establish for the algebraic dual. We will show the algebraic dual of R n star is nothing, but R, because both are identical.

So, basically, require to prove is that, R n dash, which is already equal to R n star is nothing, but R n, clear. That is our aim. R is isomorphic to R n. This dual is isomorphic to R n. So…So, we have to establish a mapping from R n star to R n, which is one-one onto and preserve the norms. So, that is ok. Now, let us take, let x belongs to R n and f is a point in R n dash. So, it is a bounded linear functional. So, f will be a point in R n star as a (( )) one. x is in R n. So, x can be written as sigma x i i e i, i is 1 to n, where, where e 1, e 2, e n, this is the basis for X, because X is finite dimensional, so, basis for R n; I am sorry, this is basis for R n; because we are, it is dealing with R n; is a finite dimensional. So, e 1, e 2, e n will be basis for this. f is a linear, bounded linear functional. So, it is a same as the linear functional, linear functional.

(Refer Slide Time: 19:47)



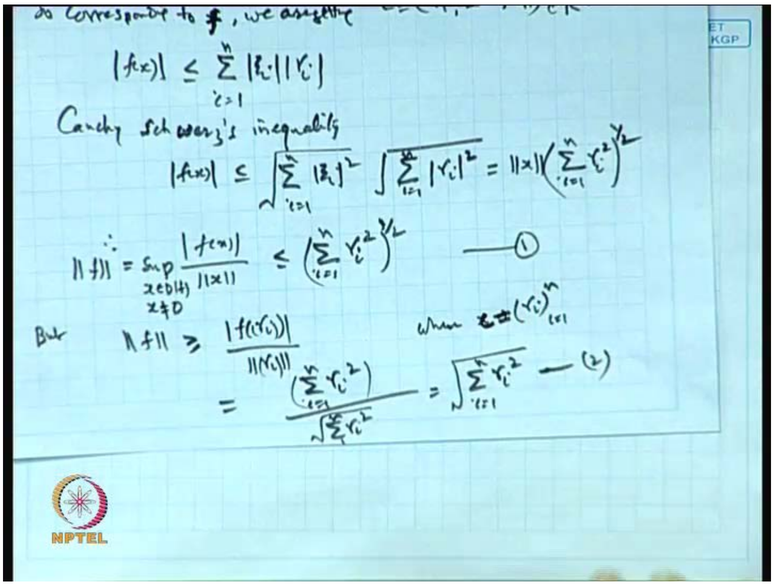
So, we can write the f of x, so, we can say, f of x equal to f of sigma i is 1 to n, x i i e i, and that will be the same as sigma i is 1 to n, x i i f of e i, which is, I am writing gamma i, where gamma i is equal to f of e i, gamma i is f of e i, is it correct or not. Now, if I take mod of x, then, mod of x will be less than equal to sigma mod of x i i mod of gamma i, i is 1 to n. Now, apply the Cauchy Schwarz inequality. So, by Cauchy Schwarz inequality, we can say, mod of f x is less than equal to sigma mod x i i square, i is 1 to n, under root, into under root sigma i equal to 1 to n mod gamma i square. Is it ok, by Schwarz inequality?

Sigma of x n y n is less than equal to sigma x n square power half is sigma y n square into sigma y n square power half. x is a point in, x is a point in R n. So, what is the norm of x? The norm of x, we have already defined, R n is this way, clear. So, basically, this part is nothing, but norm x. So, we can say, this is equal to norm of x into sigma of this, i is 1 to n gamma i square, mod will, because positive; so, it is like this, gamma. Is it correct or not? So, take the supremum. Therefore, mod of f x over norm x, supremum over all such x will be less than equal to sigma i is 1 to n, gamma i square power half. Let it be 1; but this supremum, is it not the same as norm of f, by definition; x belongs to domain of f, x is not equal to 0.

So, it is the norm of f. So, norm of f is less than equal to this, clear; but, norm f is also greater than equal to mod of f, say, e i gamma i and let it be, gamma k, gamma i over norm of gamma i, norm of gamma i; that is, what is this is, meaning f x is sigma x i i gamma i; I am putting x to be same as gamma i, x to be same as gamma i sequence. So, let it be, sequence gamma i, where x, I am replacing by a sequence gamma i, i is 1 to n, or let it be, clear from here. Once you start with this, or in dual, yes, we have started with this R n dual; now, we have taking a point x in R n and then, when we operate f, you are getting a gamma i.

So, basically, what you are getting is… So, corresponding to each, corresponding to x, we are getting a C which is gamma 1, gamma 2, gamma n. Will it be ok or not? If x is given, then, will you not get gamma 1, gamma 2, gamma n, as a point, f of here. So, corresponding to x, we are getting x i (( )), we are getting this; corresponding to f, sorry, not x; f is, x is fixed; f because, f decide the gamma i; gamma f of e i is gamma i. So, corresponding to f, we are getting the gamma 1, gamma 2, gamma n, like this. So, this C will be a point in R n. Why? Why it is in R n, because it is an n-tuples. So, it is in R n. So, here, this norm is the, supremum taken for all f x, where x belongs to R n; therefore, we can replace this by C.

(Refer Slide Time: 19:47)



So, C is this. So, if I replace this by C, then, it becomes greater than equal to this and by definition, this is equal to sigma gamma i square, i is 1 to n over sigma gamma i square under root, and that becomes, sigma gamma i square, i is 1 to n under root, which is equal to...Let it be 2. So, if we combine 1 and 2, what you get, norm f equal to this, norm f equal to this. So, we get from… Because, norm f is greater than this; norm f is less than this. So, we are getting norm of f (( )).

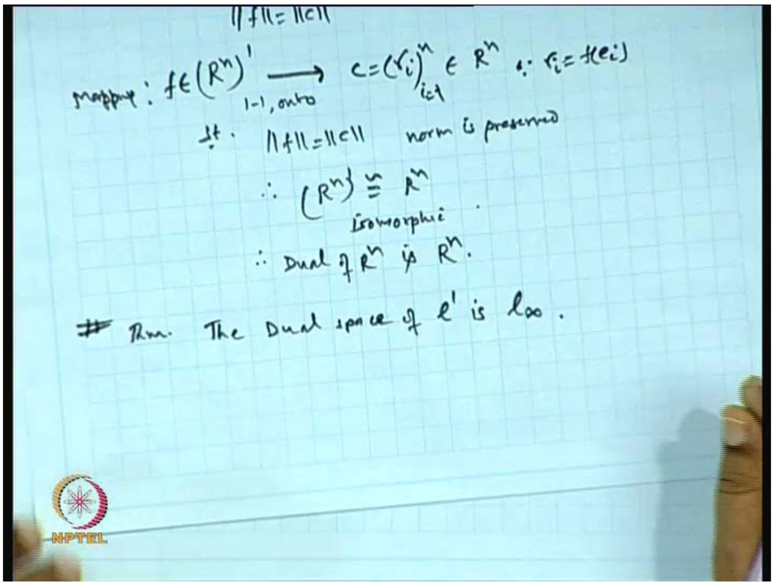
(( )) replace the (( )).

x, y and c, ok.

(( )) is greater than equal to that (( )).

Why, because, what is the f? Norm f is the sup; supremum is taken for all x. So, it means, if I replace any x, just if I remove supremum, then, this will be greater than equal to a value at particular point. Because, so many points are there and you are choosing the supremum among them. So, supremum will be the largest value. So, it will be greater than or equal to the value of the function at a particular point and that particular point is nothing, but C. So, we are getting it. So, that is it. So, we are getting this, clear, and this.

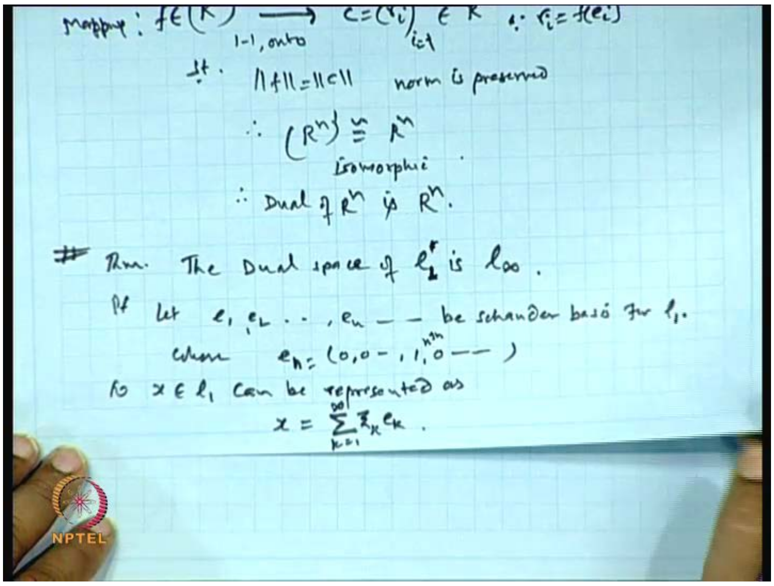
(Refer Slide Time: 27:26)



So, now, what we get it from here is, the norm f equal to norm c. So, now, let us combine this. We have started with the R n dual; an element we have chosen in R n dual, which gives you a element C, which is gamma i, i is 1 to n and element of R n, such that, norm of f equal to norm of c. So, norms is preserved, is it not. Norm is preserved. A mapping, this is a mapping, which transform f to C; corresponding to f, we get C, because gamma i is equal to f of e i. So, gamma i depends on f. So, corresponding to each f, we can get the gamma i, that is, we can get C; vice-versa, if I get the C, it means we are getting gamma i. Gamma i means, the corresponding function is known; because gamma i is known, this is known, so, corresponding functions.

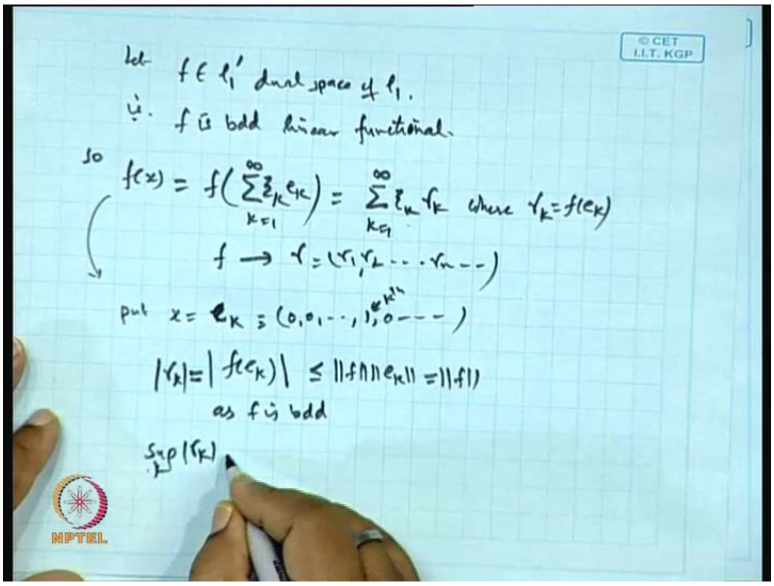
So, we can always get a one to one correspondence; this is one-one and onto mapping, which preserve the norm C. Therefore, these two spaces are isomorphic; once they are isomorphic, so, the dual of this R n is R n, is it clear. So, that is all. Now, second, we… The next result, the dual of space of l 1 is l infinity; l 1 or we can write, I think, l 1 we have defined that way; oh, so, let it be this; not this; dual of l 1 is l infinity.

(Refer Slide Time: 27:26)



So, again, the same trick is there. We will identify the mapping, which is one-one onto and preserve the norms. Then, this will... So, l 1 dual will be l 1. So, let us take, let e 1, e 2, e n and so on, be the Schauder basis for l 1. Let e 1, e 2, e n, be the Schauder basis, where, what is e n? e n is 0 0 0 1 0 0 0; this forms a Schauder basis for l 1. So, any element of l 1...So, x belongs to l 1, can be represented… nth value is 1. This is the nth point and rest are 0; represent it as sigma x i k e k, k is 1 to infinity. This is the representation of this, ok.

(Refer Slide Time: 31:25)



We wanted to, dual of l 1 is l infinity. So, let us pick up an element in l 1 dual. So, let f belongs to l 1 dual, dual space of l 1. So, f is a bounded linear functional; that is, f is a bounded linear functional, bounded linear functional. Therefore f of x, we can write f of sigma x i k e k, k is 1 to infinity, this will be equal to sigma x i k, k is 1 to infinity f of e k, so, which I am writing gamma k, where gamma k means, it stands for f of e k. So, again, what we have seen is that, this gamma k is uniquely determined by f. So, if f is known gamma ks are known. Therefore, the corresponding sequence gamma 1, gamma 2, gamma n is known. So, for, f will give you the gamma, which is gamma 1, gamma 2, gamma n, is it ok, clear; or we can say C.

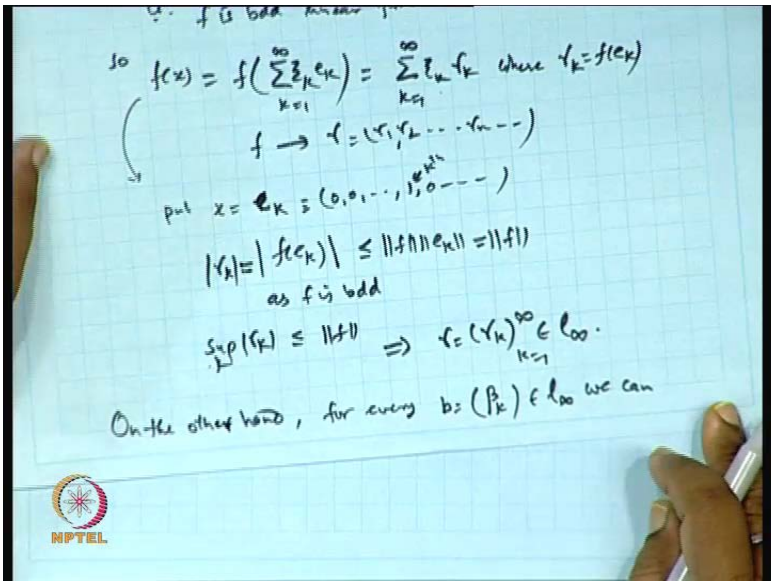
Sir, upto infinity?

Upto infinity, ok.

So, for each f, we can identify the gamma. What is this, whether it belongs to l infinity or not, that is to be identified. Now, if we take from 1, in this case, put x equal to gamma k, e k, x equal to e k. So, when you take x equal to e k, the image of this f of e k, mod of this, this is equal to what? When x is equal, rest of the thing will be 0. So, here only, you are getting what? By definition, this is less than equal to norm of f into norm of e k, as f is bounded; by definition of bounded linear, f of x is less than norm f into norm, but e k norm is 1. So, basically, this is norm of f, clear.

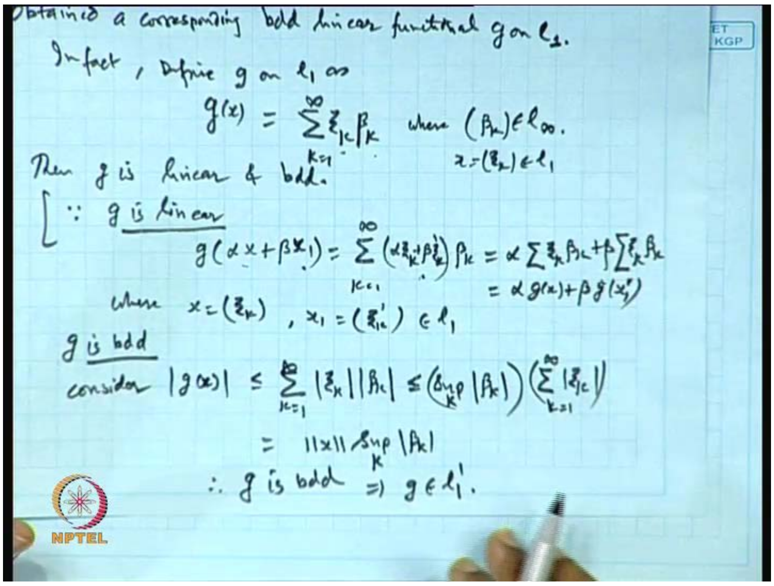
And, what is this? Is it not the same as mod gamma i? You take, if x equal to e k, so, x i k will be 1, only when, when x i k is equal to...Means, this value will be only what, gamma k only, because x equal to x i 1, x i 2, x i n only; means this x, x is e k means, this is equal to what? This is equivalent to 0 0 0 and 1 here, 0 0 0; so, 1 is the kth place and rest is 0. So, x i k will be 1 and rest will be 0. So, only you are getting gamma k, clear.

(Refer Slide Time: 31:25)



So, gamma k will be this. Therefore, supremum of gamma k, over k is finite. This shows that, a sequence gamma, which is gamma k, k is 1 to infinity, must be a point of l infinity. The l infinity is the class of those sequence, which are finite; this is bounded. So, it is finite; this is finite. So, we have established one thing that, for each f, we can find a point in l infinity; that is, every bounded linear functional, we can identify a point in l infinity. Now, let us take the converse. If a point in l infinity is given, then, with the help of that point, we can also identify a point in dual; then, there will be a one to one correspondence, ok.

(Refer Slide Time: 36:37)



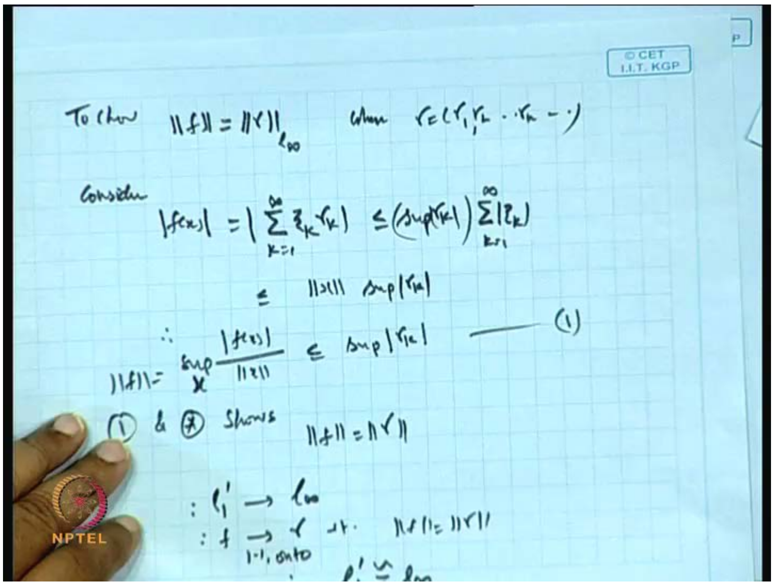
So, let us take the… On the other hand, on the other hand, for every sequence or every point B, which is beta k of l infinity, we can obtain, we can, it is ok, this, obtained a corresponding, we can obtained a corresponding bounded linear functional g on l, on l 1. In fact, if we define l 1, define g of x, define g on l 1, as, as g of x is sigma x i k beta k, k is 1 to infinity, then, this g is linear and bounded. Why, we have started with beta k, a point in l infinity, where the beta k is a point in l infinity; x is a point in l 1. We are defining a functional g on l 1, as g x equal to this and we claim that, this g will be linear and bounded.

Why, because g is linear follows from this alpha x plus beta y alpha x plus beta x 1; this will be equal to what, sigma k equal to 1 to infinity, only change is come here, alpha x i k plus beta x i k dash and beta k, is it not. If x dash equal to, x 1 is equal to x i k dash, so, this is, where x is x i k, x 1 is x i k 1 dash, both are the points in l one. So, alpha x plus beta x i 1 means this point, this sequence multiply by beta k. So, this can be written as, which is less than equal to or which is equal to, alpha into sigma beta k plus beta sigma x i k beta k.

And, that is equal to alpha into g x plus beta into g x dash x 1. You please check, it is ok? So, g is linear, means replace this x by a linear combination; correspondingly, the change will come here, that is this one and then, just addition. So, it is...Now, g is bounded. So, consider the mod of g x, mod of g X, this will be less than equal to sigma k equal to 1 to infinity, mod of x i k mod of beta k; but what is beta k, beta k is in l infinity. So, supremum of beta k is finite. So, it is further less than equal to supremum of mod beta k into sigma of this part; k is 1 to infinity, ok.

But this is nothing, but norm of X and this is a constant. So, this will be equal to norm of x into this supremum value over k. So, g of x is less than equal to some constant time norm X; therefore, g is bounded. And, this is a bounded linear functional on l 1. So, this implies, g belongs to l 1 dual; because it is a bounded linear functional, so, it must be a point of l 1. So, corresponding to a point in l infinity, we can always find a point in l 1 dual. So, there is a one to one correspondence.

(Refer Slide Time: 41:50)

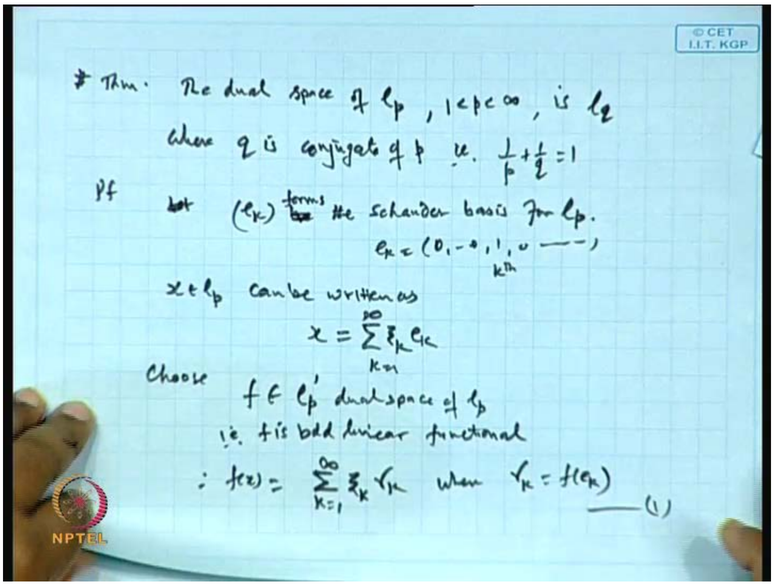


Now, if this mapping preserved the norms, then, both are identical. So, next, we see the norm. What is the norm of…To show norm of f equal to norm of gamma, where this is the norm gamma, where gamma is, gamma 1 and so on; this preserve the norm. So, this is the norm on l infinity and here, it is the norm of dual, l 1 dual; so, norm of l 1 dual. Now, how will we go? Consider mod of f x. This is equal to sigma x i k gamma k, k is 1 to infinity, which is less than equal to supremum of gamma k into mod of gamma k, of course, into sigma of mod x i k. But this is the norm. So, this is less than equal to norm of x into supremum of mod gamma k, divide by norm. Therefore, mod f x over norm x supremum over x is less than equal to supremum of mod gamma k, clear.

And, earlier, we have seen, also this part. This part we have seen, gamma k, mod of gamma k is less than equal to norm. So, these parts we have seen, say 1 star. So, if I take star and this combination, because this is our norm f, so, 1 and star will show, shows that, norm of f equal to norm of gamma, clear. So, a mapping can be defined from l 1 dual to l infinity, a mapping can be defined from l 1 to dual, which transfer f to gamma, such that, norm of f equal to norm of gamma; it is one-one and onto; so, these two spaces are isomorphic. Now, these are all isomorphic spaces. Therefore, dual of this l 1 is l infinity. Now, we go for l p.

Now, in case of the l p, the lines of proof is the same, except the part, where we wanted to show the gamma, because here, gamma 1, gamma 2, gamma n, unless the supremum is finite, you cannot say, it is point l infinity. So, the corresponding to f, when you are getting a gamma, that sequence has to be shown in l p, is it not. l q, l p or l q, whatever; dual of l p is l q. So, we have to prove that, this sequence belongs to l q. So, that portion only, the proof will differ and rest of the things will be the identical. So, I will just give in a nutshell, what is the outlines of the proof; otherwise it goes (( )). So, next result is…

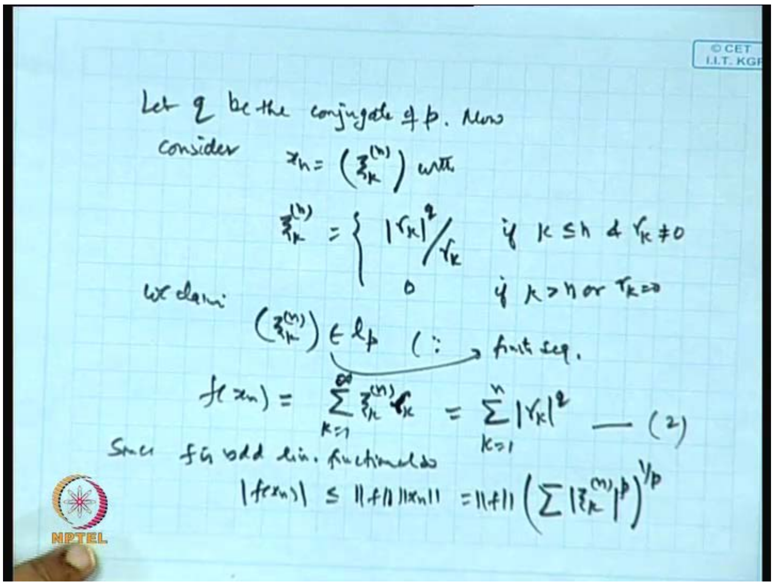
(Refer Slide Time: 46:19)



Theorem. The dual space of l p, l p, where 1 is less than p less than infinity is l q, where q is the conjugate of p; that is, 1 by p plus 1 by q is 1. So, that is our...Proof is same; lines of proof, first, e 1, e 2, e n be the Schauder basis over a…Let e k be the Schauder basis for l p or e k be the Schauder… Let k… This is true, ok. So, e k is the Schauder basis for l p. l p has a Schauder basis e k, where e k is 1; e k is defined as 1 0 0 0 1 0 0 0; kth place, it is 1; rest are 0. So, e 1, e 2, e n, these are, this forms a Schauder basis for, forms a, forms, in place of be, let it write, forms Schauder basis for. So, any element x belongs to l p, can be written as, x as sigma x i e k, k is 1 to infinity. Now, let us choose a point in l p dual, dual space of l p. So, this is a bounded linear, f is bounded linear functional. Let us see.

Therefore, when you define the image of x under f, then, it is automatically, because of the linear property, it will come like this, k equal to 1 to infinity x i k gamma k, where gamma k is the f of e k. Again, this gamma k can be determined uniquely by f and we get this one. So, let it be 1. Now, from here, we wanted to show this sequence gamma 1, gamma 2, gamma n, belongs to l q. So, here this trick is that, we have to choose a particular values of x in l p. So, how we will choose? Let q be the conjugate of p, of p and consider now, now, consider x n sequence as x i, x i k n with x i k n is equal to mod gamma k power q, divided by gamma k, if k is less than equal to n and gamma k is not 0; otherwise, you put it 0 or gamma k equal to 0.

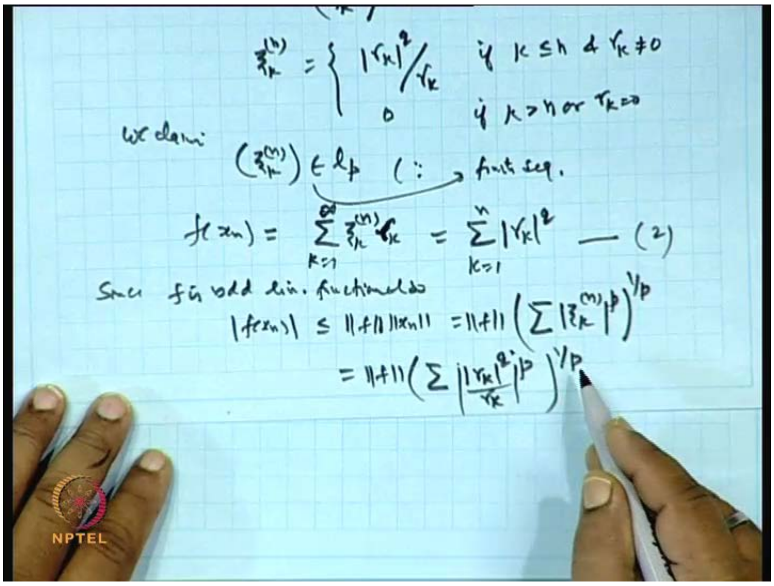
(Refer Slide Time: 49:32)



Let us define this. We are choosing this sequence x n, clear. It means, first n terms are like this; the first term is gamma 1 power q by gamma 1 and so on; other term like this and rest after this, it is 0. So, we claim that, this sequence x i n is a point in l p. Why, because these are finite terms only. So, sum will be finite, is it not. Because, this is a finite sequence, this is a finite sequence; only finite number terms are 0, non-zero; rest are 0. So, sum of this series will always be convergent. So, whether power p or any number you get.

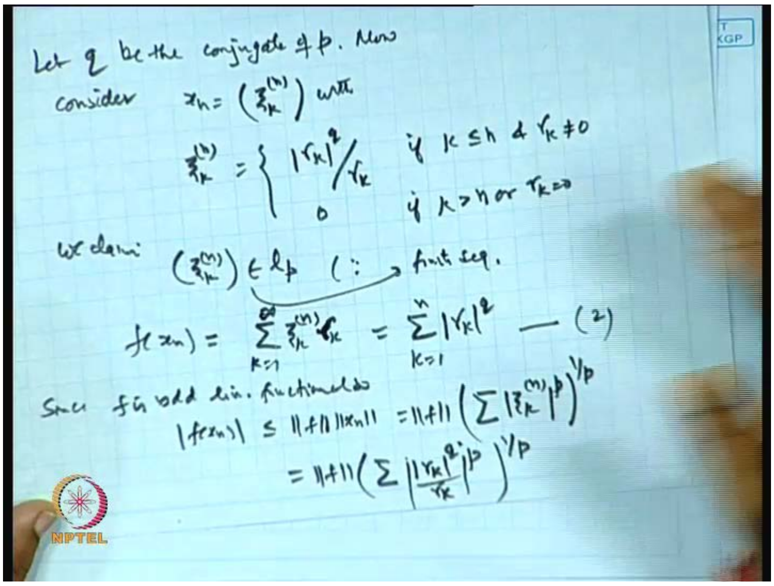
So, it is a point of l p. Therefore, we can replace x by this number. So, x f of x n, we can write it now, sigma x i k n e k, gamma k, sorry, this is gamma k, k is 1 to, k is 1 to infinity, but 1 to infinity means, it is reduced to 1 to n only; because up to n, and rest are 0. So, it is k equal to 1 to n and then, replace x i n by this. So, gamma k gets cancelled and we get, gamma k power q, is it clear. Let it be equation 2. Now, since f is bounded linear functional, so, by definition, mod of x n is less than equal to norm of f into norm of x n and this is equal to norm of f, norm x n means it is a point in l p; so, norm of l p will come x i n power p raised to the power 1 by p, ok.

(Refer Slide Time: 49:32)



But this x i k n, when you substitute this value, you are getting norm of f sigma mod gamma k power q p minus q and then, divided by this. So, gamma k over gamma k and raise power p; power p and raised to the power 1 by p.

(Refer Slide Time: 53:39)



So, this becomes q p minus p or q p minus p. So, p can be taken out, q minus 1 into p. So, what we get it that, this is equal to norm of f sigma mod gamma k power q minus 1 p raised to the power 1 by p; but q minus 1 into p is nothing, but sigma gamma k, by definition 1 by p plus 1 by q is 1, this is q power 1 by p. And the right hand side f of x n, this is equal to, this part is equal to sigma of gamma k power q. So, divide by this. So, we get sigma mod gamma k power q raised to the power 1 minus 1 by p is less than equal to norm p and this is 1 to n, let n tends to infinity, this will come to the gamma k power q power 1 by q, q is less than equal to. So, this shows that, gamma, which is gamma k is in l q and rest of the things will be the same clear; and rest is, is same.

Sir, only one point (( ))

How can…

In the previous slide…

Here?

Sir, that equation.

2?

This is…

How can you…

Be a point in l p.

How to prove this…

No, this is because, we wanted to show the gamma k in that l q. So, this is a particular case of the point; f can take any image of any point in l p. So, we wanted to such an x, so that, it is in l q, ok.

Sir, we could take (( )) different type of x…

Then this can be reduced to that form. This is so that, this can be, because one can identify that sequence, thank you.

We can discuss, you can come here and show.