Functional Analysis

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Module No. # 01

Lecture No. # 19

Tutorial - I

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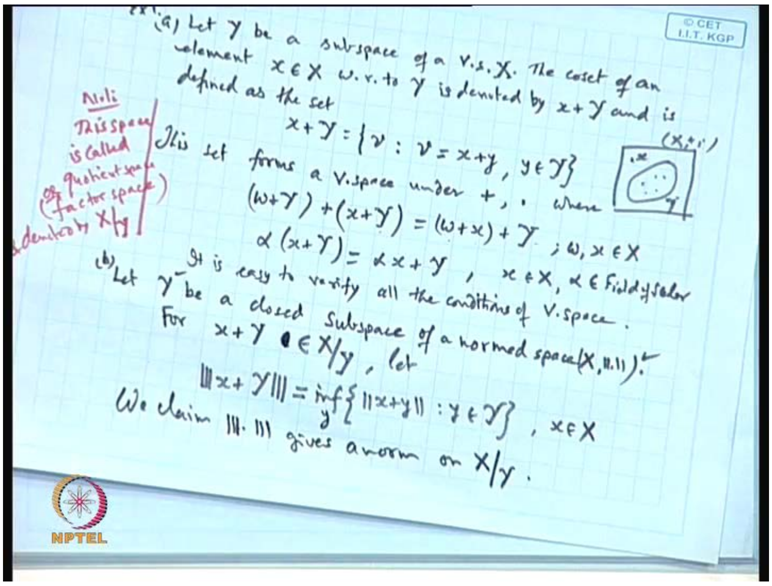

The problem was, let Y be a subspace of a vector space X. The coset of an element x belongs to capital X, with respect to capital Y, is denoted by x plus Y and, in fact, x plus Y, with the collection of those points v, which can be written as x plus y, where y is an element of this. So, what is the idea is that, if you take a element x from capital X, which is a vector space, and Y be a subspace of this vector space, then, find out the combination of this element x, with every element of Y; linear combination and this collection will form a set.

Now, this, not only forms a set, it is also a vector space. So, we defined suitably the operation, addition and the addition and scalar multiplication is defined like this. If you take the two point like, w plus Y and x plus Y, then, the addition is defined as w plus x plus Y. So, this will be an element of capital X, because capital X is a vector space and it is closed with respect to addition and scalar. So, again, it is of the form of x 1 plus Y, that is, a point of this class. Then, alpha x plus Y, this will be alpha x plus Y; again this will be a point in capital X. So, it is a point of x plus Y. So, it is closed with respect to addition and scalar multiplication; hence, it forms a vector space. So, that is why it is so, it is easy to verify the conditions of all vector space.

Now, this particular vector space…

Sir, is it possible to get such a…

Such a…

I mean, such a equation, is it possible to get?

Which equation?

w plus w, w plus y…

It is possible. Why, there could, suppose, you take the w plus Y and x plus Y. What is that w plus Y? The w plus Y, this will be the point w plus small y, you have this type; and x plus small y. So, you have taken w plus x and Y 1 plus Y 2. But Y 1 plus Y 2, it will be a point of Y. So, it will belong to Y, clear. Expression is, the first element should be the point in X and rest of the elements are a combination, with the elements of Y. So, basically, this will be w plus Y 1; this is x plus Y 1 Y 2. So, we get w plus x, plus Y 1 plus Y 2 becomes another third element, which is an element of Y. So, we can get it.

Now, this space, particularly, is called the quotient space. This space, we call it as a quotient space. Quotient… This x plus Y, this vector space, and or factor space, which you are talking about, some authors say factor spaces also; like quotient space and denoted by, and denoted by X oblique Y, X oblique Y, ok. So, whenever the X oblique Y means, it represents the quotient space. X is a vector space and Y is a subspace of this. Now, with the help of this, we have got the new space, which is called the quotient space.

Sir, is it factor space or factored?

Factor, f a c t o r.

So, that we will be take later. The factor space, clear. Now, this space forms a vector space. Now, we wanted this space to be a normed space. So, we have to modified our example accordingly. Now, what we did, let us suppose, Y is a closed subspace of the normed space X; Y is a close subspace of normed space X; then, for any point x plus y in the space, then, for x plus y in the quotient space, belongs to capital X over Y or in the quotient space X over Y. Let us introduce this operation as infimum of norm of x plus y, norm of x plus y, where y is a point of Y. This is our, X is a normed space; Y is a closed subspace of normed space.

Now, x is a point in capital X; y is a point in capital Y; so, but it is a subspace. So, it will be a point of capital X again. So, norm of x plus y will be well defined, because x plus y will be a point of X and the norm is introduced as the norm of X. So, this is a well defined thing and infimum is taken over all such y belonging to capital X; x is fixed, infimum. Now, what we claim is that, this forms, we claim that, this gives a norm on X over Y; this is our case. It means all the conditions of the normed space must be satisfied, is it not. I have used the triple notation, because this norm I am using here; so, that is why, in order to avoid the confusion, I am using this thing.

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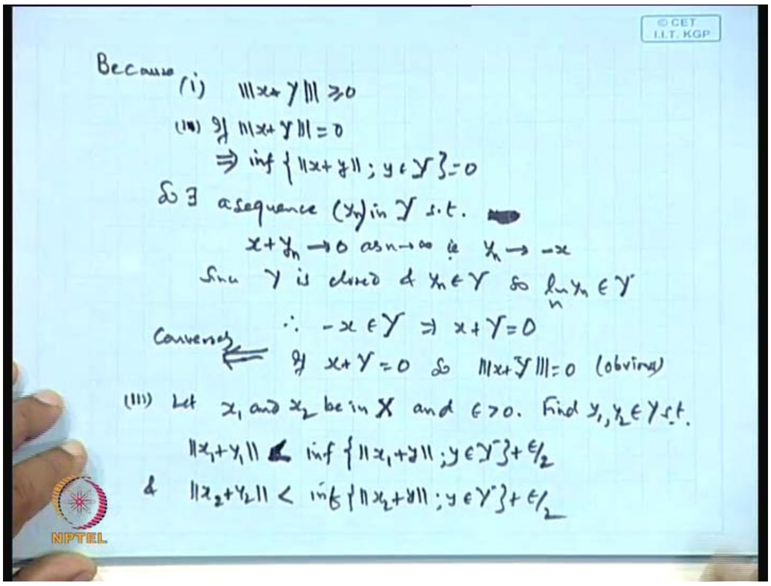

Now, how does it form the norm? The first thing is, obviously, this is greater than equal to 0, clear; because number one, this is greater than equal to 0; because the definition says that, the norm, infimum is taken over these values, is it not; all the values. Norm is a real nonnegative quantity. So, it is greater than equal to 0. So, infimum may be, at the most 0. It cannot be less than 0, cannot be negative. So, it is greater than equal to 0. Second part, if it is 0, if it is 0, then, what we want to show, x plus y must be 0. Now, what this mean by this? It implies, by definition, infimum of norm x plus y, where the y belongs to capital Y, is 0; the infimum is taken over y. It means, there must be a sequence of the points in Y, y 1, y 2, y n, which is combining with x and the norm value is tending to 0, then only infimum.

So, there exist, there exist a sequence y n in Y, such that, norm this x plus y n, such that, x plus y n, this must go to 0, as n tends to infinity, is it not. It means, that is, y n, x tends to minus x or you can say, y n tends to minus x, is it ok or not. Achcha. Now, y 1, y 2, y n, these are the points in Y; Y is closed, Y is closed. Since Y is closed, and y n, these are the points in Y. So, limiting value of y n must be a point in Y, because it is closed. So, limiting value of y n is nothing, but minus x. Therefore, minus x belongs to capital Y, is it ok. When minus x belongs to Y, then, it means, minus x plus y must be 0; for some sequence it will be 0, is it not. So, this implies x plus capital Y is 0; you follow me? Clear?

Conversely, if we take at this side, if x plus y is 0, then, automatically, this thing will be 0, because what is the infimum? x plus y is what, collection of all the sets. What is x plus y? Collection of all the set, is it not. So, it means, we, this point, norm will be 0. So, there nothing to prove obviously; obvious, ok. So, one. Second condition is, this is the second condition. Third condition or let it be the norm of x plus y is less than norm of x 1 plus x 2 is less than equal to norm x 1 plus norm of x 2. So, let us take…Let x 1 and x 2, x 1 and x 2 be in X; let us take the two point x 1 and x 2 in X and epsilon greater than 0; then, find y 1and y 2 in capital Y, such that, norm of x 1 plus y 1, which is less than equal to or strictly less than, strictly less than infimum of norm of x 1 plus y, y belongs to capital Y, plus epsilon by 2. And, norm of x 2 plus y 2 is strictly less than infimum of norm of x 2 plus y, y belongs to capital Y, plus epsilon by 2. Let me see, y, what I am doing is, I wanted to show the x 1 plus capital Y, x 1 plus x 2 capital Y, under the condition, is less than equal to x 1 plus y plus x 2 plus y.

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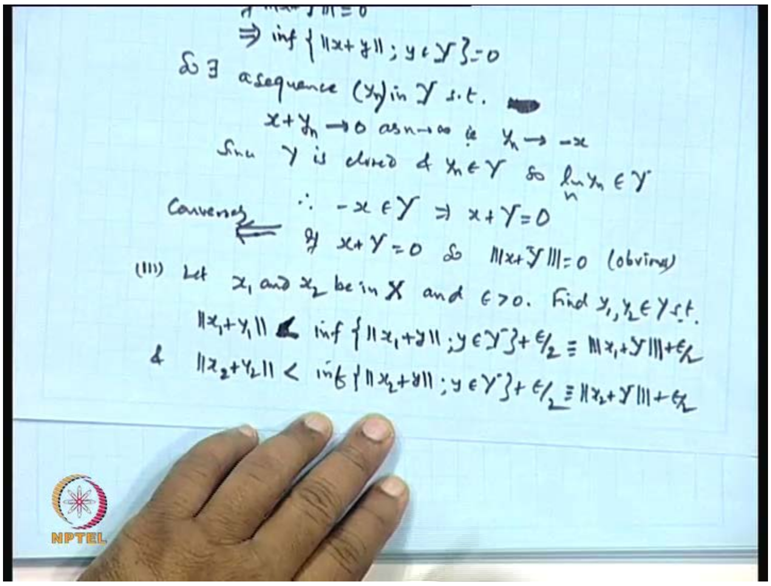

So, for that purpose, I have taken x 1 and x 2 are the two point in X, clear. Now, x 1 is in X. So, we can find out this infimum; an infimum is nothing, but what? This is equal to x 1 plus capital Y, is it not and plus epsilon by 2. So, if I remove the epsilon, this infimum, then, this quantity will be always, be more than this; but for certain sequence, it will be less than this plus epsilon by 2. So, that is why I have written, x 1 plus y 1is less than this number plus epsilon by 2, agreed. Similarly, this quantity is less than this number plus epsilon by 2, agreed.

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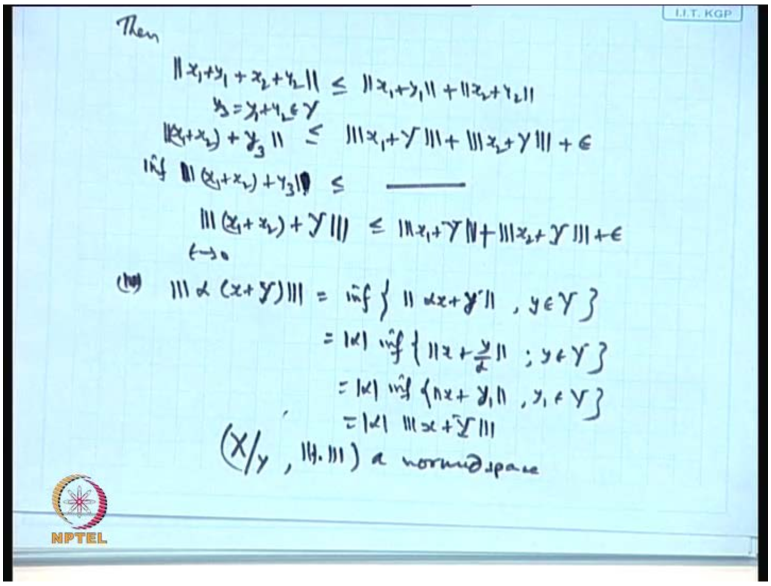

Now, consider norm of x 1 plus y 1, x 2 plus y 2. Now, this will be equal to, less than equal to norm of x 1 plus y 1plus norm of x 2 plus y 2; but by this one, x 1 plus y 1is less than this number; x 2 plus y 2, is less than this. So, we can write, this is less than equal to triple of this x 1 plus Y plus triple of this, plus epsilon; and left hand side, what you can write is, you can write this x 1 plus x 2; y 1 plus y 2, can you say this is another element y 3, belonging to capital Y. Is it ok now, where the y 3 belongs to capital Y. Now, take the infimum over all y 3s. So, take the infimum. So, infimum over all such y belonging to this. So, when you take the infimum of this, it is less than equal to this number, because right hand side, it free from any, this norm is a real number, this is real number; it is free from any y.

So, infimum of this, will it not be the same as the norm of, sorry, not, this is double norm because, once we take infimum, then, now, it will come, triple norm x 1 plus x 2 plus capital Y; this infimum value is denoted by this, ok. Infimum of this thing is less than equal to x 1 plus capital Y plus x 2 plus capital Y plus epsilon; but epsilon is arbitrary small. So, let epsilon is tending to 0, arbitrary, small; we get this condition that, norm, this mod of this triple under x 1 plus x 2 y is less than equal to triple x 1 plus y plus triple x 2 plus y. So, condition, that second condition of the norm is also satisfied, ok. Now, next condition I think, third or fourth maybe, norm of alpha x. So, consider this alpha x; alpha of x plus y; consider this thing, where this is equal to infimum of norm of alpha x plus y, by definition; y belongs to capital Y; by definition, because, when it is multiply alpha, it is equal to getting alpha x plus y.

Now, take alpha outside. So, when you take alpha outside, because of the norm, mod alpha will come. So, you are getting, this thing is mod alpha infimum of norm x plus y over alpha and then, y belongs to capital Y, ok. But y is a point in capital Y, alpha is a constant. So, Y is a subspace. So, this will be again a point in Y. So, infimum of taken means, it is same as infimum of norm of x plus, say y 1, where the y 1 belongs to capital Y and which is the same as triple of x plus capital Y, ok. So, all these conditions of the norms are satisfied. Therefore, this space X by Y under this is a normed space. A typical examples we have; it is not a routine one; it is not routine example, it is typical, but it requires that, is it clear. Any doubt, please say, ok.

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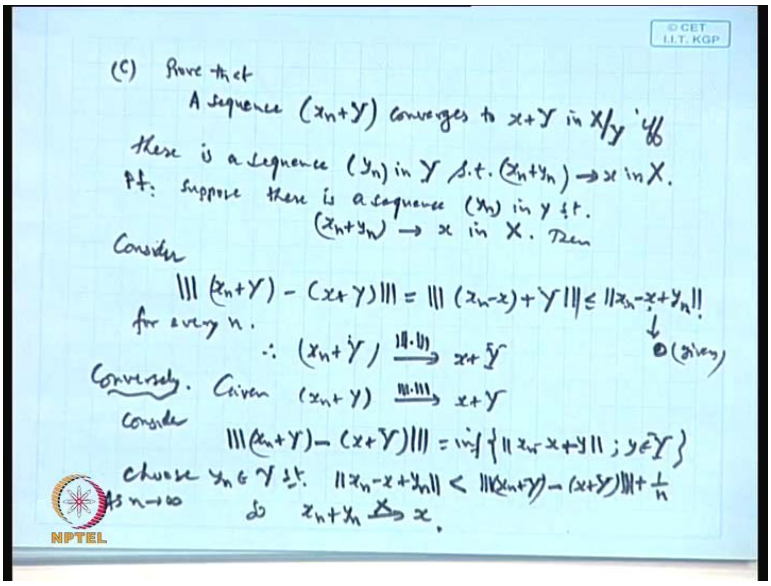

Now, next part, the same thing, same question, we put it another part c; the part is a sequence I can approve that, a sequence x n plus y converges to x plus y, in the quotient space x over y, if and only if, if and only if, there is a sequence, there is a sequence y n in capital Y, such that, x n plus y n, this sequence converges to, converges to x in capital X ok. So, this result shows how a sequence in this quotient space converges to a point. So, normally, when we say the x n converges to x, the difference between x n minus x goes to 0 under the norm; now, here is something, because this space x n plus y is a point; x n plus y is a point. So, what you say the, if you take a sequence x n plus y means, take the point x 1 from x, find out the all the combination; x 2, another combination like this. So, we are taking x n plus y as a sequence.

Now, this sequence will converge to a point x, if and only if, the sequence y n, there is a sequence y n, where the x n plus y n will go to x, in this. So, that is, combination. Let us see the proof. So, suppose, this is given; suppose, there is a sequence y n in Y, such that, x n plus y n goes to x in capital X, converges to x in capital X. Suppose, this is given, clear? Then, what we want to show that, this converges to this, in the norm of X over Y. So, consider triple one x n plus y minus x plus y. Now, if this goes to 0 means, this will go to this; n, n tends to infinity under this norm, ok.

Now, this can be written as x n minus x plus y, is it not. Clear? At the, subtracting this. Now, is it not less than equal to norm of x n minus x plus y n for every n, do you agree with this? Why, because the definition of this says...So, remember the definition of this, this is the definition, x plus y is infimum of this. So, if I remove infimum, all the values will be greater than equal to this. So, that is why, we are getting the, and this is true for each n. So, there will be a sequence y n y. Now, it is given that, x n plus y n goes to x in the, this is the norm of x and x n plus y n goes to x, as n tends to infinity.

So, basically, the right hand side tends to 0. So, this part tends to 0, it is given, is it not. We have assumed this. Suppose, there is a sequence y n such that this can... So, this is assumed; this is given. Therefore, can you not say that, this thing, x n plus y, this sequence converges to x plus y in the norm of this; proved, ok. Now, let us see the converse part. Given that, this sequence converges, this sequence converges to this, in the norm of this. So, given, x n plus y converges to x plus y under this norm. We are required to show that, there will be a sequence y n, such that, x n plus y n goes to x in the norm of x. So, let us see.

So, consider norm of x n plus capital Y minus x of capital Y. This one. Now, this is by definition, infimum of norm of x n minus x plus y, where the y belongs to capital Y. This is just by definition, is it not, ok. Now, if I choose the y n suitably here, then, this entire thing can be made less than this plus some number epsilon. So, pick up the y n, choose y n, belongs to capital Y, such that, this number y n will be less than, strictly less than this one, infimum of this or this value plus 1 by n, where 1 by n is a epsilon number; it is clear. Now, as n tends to infinity, what happens? This goes to 0; this is already given, this converges to x; it means, right hand side will go to 0. So, x n plus y n will go to x under the norm of this. So, we get x n plus y n goes to x in the norm of x; that is all and this is proof, clear.

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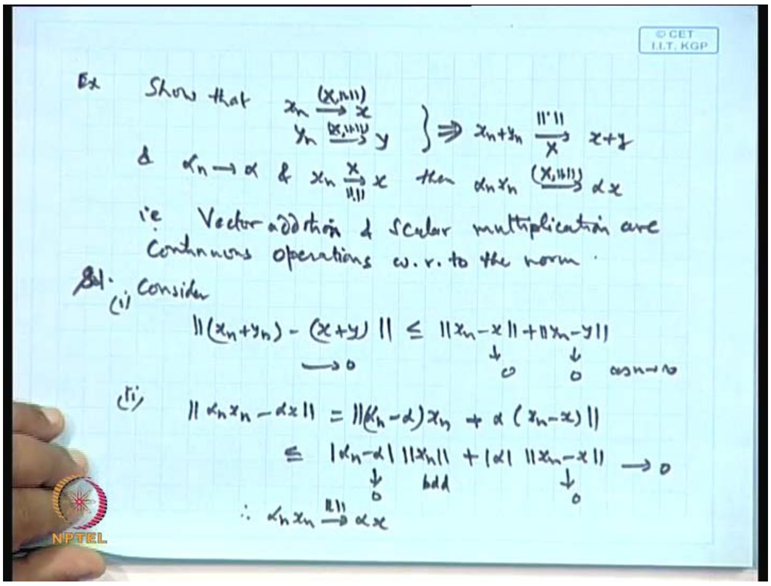

So, we get this one. So, this was the example, which I wanted to show. Now, another example is, of course, this I do not, I do not remember whether I have done it or not. If x be a normed space, it has the two operations, addition, scalar multiplication and also structure norm. The addition and scalar multiplication are jointly continuous in the normed space, whether it was done or not? No, ok. So, show that, show that x n converges to x in the normed space X; y n converges to Y in the normed space of X, implies the x n plus y n will converge to X plus Y in the norm of X, in the norm of X. And, if alpha n converges to alpha and x n converges to X in the norm of X, then, alpha n x n will converge to alpha x in the norm of X; that is, a vector addition in the normed space is continuous; a scalar multiplication is also continuous function. So, vector addition and scalar multiplication, that is, vector addition and scalar multiplication are continuous functions, are continuous operations with respect to the norm, operations with respect to the norm, clear.

And, I think, the proof is very easy, a solution. We wanted, this is given, x n converges to X; y n converges to Y. So, we wanted this one. So, consider this norm of x n y n minus x plus y; apply the triangular inequality. So, we get and then, this goes to 0, this goes to 0, as n tends to infinity; so, obviously, this will go to 0; satisfied. Then, second, consider alpha n x n minus alpha x. Now, here add and subtract; minus alpha x n; then, plus alpha x n minus x. Now, apply the triangular. So, what you are getting is, mod alpha n minus alpha norm of x n plus mod alpha norm of x n minus x. Now, alpha n goes to alpha, so, this will go to 0; this will go to 0, but x n is a sequence, convergent sequence.

So, it is bounded sequence. So, this will be a bounded sequence. Therefore, norm will be a finite, some value less than equal to n. So, entire thing will go to 0; hence, we say this tends to 0. So, alpha n x n will go to alpha of x in the norm of X. So, that is all. Very simple, yes, simple.

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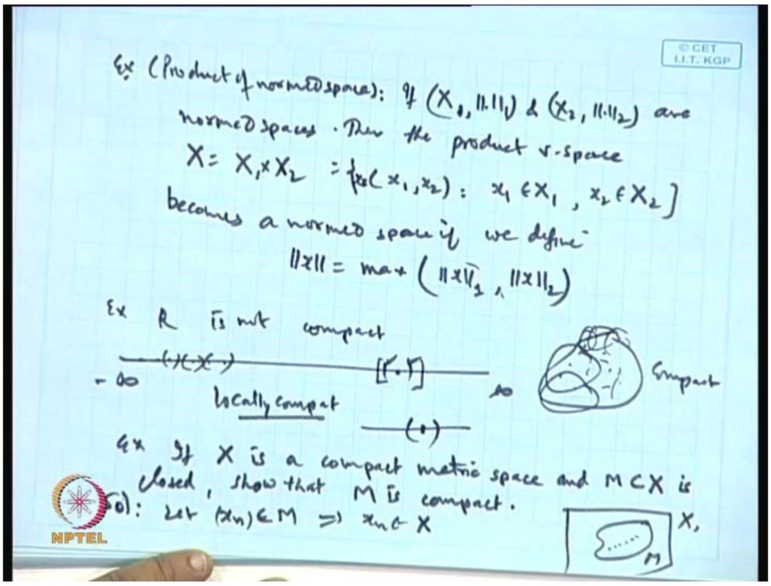

Product of the two norms. If the two norms are there, one can also define the norm on the product of the normed space. So, this, product of normed space, normed space. If x 1, x 2, x 1 norm 1, x 2 norm 2, these are the two normed spaces, normed spaces, then, the product vector space, vector space X, which is x 1 cross x 2, means, every element of this will be of the form, ordered pair type, x 1 comma x 2, where the x 1 belongs to capital x 1 and x 2 belongs to capital x 2; cross product of the x 1 x 2, product of this. This becomes a normed space, if we define, norm of x is the maximum of norm of x 1 comma norm of x 2, where x is this; x is this element; this is our x. I think, this is obvious; it is very easy, is it not. Why, because, x 1 and these are already given to the norm.

So, it satisfy all the condition of norm; this also satisfy all the condition of norm; and, maximum will be one of them; either this will be maximum or that will be maximum. So, all the condition of the norm will be satisfied. So, basically, there nothing to…So much…One can do it. I think it is clear.

Sir, x is generated from, means, what point of x 1 (( )) point of x 2; (( )) I am picking one point of x 2…

No, what you are saying is…

Product, no?

Yes, product is x 1 cross x 2.

x 1 cross x 2, no?

So, x 1 cross x 2, so, what is this product? Say, x 1 is the collection of the points; x 2 is also collection of point. Pick up the element of x 1; say, I am taking x 1 and let us combine all the elements of x 2. Then, pick up another one, combine all these. So, basically, it is the combinations; a cross product, a combination; a set of ordered pair; it will be a set of ordered pair, where the first element must be the point of x 1; second element should be the point of x 2, is it not. So, that beats it.

Now, we have shown one compactness, is it not; one compact set we have defined; the compact set, if you remember, a metric space, x is a metric space; m be a subset of this; a comp m is said to be compact, or in general, a compact set is defined, if every sequence has a convergent subsequence, or every open cover has a finite sub cover, is it not, compact.

Sir, it is a finite sub cover?

Finite sub cover. Now, the suppose set is compact; I say, this is a compact set. What you mean by this? It means, if I draw the walls around each point of this, there are infinitely many walls will be there, which can cover this, is it not. But, why do you require this infinite number wall? If only finite number wall are sufficient to cover it, then, we say it is a compact set. So, that wall, if we chosen in the form of opens covers, then, we say, if every open cover has a finite sub cover, then, we say, the space is compact; the set or space is compact. Similarly, open cover, we mean, a collection of the open sets, which is closed under union and all these condition. So, that will be the, means, collection of the open cover, union is also, is an open, ok.

That, if it is responsible to close every point of this or includes all the points union cover, then, we say it is a open cover; means, if we pick up any point of the set, it must be the point in one of the collection of that open sets, clear; then, we say it is an open cover for them. If there are infinitely many, if we take any open cover it and out of this, if we have the finite cover, which is also suitable to cover the entire set, then, we say the set is compact. Equivalently, we can say, a sequence, if a sequence has a convergent sub-sequence, then, we say it is a compact set. Every sequence has a convergent subsequence; then also, it is a compact, is it not. So, this compactness can be defined there.

Now, if we take the real line R, it is not a compact. Why R is not compact? Why it is not compact because, this is our entire real line minus infinity to infinity; these are the points. What are the open cover, is an open intervals? How many open intervals of… Infinite number of open intervals is required. But can you find out, out of an infinite number of open interval, only finite numbers of open interval, which are responsible to cover it. So, such a, is not compact, but it is a locally compact, locally compact. Locally compact means, that a set is said to be locally compact, if around each point, one can find out a neighborhood, which is compact. So, real line is a locally compact; take any point, you can find an open interval; then, it is closed, compact. Why, because, one can get it like this, is it not; closed interval.

Because boundary is now fixed; these points are fixed now; but in case of the real line, it is not at all fixed. So, that is why, it is a locally compact, not a compact. So, I am not going to give that one; this is a very interesting example here is, if X is a compact, if X is a compact metric space and M is a subset of X, is closed; then, show that, show that, M is compact; as a convergent subsequence of T. Now, how to show this? X is a compact metric space. It means, that is has a property that, every sequence has a convergent subsequence; means, limit point belongs to it. We wanted to show M to be compact, where M is given to be a closed set, clear. So, this is our X; here, it is M; this we wanted to show, compact. So, we should choose a sequence in a… So, let, let x m n or x n be a sequence in capital M; this is a sequence in capital M; so, obviously, this x n will be a point of x also, ok.

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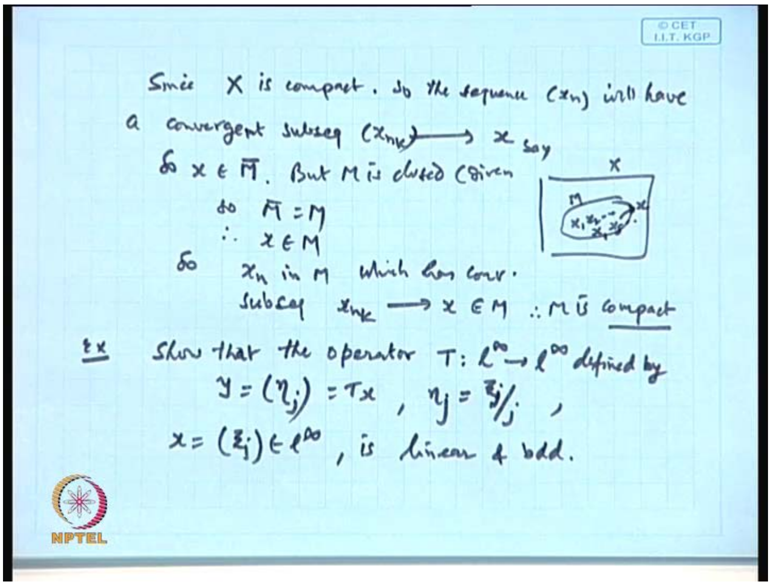

But X is compact. So, every sequence will have a convergent sub-sequence. So, we get, since X is compact, so, the sequence x n will have a convergent sub-sequence, say x n k, which converges to the point, say x; is it clear now, ok. So, you are choosing the sequence x 1, x 2 and so on, a sequence in M. It has a convergent subsequence, say x 1, x 5 and so on; the limiting value is x. Now, this x may be a point of X or, or a point of M.

But, obviously, x belongs to closure of M, because all the sequence points are in M. So, x is the limiting value. So, it is the point of the closure of M; but M is closed; it is given. So, M closure is M; therefore, x will be a point of M.

So, what you can get, you are, what you are getting is, so, we are taking a sequence x n in capital M, which has a convergent sub-sequence x n k, whose limit point x belongs to M; therefore, M is compact. It is interesting or not? Now, let us take an example now. On operator, we have discussed the linear operators. So, just one example, in fact, related to others, problem will come from this example; the example is very simple. What is this example show, show that, show that the operator, show that the operator T from l infinity to l infinity, defined by y is equal to eta j, which is T x, eta j is x i j y j, x is x i j belongs to l infinity, is linear and bounded, is bounded.

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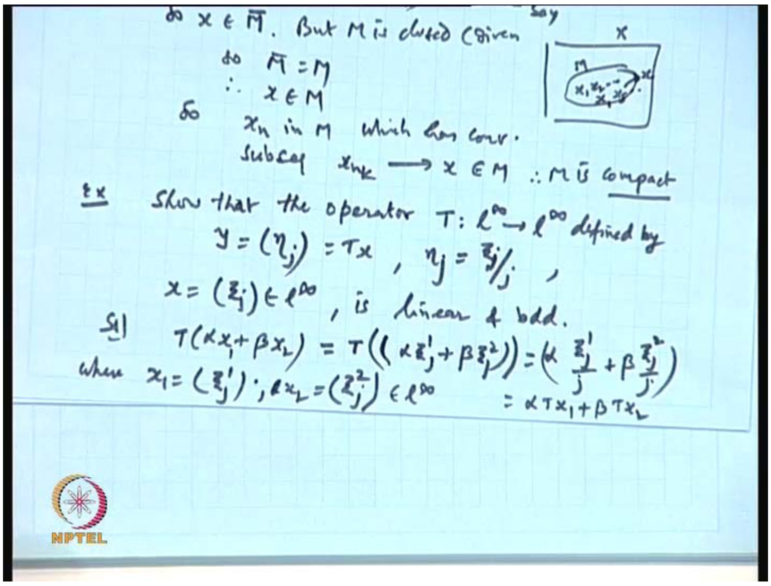

So, if we look this problem, the solution is simple. We want this operator to be linear and bounded. So, for the linear property, we have to write alpha x plus beta x 1 plus beta x 2, is it not. If I say, this is alpha of T x 1 plus beta of T x 2, then, T becomes linear, where T x is defined in this fashion by, ok. So, suppose, x 1 and x 2 are the two point, where x 1 is, say x i j 1, x, x 2 is x i j 2. Suppose, I am taking these two sequences in capital l to l infinity, and linear combination under T. So, what will be this T of alpha x 1 plus beta x 2? What will be this point?

T of…

T of alpha x i j 1 plus beta x i j 2, is it not; and, when you take the image of this, then, image will come out to be the alpha x i j 1 y j plus beta x i j 2 y j, this sequence, is it not. And, that can be written as, again, can you not take it alpha of this plus beta of this, which is equal to alpha of T x 1 plus beta of T x 2. You can break up this again; alpha, this sequence, plus beta, this sequence, which is equal to…So, T is linear. Now, to show T is bounded. So, to show T is bounded, what we have to show, because l infinity is a normed space. So, when T is an operator from l infinity to l infinity, and if we want to show it is a bounded, you have to prove that, norm of T x should be less than equal to some constant times norm of X, is it not.

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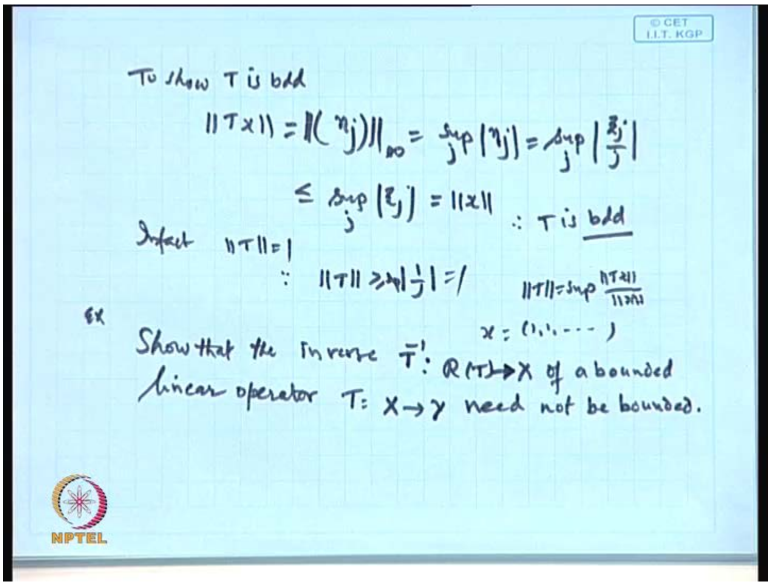

So, let us take the norm of T x. What is this norm of T x? It is basically, eta j, because T x is equal to y eta j and norm of this; but norm is nothing, but the infinity norm; because, it is a point of l infinity. So, basically you are taking the supremum of eta j over j, is it ok or not. But, this will be equal to supremum of x i j y j over j; can you not say this is less than equal to supremum of mod x i j over j.

Sir, this is…

Yes.

Maximum for…

x i j y j.

Mod?

Mod because scalar quantity j. So, it will be supremum j and since it is divided by j, so, if I remove j, it is less than equal to this, clear. One. So, this is nothing, but the norm of X. So, norm of T x is less than equal to norm of X. So, T is bounded. In fact, it norm will be 1. In fact, the norm of T will be 1, because if I take greater than equal to this, I can choose such a sequence, where the norm becomes 1. So, it will be, the norm will be 1. Take the x i j to be 1 1 1, because norm of T is greater than equal to this norm;

supremum is taken over eta j; take this, mod 1 by j supremum; x i j to be 1 1 1. So, this is equal to 1. So, we can take. So, in fact, we get the norm T. So, this is a linear and bounded.

Now, the interesting question comes here.

1 by j, oh, what you are saying is norm of T. What is the norm of T? This is equal to supremum of norm T x over norm of X, is it not. I take x to be 1 1 1 1 1. So, what is this is, norm is 1 and here, it is what, 1 by j, mod. So, that supremum will be 1, ok. Because 1 1 1 x i j will be 1. So, 1 by j supremum is 1 only. So, you have taken this 1 and this is 1, so, you are getting 1. The interesting question is, show that, show that the inverse, inverse, T inverse, from range of T to X of a bounded linear operator, operator T from x to y, need not be bounded, ok.

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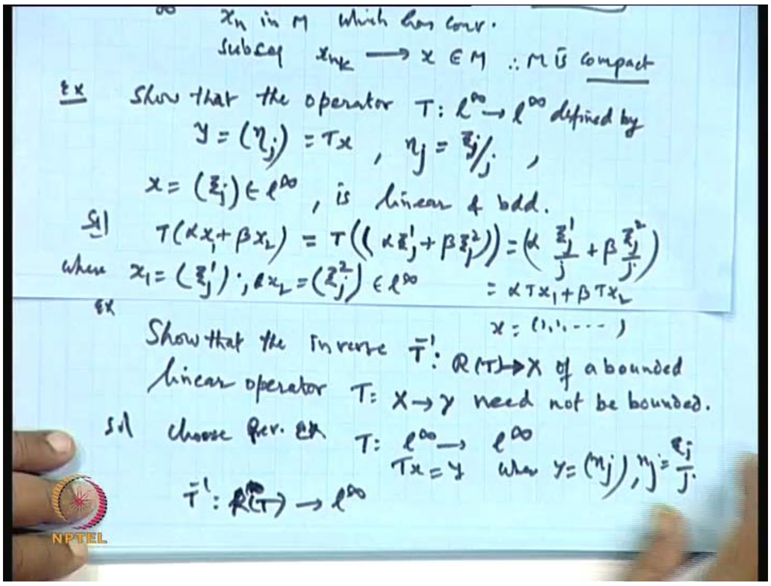

Now, let me see, what is this question? The question was, show that the operator T defined by this is linear and bounded. What this question says, show that the inverse operator, T inverse of a bounded linear operator need not be bounded. It means, there are the operators which are linear and bounded, but when you take the inverse of that, inverse operator, if it exists, then, it need not be, it may not be a bounded operator; means, continuity might be felt, ok. So, we have to give a suitable example, where the one side, it is a bounded operator, directly it is bounded from x to y; but when you take the inverse side, T inverse, then, it is unbounded, and the example is nothing, but the previous one.

If I choose the same example here, the T from l infinity to l infinity, where the T x is defined as y j by, which is eta j of this form; then, the inverse operator of this thing will be unbounded. Why? Choose previous example, previous example. T is a mapping from l infinity to l infinity, such that, T x is equal to y, where y is equal to eta j and eta j equal to x i j y j. So, what will be the T inverse? From l infinity, from range set to, from range set to l infinity such that, such that…

Sir.

Yes, please.

T inverse is a mapping from R t to range set of this.

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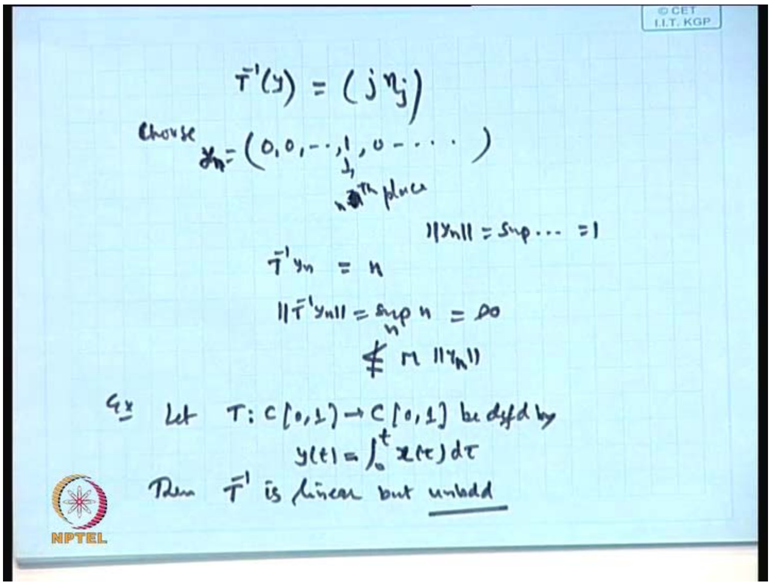

Range set become domain now to l infinity, such that, T inverse y, T inverse y will be X. So, this will be equal to what, j times eta j; j times of eta j. eta j is a point in this and then, ok. Now, if I choose a sequence like this 0 0 1, 0, 0, 0, 1 at the jth place or nth place, nth place and rest are 0; suppose, this is a x n or y n, I can choose y n, that better, y n. What is the norm of y n? because, it is the supremum of this thing and that will be equal to 1. What is the inverse image of y n? That is only n, is it ok or not. So, norm of T inverse y n is nothing, but the supremum of n over n, which is infinity. So, it cannot be less than or equal to norm of M times norm of y n; M is constant.

Sir, the whole operation is due to the, in previous example (( )) denominator.

So, it is coming to be the numerator.

Yes, is it not? So, in fact, the, (( )) the range, but corresponding inverse point is given something there. So, that is what is. So, it is a good example, unbounded and it is good. Similarly, there is also another example which I will say.

Let T from c 0 1 to c 0 1, we defined by y t equal to integral 0 to t, x tau d tau. Then, the T inverse, T inverse is linear, but unbounded. So, this you do it. I will give a hint, little bit. What is the inverse operation is the differentiation of y; T inverse is the differentiation, continuous derivative of y. So, if you take y equal to T n, the derivative differentiation is not a bounded operator; n will come. So, this is clear. So, this you do it.

Thank you.