Functional Analysis
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Lecture No. # 20

Tutorial-II

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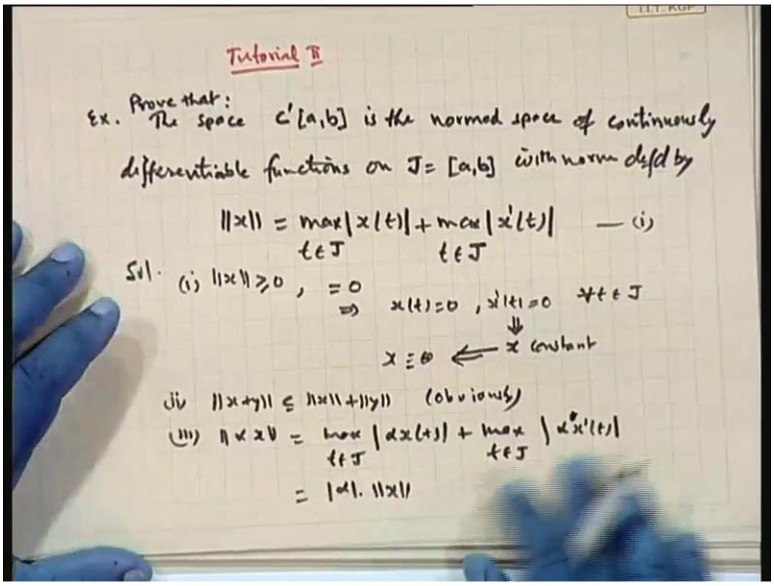

So, we will again continue the same, same problems and particularly, based on the linear function. The first problem, suppose the space c a, b, the space c, a b we have already discussed, set of all continuous functions defined over the closed interval a, b. Now, we are taking the c dash a, b. c dash a, b is the set of all that is, is, is the normed space, normed space of continuously, of continuously differentiable functions defined on J, which is a closed interval a, b with norm defined by norm of x is maximum of mod x t, t belongs to J plus maximum of mod of x prime t, where t belongs to J. So, the problem is, prove that the c a, b, I will just write like this, prove that or show that the space c dash a, b is the normed space of continuously differentiable functions on the closed interval a, b with norm defined as this. Clear?

So, in this, this space is different from the c a, b. c a, b we are taking only the continuous functions defined over the closed interval a, b and in that case the norm was given in terms of the maximum mod x t or may be the integral form a to b x t d t. But here not only those continuous functions are there, even we are including those functions means, instead of taking those continuous functions, for all the continuous functions we are taking those continuous functions whose differentiation is also a continuous function, whose derivative is also a continuous function. So, set of all continuously differentiable function is defined, that is why we are adding this term; also, in order to get all the properties of the norms satisfying. If I do not put this term you will see that all the condition of the normed space will not be satisfied.

Let us see, the first part norm is, obviously, greater than equal to 0 because each one is mod. So, and this is a continuous functions on c a, b. So, obviously, this things derivative, etcetera will give the nonnegative quantity, mod of this is there we get. Now, if it is 0 what happens? The right hand side of this expression is 0, so maximum of x t 0 and maximum of x prime t is 0. So, this will imply, that x t will be 0, x prime t will be 0 for all t belonging to J because once the maximum value is 0, then for all values of t it will be 0 modulus, is given absolute value. Now, x t is 0 means, sorry, x prime t 0 implies the function x must be a constant function and since first condition shows x t is 0, 0 is also constant. So, it can take the value only when x is 0, so x must be identically 0. So, norm of x equal to 0 will imply x is equal to 0. Is it ok? So, I will write theta clear 0. Is it ok?

Second condition, norm of x plus y is less than equal to norm x plus norm y. I think obviously, it is true, is it not? So, I will not write this way, this is norm x plus norm y, one can obviously see and similarly, the third condition, when alpha is a constant, then you are getting this is maximum of alpha x t, t belongs to J plus maximum of alpha x prime t, this alpha x prime t and t belongs to J. So, we can take mod alpha outside and remaining thing is norm x. So, all the conditions of this are satisfied. Therefore, it forms a normed space. Clear? Is it ok? So, and this space is interesting, is very interesting space and we will see many application in, you will see, this type of space.

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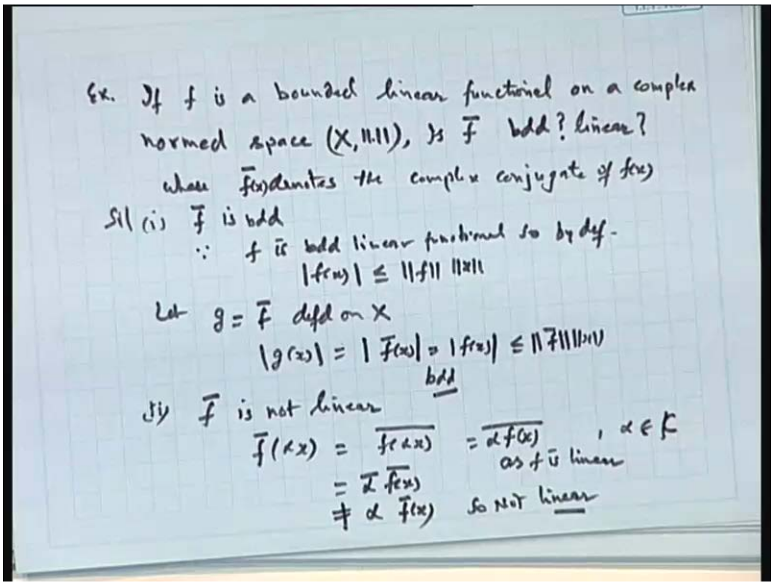

Now, another problem, if f is a bounded linear functional on a complex normed space, complex normed space, capital X, then each f bar is f bar, where f bar is a complex conjugate of f. Is f bar bounded? Is f bar linear, where f bar denotes the complex conjugate f. f bar x means complex conjugate of f x. So, what is given is, f is a bounded linear functional on a complex normed space. It means, when you take x point in x, the f of x that will be a linear, that will be a scalar quantity and belongs to a complex set. So, we can talk about the conjugate of f.

Now, the question is, in place of f if I take the conjugate of the function and then operate on x, whether that function remains linear, remains bounded? So, first part is we claim f bar is bounded. Why, what is the condition of the boundedness? Because f is bounded, f is bounded linear functional, so by definition mod of f x is less than equal to norm f into norm of x, this is by definition. Clear?

Now, I replace g, suppose g is equal to f bar. Now, we wanted to test whether this function is bounded or not defined on x. So, let us consider the mod of g x, that will be the same as the mod of, mod of f bar x, but this f bar x, because it is a conjugate value, that will be the same as the mod f x. So, we get mod f x mod of z and mod of z bar will be the same. So, this will be again less than equal to norm of f into norm of x or this norm f, you can just put it the star also bar, there is no problem. So, we can say this is less than equal to g in the normed space. So, it is bounded. Ok?

Now, what about the linearity, the f bar, will it be linear? f bar is not linear. It means, the condition of the linearity, that is, f of alpha x plus beta y should be equal to alpha f x plus beta f y fails or may be one of the condition fails, because there are two condition. One is f x plus y is equal to f x plus f y; another one, f of alpha x is alpha times f x. So, if one of the conditions fails, then you say f is not linear.

So, I take f bar alpha x. Now, is it the same as alpha x conjugate basically? f bar means a function f defined on some point and then taking the conjugate, that is the f bar. So, alpha x function is defined and then takes a conjugate, but f is linear. So, we can write alpha times of f x conjugate as f is linear. By the property of linearity this alpha can come outside since alpha belongs to as a point in c. So, when you take the conjugate of this, it will be equal to alpha conjugate into F. Clear? It means f bar of alpha x is not equal to alpha time f bar x. So, it is not linear. So, function f is bounded and linear, but the conjugate of this will remain bounded, but not linear. So, that is the interesting.

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Which one? f bar of alpha x, this is ok, then we are not touching this one bar, we are applying the property of the linearity of f, f is linear. So, alpha can be taken outside and then this will be alpha bar into this one. So, this one, but actually we wanted the alpha times of f bar x if it is linear. So, it does not satisfy that condition. Therefore, it seems to be a linear. So, this completes.

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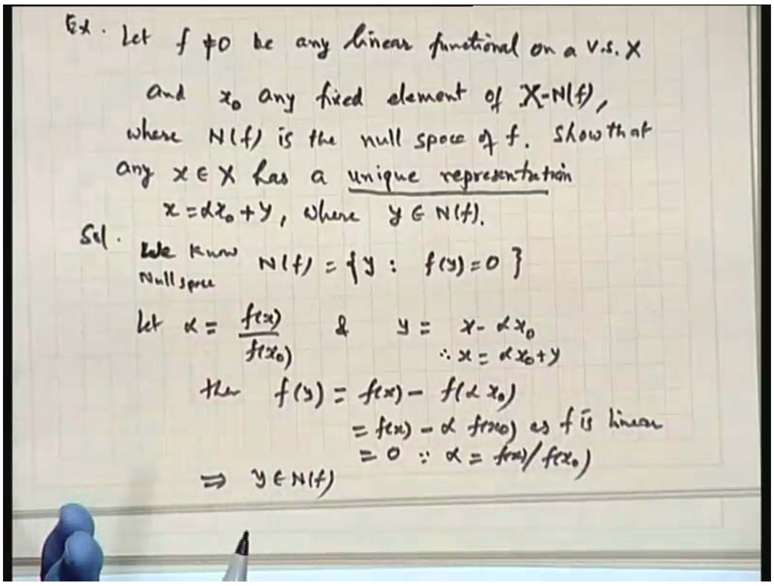

And another example is, let f, which is not equal to 0, be any linear functional on a vector space X, on a vector space X and x naught, and x naught, any fixed element, fixed element of X minus N f where N f is, where N f is the null space of f, of f. Show that any x, show that any x belonging to X has a unique representation, representation as x equal to x naught plus alpha, sorry, alpha x naught plus y, where y belongs to N f. Let us see what is this?

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f is the any non-linear, f be any linear function on vector and x not be any fixed element of X. So, x naught is given in X minus N f. It means the value of the function at a point x naught is not equal to 0. Clear? (( )) what we wanted is, that we wanted to show, that any element X has a unique representation of this part where alpha is a point in complex, in the field of X, alpha belongs to the field of a scalar of X and y is a point of N f this well.

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So, let us choose null space is the set of those points or we know the null space N f is the set of those points y, such that f of y is 0. This is the null space for this. Now, if we take alpha to be x, f x divided by f x naught. Suppose, I take this alpha, x be any point and y, I, I am choosing to be x minus alpha x naught because x naught is given, x you are choosing an arbitrary point in x. So, these two things are known. If I take alpha in terms of f x and f x naught, this is also known. Clear? Because f is given, f is given, so everything we can now pick up.

If I construct the y as x minus alpha x naught, then what is the image of y under f? This is equal to f x minus f of alpha x naught, but f is linear. So, it can be as f x minus alpha times of f x naught as f is linear. Clear? f is linear, but alpha we are choosing f x over f x naught. So, this entire thing will be 0 because alpha I have chosen to be f x over f x naught. Is it clear or not? It means, the image of y under f is coming to be 0. So, this shows that y must be a point of Nf.

So, what we have proved is, then any element x from here, x can be written as alpha x naught plus y, any element x can be expressed in this form where y is a point in N f. Is it ok? But what we want is this representation should be unique also, means, we can express x 1 in one and only one way, in this way. It means, if we have the two representation, then that representation will not give the two different values, that is, if I take x as a alpha x naught plus y 1, another representation x is alpha x naught plus y 2. Then, these things should not be the same; we will give the same, same thing.

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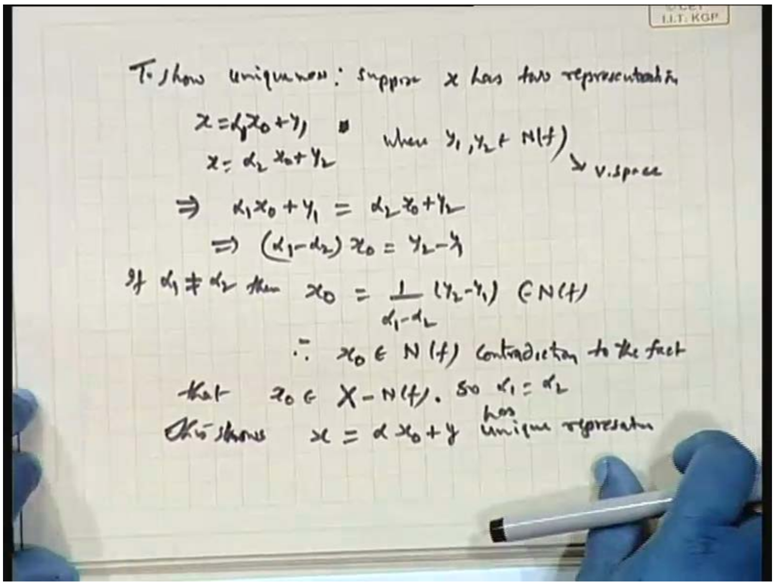

So, to show the uniqueness, uniqueness, suppose it has two representations. Suppose x has two representations, one is, x is alpha 1 x naught, alpha 1 x naught plus y 1, another representation is alpha 2 x naught plus y2. Suppose, it has the two representations like this, where y 1 and y 2, these are the elements of Nf and alpha 1 alpha 2 belong to the field of a scalar space. Clear? So, from here we get alpha 1 x naught. It means, we should have a contradiction alpha equal to alpha 2 x naught plus y 2. Now, this will give alpha 1 minus alpha 2 x naught equal to y 2 minus y 1. Clear? Is it ok?

Now, from here we conclude, that alpha 1, if y 1 and y 2 belong to what, if alpha 1 is not equal to alpha 2, what happens? If alpha 1 is not equal to alpha 2, then what we get? We get x naught equal to 1 by alpha 1 minus alpha 2 y 2 minus y 1, but y 1 and y 2 are the point in null space and null space. This is a vector space we have shown earlier. So, this element will be a point of Nf and this is nothing, but a scalar. So, entire thing belongs to Nf.

So, what conclusion is? So, therefore, x naught point, the x naught point, this is an element of Nf. Is it ok? But it is a contradiction, contradiction to the fact, that x naught belongs to X minus Nf. In the result, in this example, the statement itself, x naught is any fixed point of x minus Nf, so x naught belongs to x, but not in Nf. But here we are getting x naught belongs to the Nf. So, a contradiction and this contradiction is this, because our wrong assumption, that x has two different representations. It means, this shows, this shows, that x can be expressed only in the form of alpha x not plus y. This y may be changed, no problem, but this alpha will remain because x naught is fixed. So, alpha of x naught will remain, then x linear combination with the elements of y. So, it is unique, has unique representation, means, alpha must be, so here, so, or so alpha 1 should be equal to alpha 2, this must be the same. (( )) Is it correct? (( )) And next…

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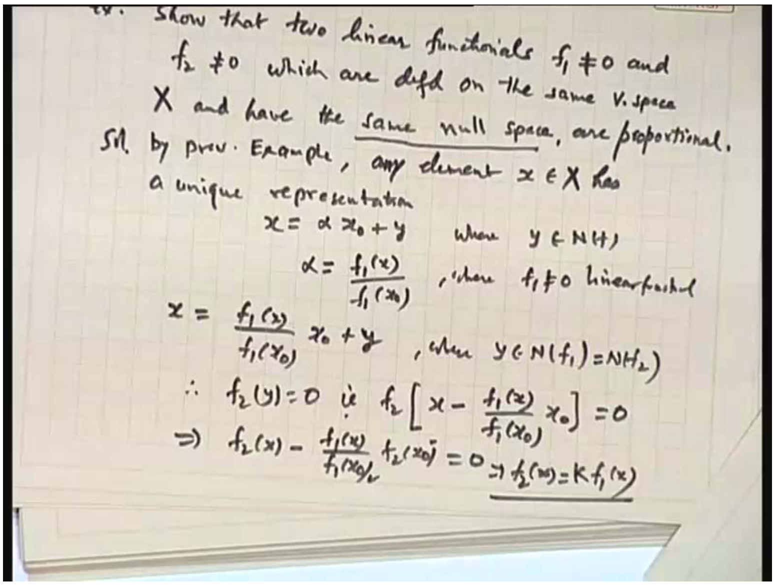

Show that two linear functional f 1, which is not equal to 0 and f 2, which is also not equal to 0, it is non-zero linear functional, which are defined, which are defined on the same vector space, vector space capital X and have the same null space, null space, are proportional, are proportional, are proportional. So, this is what we want. Let us see how this works?

Show, that the two linear functional f 1 and f 2, which are defined on the same vector space x, have the same null space and have the same null space are proportional. It means, f 1, f 2, X can be expressed as a constant times of the f 1 x, then only, is it not? This we wanted to show. Now, prior to this we have proved one result, one example we have done, that if f is any non-zero linear functional, then any element x and x naught is a fixed element of this, then any element x can have a have unique representation like this. So, we will take advantage of this, because f is giving to be non-zero. Here, f 1 and f 2, both are given to be non-zero, so we can take one function, say f 1 first, and then any element x can be expressed uniquely in this form, alpha x naught y.

So, let us take solution. So, by previous example, should I write by previous example, we can say any element x, any element x belonging to capital X, is it not, has a unique representation in the form of x as some alpha x naught plus y, where y is an element of

Nf. And if you remember, in derivation alpha is taking as f 1 x over f 1 x naught. Is it not? Clear? So, any, any element x has unique representation of this, where f 1 is not equal to 0, are linear functional on x. So, basically we are writing x to be f 1 x over f 1 x naught into x naught plus y.

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Nf 1, we have y belongs to Nf 1, is it ok? Now, what is given in the question? Two linear functionals are given, both are non-zero. I am picking up only one linear functional and applying the previous example and writing the expression x in terms of f y, is it ok, where this element belongs to y. But another condition is given, that f1 and f2 have the same null space. It means, Nf 1 will be the same as Nf 2. So, the value of y under f 2 must be 0; so, the value of y under f2 must be 0, that is, what is the value of this y is nothing, but the x minus f 1 x x naught into x naught, this must be 0. Is it ok? Or from here can you say f 2 x minus f 1 x f 1 x naught into f 2 x naught is 0. Is it ok?

Because f is linear, f2 is linear, what are the constant norm values? f 2 x is a variable because x is variable, so it will have a different value depending on x, but this has a fixed value; this has a fixed value. So, basically, f 2 x can be written as constant times of f 1 x. So, this implies f 2 x is equal to constant times K of x. So, are they not proportional? So, f 1 and f 2 are proportional. Clear? It is a good example? So, this gives a conclusion like this. Suppose on the vector space x or normed space x if we are having the two different linear functional, then if their null space are the same, then they are proportional.

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Linear vector space

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Actually, the vector space means, the set x together with the operation, addition and multiplication, then this is called the vector space or sometimes we call the linearly space. But linear vector space we mean, when this addition and scalar multiplication, they are jointly continuous, jointly continuous, then space is said to be a linear vector space, linear vector space. For the vector space, the continuity of the addition, scalar multiplication is not required, simply we test it whether it is closed with respect to the operation, addition all the five pole property with the scalar multiplication of five pole property, then it is a linear space. Clear?

But if on this structure if I define some norm or some other operation in which the addition and scalar multiplication becomes continuous, for example, if I define the norm, then what happens? The norm becomes continuous with respect to addition, as well as, scalar multiplication. So, when the addition and scalar multiplication are jointly continuous, then such a structure we call it as a linear space. It means, it should be a linear space, vector space. It should be continuity and scalar multiplication must be jointly continuous.

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In fact,…

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Continuous, we mean, I think this we have discussed in last class. Normally, we say the normed linear space, when you say linear vector space it has meaning norms, meaning unless you use some norm.

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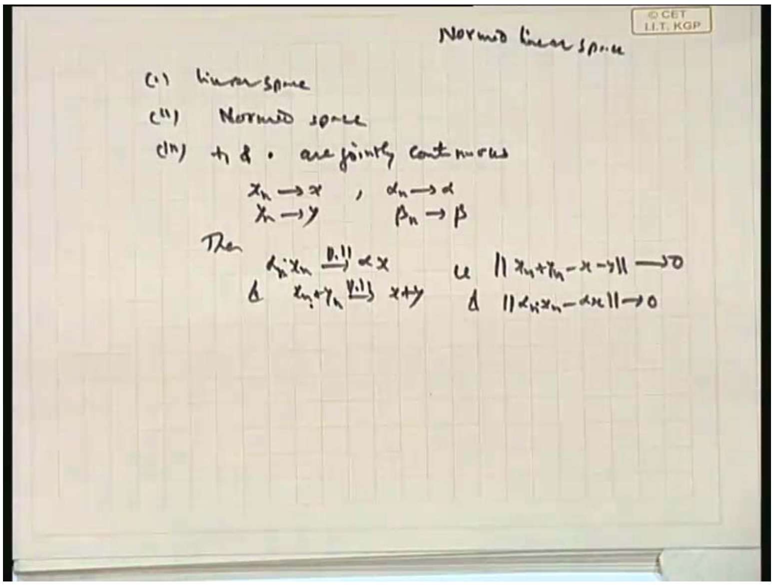

So, when we say the normed linear space, normed linear space, it means, it is a linear space, it is a normed space. And then, third is the addition and scalar multiplication are jointly continuous. Jointly continuous means, if x n converges to x, y n converges to y, alpha n converges to alpha, then and beta n say converges to beta, then alpha n x n must go to alpha x dot product and x n plus y n must go to x plus y under the norm. Then, we say this norm is a continuous function with respect to addition and scalar multiplication, that is, that is the meaning is norm of x n plus y n minus x minus y must go to 0, then n norm of alpha n x n minus alpha x must go to 0. So, with respect to addition and scalar multiplication it is continuous. So, if a structure, which has addition, multiplication and some other suitable operation like norm may be an inner product like this, then only the concept of the normed linear space comes, then they joint. So, but we are dealing only with the linear spaces or normed spaces, jointly this will, in fact, this lead to a concept of topological vector space, that way it is clear.

Now, let us see another example.

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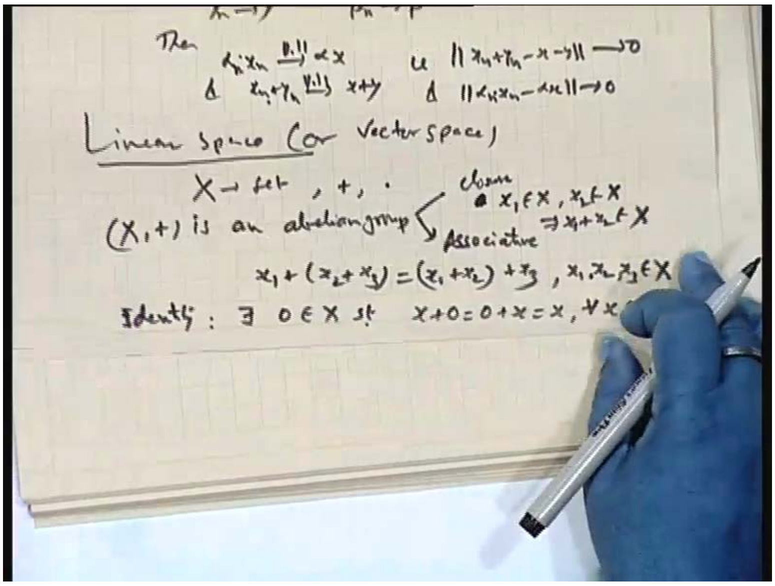

Linear space is same as the vector space, linear space. A linear space or we called a vector space, both are terminology same. It means, as x is a set and two operation addition and scalar multiplication are (( )) if X plus is an abelian group. Abelian group means, it is closure property; satisfy closure means, if x, x 1 belongs to X, x 2 belongs to X, then x 1 plus x 2 must be in x, closure property. Then, identity, associative property; associative means, x 1 plus x 2 plus x 3 is x 1 plus x 2 plus x 3, where x 1, x 2, x 3, these are the elements of X.

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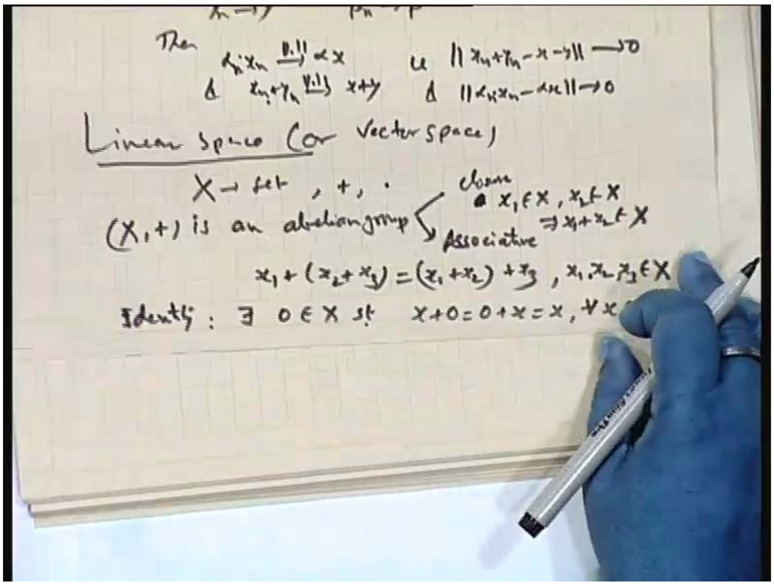

Then, third is first, second, then identity for addition, addition. There exist an element 0 belongs to x such that x plus 0 is 0 plus x equal to x and this is true for every x belongs to X, then we say, that identity element with respect to the addition exist.

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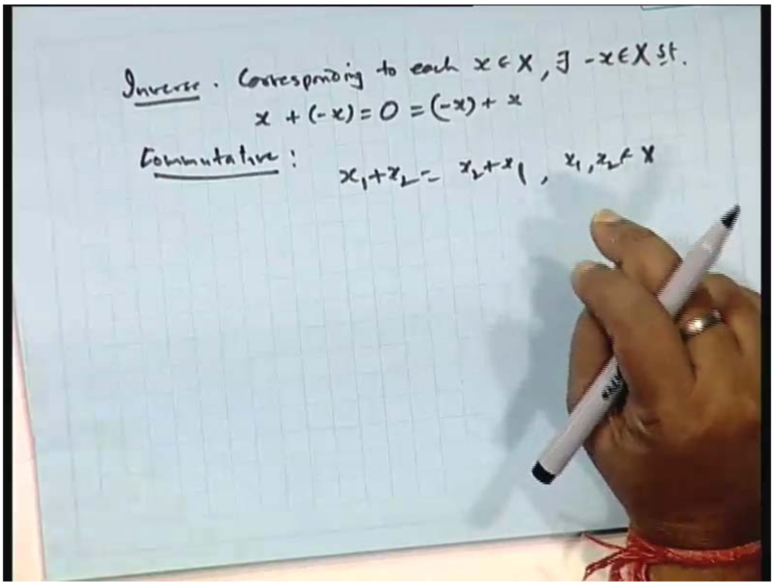

And fourth is, fourth is inverse corresponding to each x belonging to capital X, there exist minus x belongs to X such that x plus minus x equal to 0 equal to minus x plus x, then this is called additive inverse of x. Then, abelian or commutative, up to here four property, it is called a group. x plus is a group and when it takes another property commutative, that is, if x 1 plus x 2 is the same as x 2 plus x 1, then where x 1, x 2 are any arbitrary element of x, then we say it satisfy the commutative property and structure is the abelian group.

So, with respect to the first operation x must be an abelian group. Then, second one is, so this is the first A condition. So, there are five conditions are required to satisfy to be abelian.

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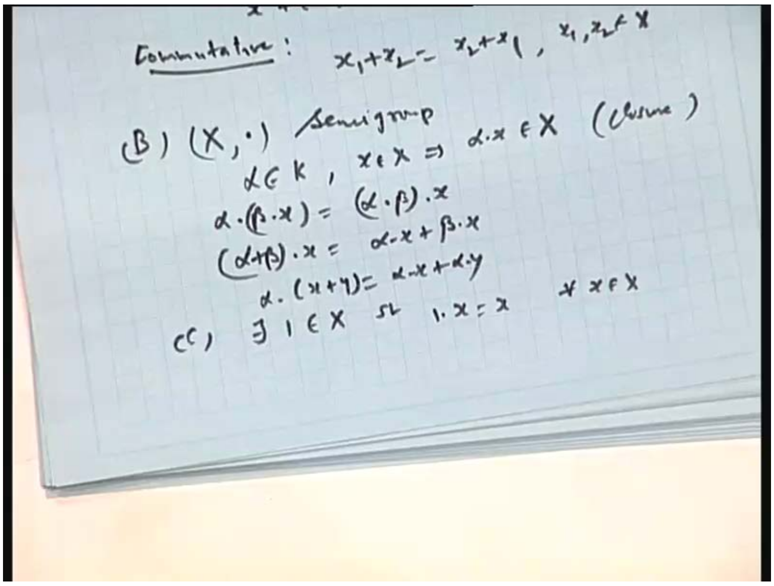

Now B, X with respect to multiplication this is a scalar multiplication, is a semi group, that is, if alpha belongs to the field k of scalars, x belongs to capital X, then alpha dot x must be a point of X. So, closure property, this is the closure property for a scalar multiplication, then alpha dot beta dot x, this will be equal to alpha dot beta dot x, then alpha plus beta dot x is alpha dot x plus beta dot x. This is true, then alpha dot x plus y is the same as alpha dot x plus alpha dot y.

Then, another property is 1, there exist 1, see there exist a 1 belongs to X such that 1 dot x is equal to x for all X scalar multiplication, identity for multiplication. So, this property has a semi group like. So, basically there are ten property satisfied with this. So, these properties we call it as a (( )). Is it ok or not?

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All, all the properties, yeah, yes

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Any vector spaces

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Linear has no meaning, linear space is the same as the vector space.

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Linear vector space is different than the linear space or the vector space I told you.

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That, that just now I earlier told when the operation, addition and scalar multiplication, they are also connected with some other operation, which is jointly continuous, then it is called the linear vector space. Now, maybe some books, some books use the word linear space, also as a linear vector space in some books, but normally we use like crazy here in, in this book linear space and linear vector space is different conserved. But in some books, linear space and linear vector space they consider to be the same, they satisfy this format, so depending on the books.

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Yes, exactly.

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Normed linear space norm is there.

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Norm, under the norm it is continuous.

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Yes

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And then jointly continuous

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Then, it is a normed linear space, but if the some books is writing only the linear vector space it means, he is meaning with the linear space only, is mean, is mean, he means only the linear space. So, you just check the book what are these, whether any…

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But whether any operation, addition and scalar multiplication, apart from this any other operation he has taken?

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[fl] whether addition and multiplication, whatever the two operation is there, any third operation in that? If not, then it is the same as the linear space.

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So, he means only the linear space. That is all, clear. So, this…

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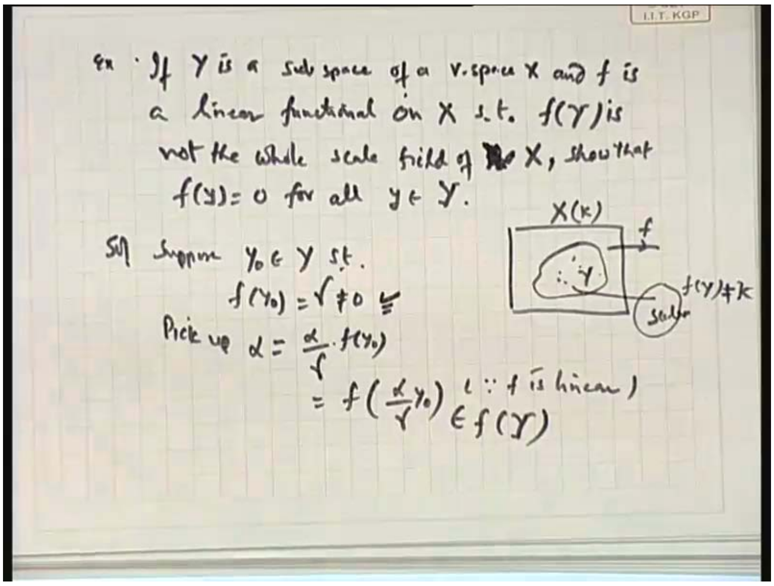

So, we were discussing the, some problem. Now, let us take this problem. If y is another problem on the linear, if y is a subspace, subspace of a vector space X and, and f is a linear f, is a linear functional f, is a linear functional on X such that f y is not the whole scalar field, is not the whole scalar field K, is not the whole scalar field of K. So, that field of k, sorry, field of X, field of X, so that f y is 0 for all y belongs to capital Y, all y belongs to capital Y.

So, what is this? If y is a subspace of a vector space, this is our vector space X over the field of scalars, say k, k is a field of the scalar, y is a subspace of this vector space and f is a linear functional defined on X, then it will have the image f of y, each element of y will be mapped. So, this is the f of y. Now this is a linear functional. So, this will be the set of scalars basically scalars belonging to k.

Now, what he says is if f I y is not equal to K, that is, it is a subset of, proper subset of K, then f y will be 0 for all Y belongs to this; y is a subspace of the vector space and f is a linear functional on X such that f y is not the whole scalar field of X, that is, image of y under f is not equal to K. There are some more scalars are available, which are not in f y, then what he says is that the value of element y under f should be 0, that image of such a function, such a function f will take all the points elements of y to be 0. So, that is what he says.

Let us take this, let us see this, suppose, suppose for some point it is not 0. Suppose y naught is a point in Y such that f of y naught is say gamma, which is not equal to 0. Suppose, then we should read a contradiction now picked up, pick up alpha a scalar, which is equal to alpha by gamma into f of y not because f y naught is gamma. So, this consider alpha equal to this. Now f is what? f is linear, this is constant, so it can be written as f of alpha by gamma into y naught because f is linear. Now, since y naught is a point in Y, Y is a subspace capital, Y is a subspace. So, this will be a point of y. So, if it is the point of y, then each element must belong to f of Y, this belongs to f of Y.

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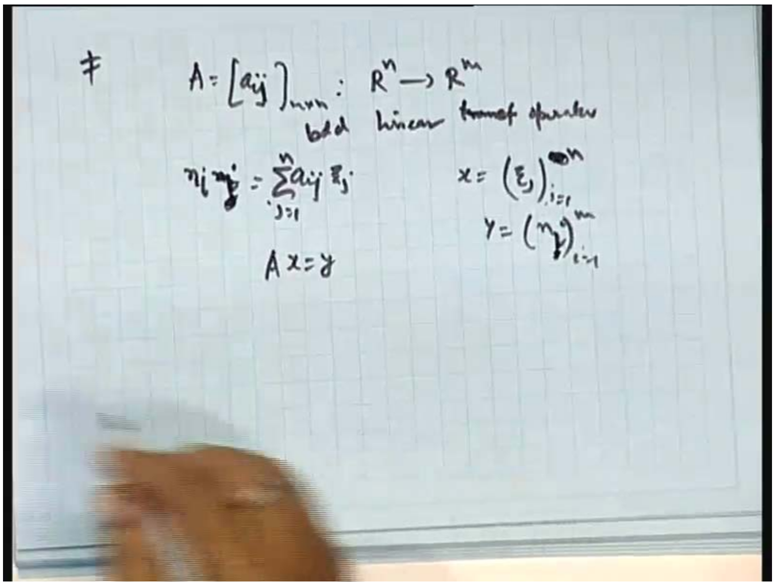

If you remember, if x is a finite dimensional case or this we know a matrix A, which is a matrix a ij of order m cross n, then this represents a transformation from R n to R m remember and this is a linear mapping, linear transformation. In fact, linear functional transformation from this matrix and bounded, also bounded linear transformation or operator sequences transformation of operator. And how to define this one is a ij and then x ij sigma j 1 to n. This is our eta j, then this elements x n where x is equal to x ij, k equal to 1 to infinity and y equal to eta i. So, this is eta i, eta i, i is 1 to, 1 to n. So, 1 to n and 1 to m, is it not. So, this is our eta i. It means, it can transfer the elements of x n to the elements of y n under this transformation and we write A x equal to y.

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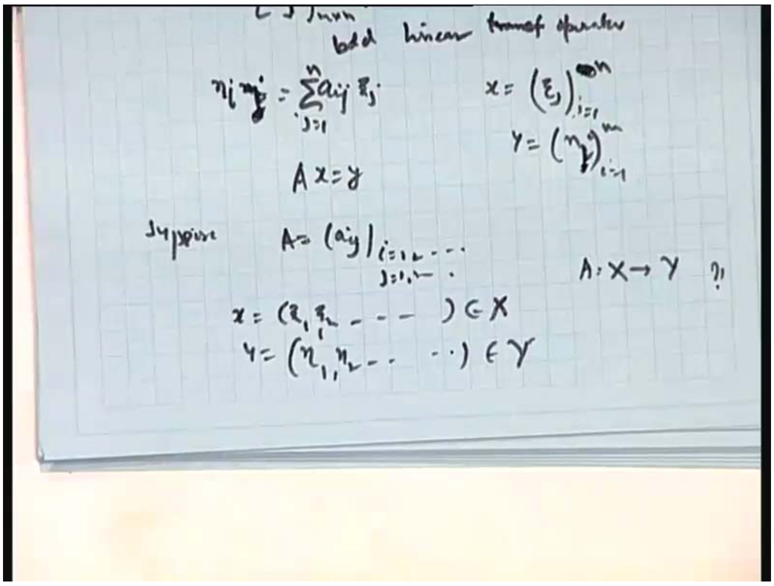

Now, this is the case when x finite dimensional, similar case we can extend to the infinite dimensional. If there is a matrix A, A is a matrix, suppose, suppose A is a matrix A ij of order infinite dimensional, where i varies from 1 to infinity, j varies from 1 to… and x and y, these are the spaces. x is x i1, x i2 and so on, is an infinite dimensional space, y is eta 1, eta 2 and so on is again an infinite dimensional (( )), then how will you operate from x to y? How to define finite dimensional case is, because this series is finite convergent, therefore the corresponding will be the convergent. So, this definition leads to the concept of j equal to matrix transformation. How did we define in a similar way?

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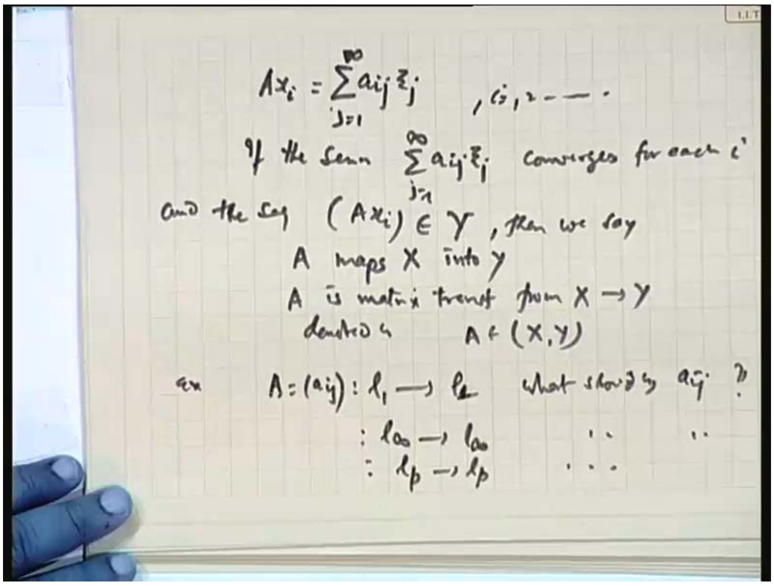

What we do is we consider this sequence sigma a ij x ij, j equal to1 to infinity. Now, this sequence we call it as Ax i, Ax i because x 1, x 1, x 2, x n are there are so, a of x i. Now, this elements, if this is convergent for each i, this sequence convergent for each i, because this is infinite series, i is 1, 2 and so on. So, if this series converges, series sigma a ij x ij, j equal to 1 to infinity converges for each i and the sequence Ax i is an element of Y, is an... Suppose it is a l infinity. So, scope of this must be there if it is l, one sigma of this thing must be there if this is in this is in Y then we say then we say A maps X into Y or A is a matrix transformation from X to Y and we denote this thing as A belongs to X Y. You follow me? Clear? Because this is a double summation a ij and i is not finite and this j is also not finite, so for each I, suppose i take a1, then a 1 j, this will give a infinite series. So, this series must be convergent for a1, a 2, a 3 and so on a i equal to 1. So, you are getting a converge sequence; all the sequence are convergent. But that sequence, if that sequence belongs to the class Y, then we say the A transfer the element of from X to Y or A is the matrix transformation from A to Y.

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A belongs to X, Y it means take the element of X. First find construct this series and this should be convergent for each i and then this sequence again take A x1, A x2 etcetera formed sequence and that should be the element of Y. Clear? So, the question is under what condition on a ij this matrix transformations A maps from X to Y when x and y are different? Suppose I take a matrix A a ij transformed l1 to l l 1, then what should be the restriction on a ij, so that it can transfer each element l1 to l1. Similarly, what should be the, should be the restriction on a ij. Clear?

If I say l infinity to l infinity, then what should be the restriction on a ij? If I say it transformed form l p to l p what should be the restriction. So, this transformation matrix a can be characterized. You can change the coefficient accordingly when we want the transformation from one space to other space. As soon as you change the space, the corresponding coefficient of the transform matrix will change and that gives you the characterization of the matrix required for transformation.

Suppose you are dealing with a problem where your domain is a space l1, but you wanted the elements should go to the l infinity. Is it not? So, in that case what type of the matrix you should use? So, that it can transfer the element of l 1 to l infinity. Clear? Similarly, if you have some range space something specified, but domain is something different, the corresponding matrix can be taken can be chosen suitably, so that you get that. So, that is gives a very interesting thing and in fact, I wanted to complete this first. So, maybe next time we will see that and that will be useful.

Thank you.