Functional Analysis

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Module No. # 01

Lecture No. # 21

Inner Product and Hilbert Space

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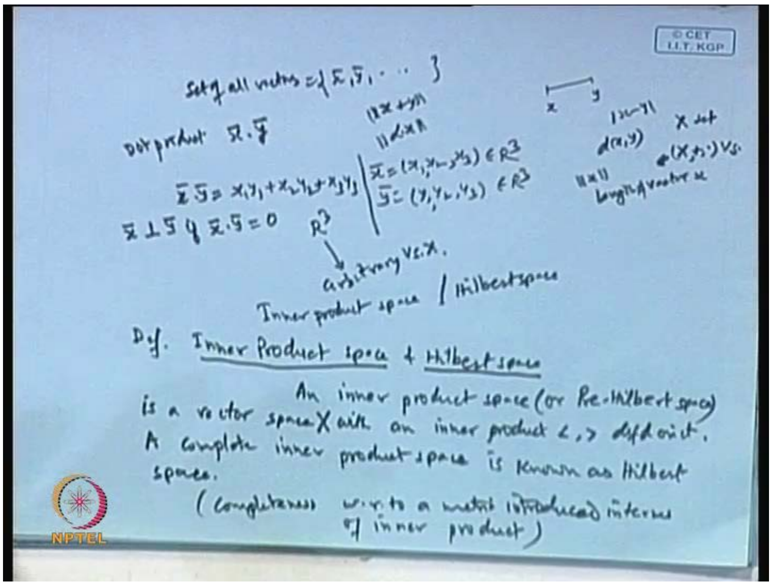

So, in the previous lectures, we have covered metric spaces and normed spaces. In the metric space, what we have seen is that, the concept of the notion of the distance between the two point on the real line is taken as mode of x minus y and this concept has been generalized to an arbitrary set X, by introducing the notion of the distance d. Then, we have, instead of taking this set, simply set X, we have introduced the operations and considering the vector spaces V and then, on this vector space, we have introduced the concept of the norm. And, this norm generalizes the concept of the length of the vector x, of the vector x.

In the normed space, we have also seen that, we can take any two points and also, we can add, we can multiply this by x. So, x and y are the two vectors; one can add in the normed space, the two vectors and introduce the norm of x plus y. Similarly, norm of alpha x can be defined. But what is missing here is that, in case of the set of all vectors, say x bar, y bar, z bar, there is a well-known concept, and useful concept, that is the known as the dot product of the two vectors, x bar dot y bar. If x bar, y bar is suppose, x 1, x 2, x 3 in say, R 3 space, and y bar is y 1, y 2, y 3, that is also in R 3 space and this dot product, in case of the three dimensional vector space is nothing, but x 1 y 1, x 2 y 2 plus x 3 y 3.

This dot products also leads the another concept, which is called the orthogonality of the two vector. We say the two vector x bar and y bar, they are orthogonal, if x bar dot y bar is zero, clear. So, this concept of the orthogonality, and concept of the dot product, so far, has not been generalized from R 3 space to an arbitrary vector space X. So, the question is, can we generalize this concept also to an arbitrary vector space, so that, the structure which you are getting will be more useful and can be, have a application, where the dot product or orthogonality plays the vital role. And, this leads to the concept of inner product space and as a particular case, when it is a complete inner product space, we say it is a Hilbert space, clear.

So, idea of introducing the inner product and the Hilbert space, basically, is to enhance the concept of the orthogonality to a general arbitrary vector space, as well as, the introducing the parallel concept or generalized concept of the dot product. So, what is our inner product space and Hilbert space? An inner product space, or we also call it as a Pre-Hilbert space, space, is a vector space, is a vector space, together with an inner product, say, I am denoting this way, defined on it, defined on it, define on X, is a vector space X, with an inner product defined on X. The inner product, we will specify, we will define below. Just let me complete. And, whereas, I told the Hilbert space is a particular case, when we put a certain restriction. So, a complete inner product space, complete inner product space is known as an Hilbert space, known as Hilbert space, Hilbert space.

The complete, we means that, we have, we should have a metric d, in terms of the inner product and under that metric, the Cauchy sequence converges. So, completeness means, completeness with respect to a metric, introduced in terms of inner product, clear. So, we are (( )). Now, entire thing which we have defined, depends on the definition of inner product; how to define the inner product? Because, once the inner product is defined, on a vector space, then, this sphere will be a inner product space and when the metric is defined in terms of the inner product and every Cauchy sequences converges, then, we say it is a Hilbert space.

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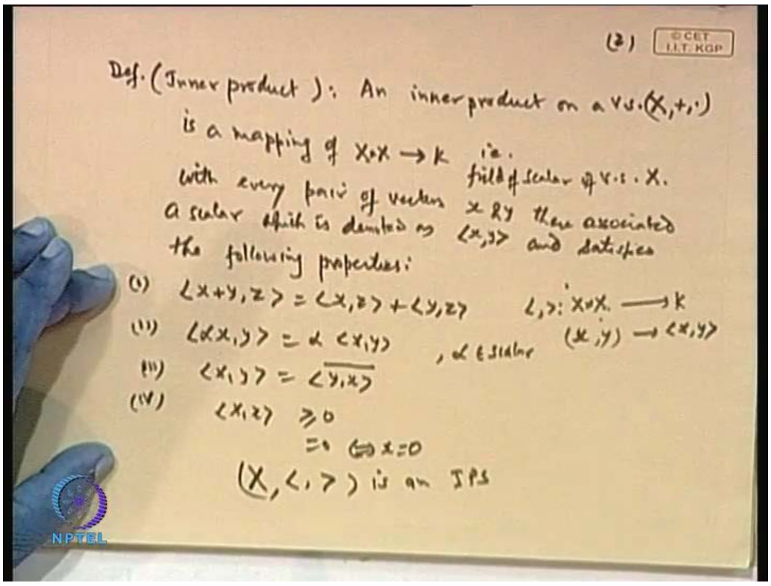

So, inner product is defined as, inner product, an inner product on a vector space, capital X, say plus and dot, these are the operation, inner product on a vector space X is a mapping of X cross X to K, K is the field of scalar of the vector space X. Because, when we say X is a vector space, then, there should be a field attached to this. So, X is a vector space over the field K. It may be real, then, we say, it is a real vector space. When K is complex, we call it as a complex. So, a mapping from X cross X to K, this mapping, we call it as inner product, inner product, it, provided the following conditions are satisfied. So, that is, a map, that is, with, for each, with every pair of vectors x and y, their associated a scalar, we denote this by, which is denoted as this x comma y, x comma y and satisfy the following property, clear.

Means, we are having, this is our X cross X. We are having a mapping from X cross X to K. So, pick up a two point x and y, clear. Say, x here and y here, and this is the mapping; so, which will associate this vector to, to the scalar x, y. Take two point x and y; x belongs to X and y belongs to X; and, under this mapping, we get a scalar x,y, which satisfy the following properties. Number one, that inner product x plus y comma z is equal to inner product of x z plus inner product of y z. Second, the inner product alpha x comma y is alpha times inner product x y, where alpha is a scalar quantity, belongs to the K set. And, third is, inner product of x y is the same as inner product of y x, conjugate of this. Fourth property, inner product of x x is greater than equal to zero and if zero, if and only, if x equal to zero.

So, if these four properties are satisfied, then, we say this structure X, under this mapping, is an inner product space I P S. Now, let us look, what is, what are these properties? The first property and second property, if I combine the first and second property, then, what we get is a remark, a conclusion, or say, remark. Remark is, if we take one and two together, then, this implies that, alpha x plus beta y comma z equal to alpha of inner product of x z plus beta of inner product of y z, clear. If I take the third condition, then, third shows that, alpha, inner product of x, say gamma y, equal to gamma y x conjugate, which is equal to gamma conjugate y x conjugate, which is gamma conjugate x y, clear. It means that, if I take a linear combination in the first coordinate, because this inner product is two coordinate, one and second; so, in the first coordinate, if I take a scalar, alpha times x and take this scalar outside, then there is no change, ok.

But as soon as this scalar is available in the second coordinate, the sign is, the value changes; it becomes the conjugate of the original one. It means that, in the first coordinate, the linear property is satisfied; whereas the second coordinate, linearity is breakdown. In fact, it is semi-linear or because, only the conjugate part is there, in place of gamma; otherwise, if we take gamma y…

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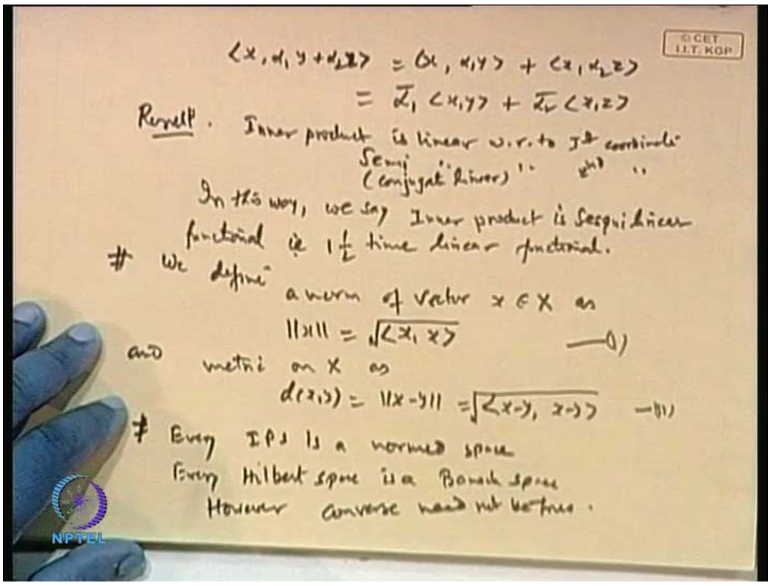

If we take here, say, inner product of x, say alpha 1 y plus alpha 2 y, alpha 2 z; then, this can be written as, x alpha 1 y inner product plus x alpha 2 z; and, that can be written as, alpha 1 conjugate x y plus alpha 2 conjugate x z, clear. It means, this alpha 1 and alpha 2 can be taken outside, but there is a conjugate sign. So, it is linear, it is not linear, but we call it, such a thing as, a semi-linear. So, if we say, the result is or conclusion is that, inner product is linear, with respect to first coordinate, while a semi-linear, with respect to the second coordinate or conjugate linear; we can also say conjugate linear, conjugate linear, clear.

So, so, so, if we combine both, with this, in this way, we say inner product is sesquilinear functional, sesquilinear functional; that is, one and a half times linear functional. Is it clear? Functional means, its value, scalar value. So, this is an example of a sesquilinear function. Now, what we have…Let us go through back, again, for this. The inner product on the vector space X, we have introduced the concept of inner product. Now, as I had spoken that, inner product, with the help of inner product, if I introduce the concept of the norm and the metric, then, it will be useful; because, this inner product will carry the extension of the dot product, as well as the orthogonality.

So, we introduce, we define a norm of a vector x belonging to capital X, as norm of x equal to a null product x x under root; and, metric on X as d of x y equal to norm of x minus y, which is inner product of x minus y, x minus y under root. We will verify that, this is a norm and this gives you the metric on X, clear. All the conditions of the norm will be satisfied, except the last one, triangular inequality; we require a Schwarz inequality, before going for establishing the triangular inequality is satisfied. So, we will see this later, and once it is norm, this will definitely give a metric, because, this is defined in terms of the norm x minus y, which is already, we spoke.

So, what this shows that, every inner product space is a normed space, is it ok; and, since the Hilbert space is a complete inner product space, so, every Hilbert space is a complete normed space; that is Banach space. So, every Hilbert space is a Banach space. The converse of this need not be true; that we will see. However, converse need not be true; that we will see there; that every Hilbert space, every normed space is not a inner product space; every Banach space need not be a Hilbert space. So, that, we will see.

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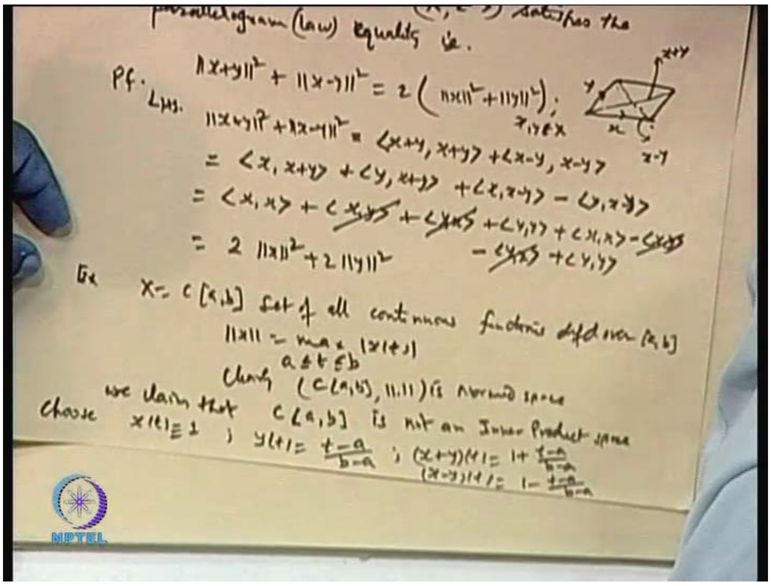

Now, before going this, let us see a general results, which we call it as parallelogram law inequality. The, we claim that, every inner product space X, this, satisfy the parallelogram law, parallelogram law or parallelogram equality, parallelogram law or equality; that is, norm of x plus y whole square plus norm of x minus y whole square is two times norm x square plus norm of y square. The physical concept of this is that, if we have this parallelogram having, say, x and y are the sides, vectors, then, this will represent the vector x plus y, while this one will represent the vector x minus y. So, the sum of the length of this diagonal, norm x plus y square plus, square of the sum of the length of this diagonal is the two times sum of the squares, square of the sides, two times sum of the squares of the sides, is it not, x square plus y square.

The proof is just simply based on the definition. If we take the left hand side, say, norm of x plus y squared plus norm of x minus y squared, then, this is equal to, by definition, norm of x plus y whole squared means, x plus y comma x plus y inner product and norm of this is, x minus y comma inner product x minus y, ok. Now, apply this. So, you take this inner product x comma x plus y, y comma x plus y, then, plus x comma x minus y, minus, you take it outside, because it is a scalar in the first coordinate, so, we get, y, x minus y.

And, this will give further, x comma x plus x comma y plus y comma x plus y comma y plus x comma x minus, because minus 1 conjugate will be the same as minus, because it is real, so, minus x comma y and minus minus plus, y comma y. Now, you will see, this gets cancelled; y x, and somewhere y x is also, x minus, minus y; so, it is y x, sorry, this is x, then y, then x x minus y; then here, minus y, x minus y; then this, we are getting x x and here x y; then y x and y y; then x, oh, x x, x minus y; then, here it is changed, minus y x is left out, is it not. So, this one. So, this gets cancelled from here, and what you are getting is, two times, x x is the inner product, is norm x square and y y means norm y.

So, this proves it, clear. So, what this shows that, if a vector space is an inner product space, then, it will satisfy, the inner product will satisfy the parallelogram law. It means, the norm which introduced by means of the inner product, that norm x is inner product x x under root, will satisfy this condition, clear. If I think other way around, suppose, a norm does not satisfy this inner product; then, obviously, we can say, that norm cannot be derived with the help of inner product; because, if it is derived with the help of inner product, then, it must satisfy the parallelogram law.

So, there are example of the normed space, which do not satisfy the parallelogram law. It means, those normed space are not a inner product space, clear. And, that gives the counter example that, every normed space, that gives an example that, every normed space need not be a inner product space. For example, if I take, say, x is C a b, set of all continuous functions defined over the closed interval a b. So, since it is defined over closed interval a b, so, it will attain its maximum, minimum value and the norm of x is defined as the maximum of mod x t and t ranges from a to b. So, obviously, C a b, with this norm is a normed space, clear. Now, we claim that, C a b is not an inner product space, is not an inner product space. It means, the norm which you are taking, does not satisfy the parallelogram law; for any x, y, this is true for any, this inequality is true for any x, y, is it not, any x, y. So, if I take a particular x and y, which violate this condition, then, obviously, this space will not be a inner product space. So, let us take the (( )) the x and y…Choose x t equal to 1 and y t equal to t minus a over b minus a. Both are continuous functions. So, what is our x plus y t will be 1 plus t minus a b minus a. What is x minus y t? 1 minus t minus a b minus a, ok.

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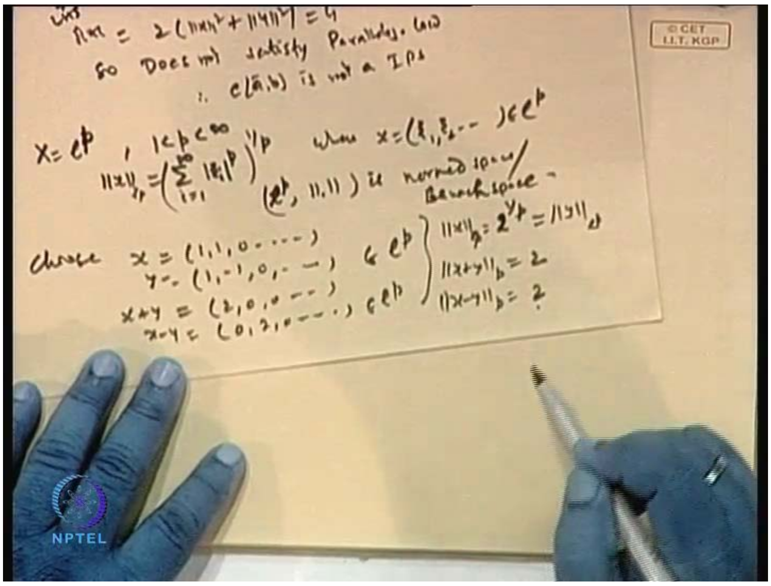

Now, the maximum value of this is defined as the norm. So, what is the norm of this? So, the norm of x under this, is nothing, but 1, because the maximum value of this. What is the norm of y? The maximum value of t minus a over b minus a, t ranges from a to b. Now, t minus a over b minus a, this function y, if I differentiate by prime t, it comes out to be 1 upon b minus a, clear. So, 1 upon b minus is positive. So, derivative is greater than zero. So, function is an increasing function. It means, its maximum value was attained at the point, t is b. So, when t is b, the maximum value of this will be 1, agreed; because, derivative y is positive, clear; y prime t is positive. So, y is increasing function and therefore, we get this maximum value is attained at the point t is equal to b.

(( )) not possible.

No, no, from the Lagrange (( )) theorem, greater than zero; so, it is a maximum. So, it is increasing function; increasing means, maximum value will attain at the end point. So, b, b equal to…Now, what is the norm of x plus y? The x plus y, again, this is 1 plus, this 1 is ok; the maximum value of this is 1. So, maximum value is means, it is 2; norm of x minus y, the maximum value will be what? What is the maximum value? We are subtracting the things. So, we have to subtract the minimum value, so that, you get the maximum value. So, minimum value attained, when t is equal to a. So, the norm of this will be 1. So, we get 1, this is 1, is it not. Now, norm of x plus y is 1; norm of x minus y was 1. So, this is 2, this is 1 and here is 1. So, what we, left hand side, this thing, this is equal to 5, while the right hand side is two times norm x square; this is equal to only 4. So, parallelogram law is not satisfied; so, does not satisfy parallelogram law. Therefore, this is not a, C a b is not a inner product space. Though it is a normed space, but it is not inner product space. So, this is one of the example.

Another example, we can say, l 2 space; X is l 2, or, we say l p. Let us take l p, 1 is less than p less than infinity. I take, and the norm of this, any point in this l p is sigma mod x i i power p power 1 by p, is it not, where x, x i 1, x i 2 etcetera, this is the elements of l p. And, we know, this l p, with this norm, is a normed space. In fact, it is a Banach space also (( )). This is also Banach space, Banach space. Now, if we choose x element as 1, 1, 0, 0, 0 and let us take the y, 1 minus 1, 0, 0, 0; both are the elements in l p, because this is finite; so, summation is finite and you get, clear. Now, if we take x plus y, this is equal to 2, 0, 0, 0; x minus y, 0, 2, 0, 0, and they are also in l p. What is the norm? Norm of x under l p, say l p, this is equal to what, 1 or 2 raise to the power 1 by p. x i 1 is 1. So, 1 to the power will be 1; x i 2 is 1. So, 1 plus 1, 2, to the power 1 by p and that is the same as norm of y a b. But what is the norm of x plus y 1 p? This is equal to 2 only, because 2 to the power p power 1 by p. x i 1 is 2. So, x i 1 to the power p power 1 by p and rest are 0; x minus y, this is equal to what, same as two.

Now, if I square these two, what you get? So, whole squared. This is equal to what, 2 plus 2, 2 square plus 2 square, that is 8 and what is the norm of x square plus norm of y square? Two times of this, which, this is equal to two times of, this is square and again this is square; so, in fact, 2 into 2 power p, is it ok or not? Clear? Now, if I want this parallelogram law or equality holds, this is only possible, when p is equal to 2; otherwise when p is different two, it will not be satisfied.

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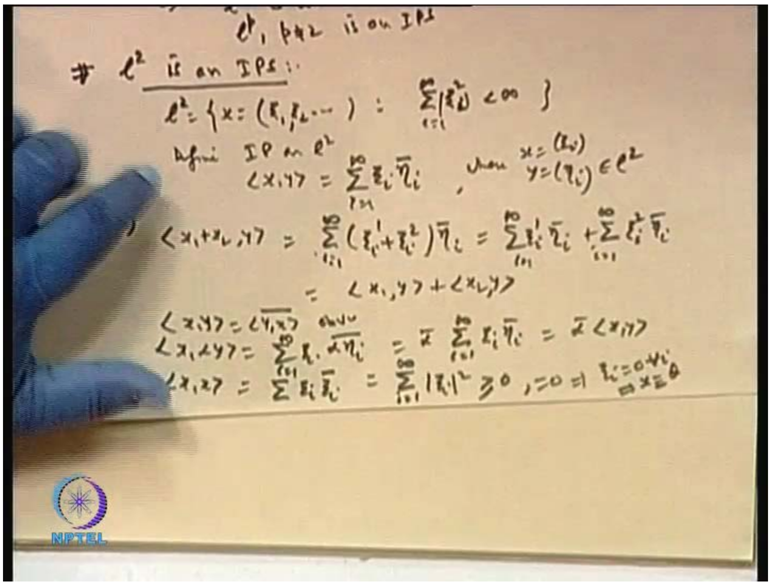

So, only when p is equal to 2, so, only when p is equal to 2, the parallelogram law holds, but while, when p is different from 2, the parallelogram law does not hold. It means that, l 2 is a inner product space, provided we satisfy, may be an inner product space, but l p, for p is different from 2, is not an inner product space; but l p is a Banach space. So, again, there is a example, where so many spaces are there, which are Banach, but not a Hilbert, or inner product, normed space, but not a inner product space, ok. Now, we will say, how this satisfy, l 2 is a inner product space. We have to introduce the notion of the inner product on l 2 suitably, so that, all the conditions of the inner products are satisfied, ok. Let us see how, inner product, that is our question, clear. So, l 2 is an inner product space, clear.

What is l 2? l 2 is the set of those sequence x, such that, sigma x i square, i is 1 to infinity is finite, clear. If it is a real space, if it is complex, take the mod; square is finite. Now, introduce, define inner product on l 2. So, inner product of x y, I am defining as sigma x i i eta i conjugate, i is 1 to infinity, where x is, x i i, y is eta i belongs to l 2. Now, this may be complex, that is why conjugate is there. Now, this will satisfy all the property of the inner product. First is, if I replace x by alpha x plus beta y, here it will change. So, you can take alpha and beta outside. If I take y alpha here, then conjugate sign is coming. So, clearly, x 1 plus x 2 y that will be equal to 1, 2 eta i bar and that can be written as 1 eta i bar x i 2 eta i bar.

So, that will be equal to what, inner product of x 1 y, x 2 y. Similarly, if we take inner product of x y conjugate is there, satisfied, is it not; obvious; inner product of x alpha y, alpha eta i conjugate. So, alpha bar is coming outside and that will be equal to…Then, what is our x x? x x is nothing, but sigma x i i x i i conjugate and that is equal to sigma i equal to 1 to infinity, i, 1 to infinity, mod of x i square. Now, this is positive. And, if it is 0, then each of this must be 0, but each one is non-negative. So, individually…So, equal to 0 will implies that, x i i will be 0, for each i; this implies x must be identically theta (( )), ok. So, this. It means, this is an inner product space, under this inner product. And, corresponding norm will be x i i square power half, that is a inner product (( )).

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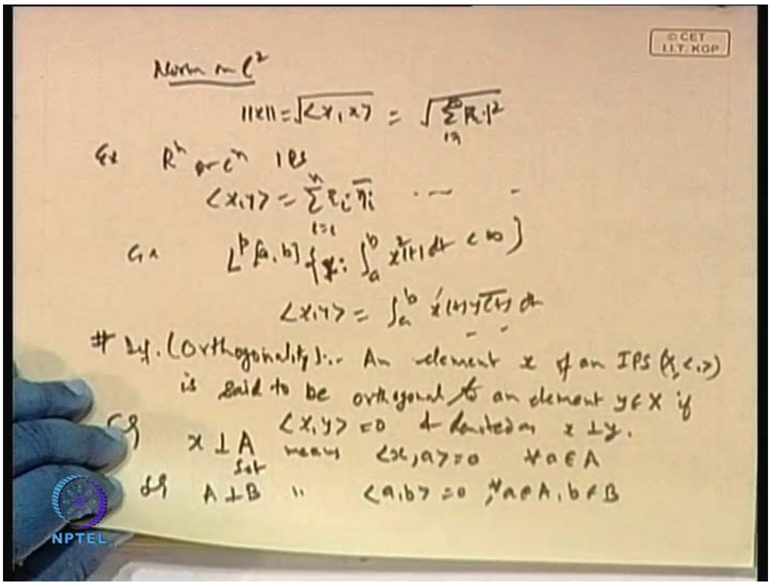

So, norm of this, a norm on l 2, norm of x is the inner product of x x conjugate. So, this is equal to what, under root sigma mod x i square, i is 1 to infinity and that is what, we had know this. So, it is a inner product. So, this is one example. Another examples also, R 3, R n, R n C n, these are also inner product space, under the inner product defined as sigma x i i eta i bar, i is 1 to n, when it is complex number and we say, all verification. There is a space also l p a b, this space also we have discussed; set of all integrated function, whose pth integral is finite. So, here, a to b, set of all functions, functions x, such that, x t square d t is finite and the inner product of x y is defined as conjugate. We can just verify (( )). So, this (( )) if it is.

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L 1 is not, small l 1 is not a inner product space; except l 2, nothing is inner product space, ok. So, it is basically, it is a very small class. The inner product, all the Hilbert space is very small class, which is a subclass of the normed space, clear. Biggest class is the metric. Then, we have excuses to find the normed spaces, clear; because, every normed space, every metric space need not be normed space, clear; and, from normed space, we have further excused as the, the inner product.

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No, we cannot. We cannot define, clear; should be, first, the space must be inner product space; then, only you can introduce the l, clear. So, these are the few examples of the inner product space. Now, concept of the orthogonality. We have, let us introduce the concept of orthogonality, in terms of the inner product; orthogonality. An element x, an element x of an inner product space, x this, of an inner product spaces, is said to be orthogonal to an element y, belongs to capital X, if the inner product of this is 0; means, an X is an set and take element x, then, find out the inner product with other elements of X; and if the inner product comes out to be 0, then, those elements y 1, y 2, y n, we say x is orthogonal to those elements, clear.

Similarly, if suppose, x is an element and A is a group of, a set of elements; we say x is orthogonal to the set collection A, if inner product of x a is 0, for every a belongs to A. Similarly, we define, x is orthogonal to, and this, sorry, here, and denoted as x is orthogonal to y. Similarly, when we say x is orthogonal to a set A, means that inner product of x a is 0, for every a belongs to A. However, we say the two sets are orthogonal; if, means, the inner product of a b is 0, where a belongs to A, b belongs to B, for every pair, for every pair, it is two. So, the concept of the orthogonality can be defined in terms of the inner product. That is (( )). Now, this inner product, this orthogonality will be used and in fact, most of these ((charm)) for the study of the inner product on the Hilbert space, is the orthogonality part. We get the projection theorem, etcetera; we know the projection, whenever any point, space is given, a point we wanted to drop the shorter distance, it is nothing, but the perpendicular distance. So, there is orthogonality (( )).

Now, we have introduced the concept of the inner product and then, a concept of the norm is there, in terms of the inner product. So, first the inner product, then, we have generated the norms; and, we have seen that, every norm is not a inner product. Now, suppose, it is given that, this particular normed space is an inner product space. So, the normed space is given, but we do not know, what will be the inner product; but it is known, this will definitely a inner product. So, can we extract the inner product, with the help of the norm of that space? Do you follow me? An inner product is given, x inner product; then, one can introduce the norm, as the norm of x, under root x, inner product x x.

Now, let us think a converse way. If the norm is given, norm of x is given, can you find the inner product x x, in terms of the norm, ok; where it is known that, this normed space is also an inner product space or the norm can be derived with the help of the inner product.

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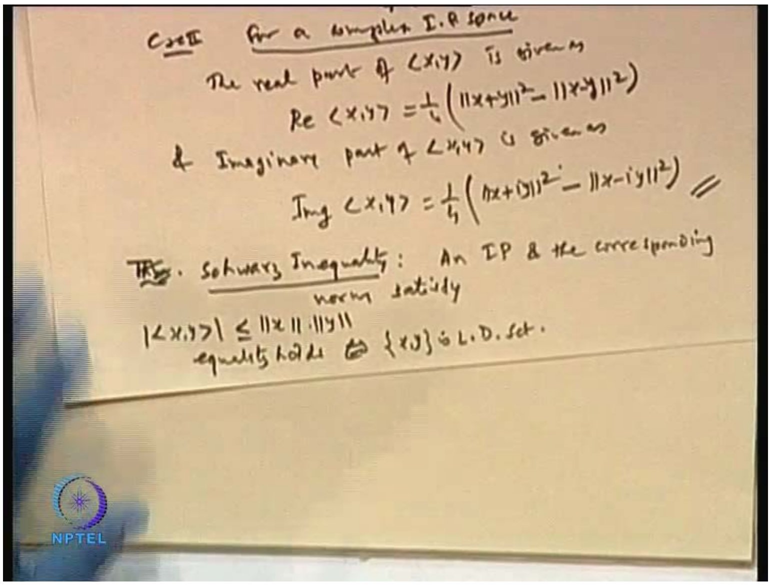

So, this is now, given by the polarization identity, identity. What is the polarization identity is that, in case of the real inner product space, case one, for a real inner product space, real inner product space X, the inner product x y, the inner product, the inner product x y is given by, is given by one fourth of norm of x plus y whole square minus norm of x minus y whole square. Means, a normed space is given, which is also an inner product space; then, one can find the inner product, obviously. This is to discover the inner product, to discover, rediscover the inner product from the, rediscover the inner product from the given norm, given norm, clear.

So, these norms are known; we can find out the inner product x y, where it is known the, this space is real space. And, if the space is complex, for a complex inner product space, the real part of the inner product, of the inner product x y is given as, one fourth of norm of x plus y whole square minus norm of x minus y whole square and imaginary part of the inner product x y is given by, is given as one fourth norm of x plus i y whole square minus norm of x minus i y whole square, clear. Now, this can be derived very easily. This norm is given and it is given that, norm is also inner product. So, use the norm x square is equal to inner product x x; open it, you will get these things, immediately; nothing, ok. So, this polarization identity will help you in rediscovering the inner product, if it is norm. Now, there is one result, which we require for establishing the triangular inequality in order to prove the norm. That result is or lemma is known as Schwarz lemma, Schwarz inequality; it is not a theorem, Schwarz inequality.

What he says is, an inner product, an inner product, inner product and the corresponding norm, and corresponding norm, satisfy, satisfy, this thing, norm of, sorry, satisfy modulus of inner product x y, modulus of inner product x y is less than equal to norm of x into norm of y, where the equality holds, holds, if and only if, x y, this set is linearly dependent set, clear. The proof is simple. The proof of this Schwarz is… So, first, we are taking, 7; this is 8.

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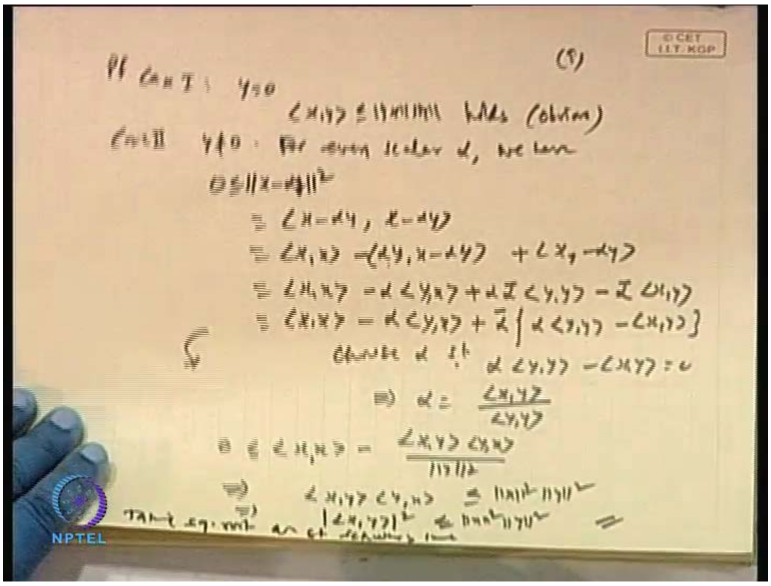

Suppose, one of this term is 0; suppose, y is 0. Then, obviously, this inner product of x y equal to, less than equal to norm of x norm of y holds, is it ok or not; because 0, it is 0. So, nothing. So, if y is not equal to 0, if y is not equal to 0, then, let us consider, for any scalar alpha, for every scalar alpha, let us have norm of x minus alpha y whole square, yes, norm of x minus alpha y whole square. Now, this norm cannot be negative. So, it is always be greater than equal to 0. Now, this can be written as x minus alpha y comma x minus alpha y, which can be written as inner product x x minus alpha, minus alpha y.

So, alpha, I am taking out; minus alpha y comma x minus alpha y; x x minus, x, first is x and then, x with this also; so, plus x minus alpha y; then, minus alpha y with this, ok. So, again, x x; here, this alpha can be taken outside, we get y x; and, when this alpha is taken outside, we get conjugate; y y and here is, when alpha is taken outside, we get, clear. Now, this is equal to conjugate plus alpha bar I take outside, alpha y y minus x y, is it ok. Now, choose alpha, such that, this part is 0; because, alpha is our own choice; alpha is any scalar.

So, if I choose alpha this, means, alpha becomes x y over… So, from here, we get, 0 less than equal to minus x y y x over norm of y square, ok. So, norm y is non-negative. So, what we get, x y y x is less than or equal to norm x square norm y square, clear; but, what is this? This is the conjugate. So, we get from here is, modulus x into x y means, x y mod square; because this is the conjugate, is less than equal to norm x square norm. So, taking the square root, we get, we get the Schwarz inequality; that is it, clear. So, this.

Now, when this equality hold...

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For other values, alpha, you can choose accordingly. Suppose, I take alpha equal to, say, suppose, 100; then, I can pick up the alpha 1 as alpha y 100, ok, so that, it gets result. We, we have to pick up alpha in such a way, so that, this condition satisfied, clear. Because, this is, what I am taking is a particular case. For any alpha, this is true; this result is true, for any alpha. So, I can choose particular alpha, such, for which this is 0, is it not; that is possible. Because, that particular alpha also satisfy this condition. So, it will not violate the assumption, which we have started. Therefore, this. And, when the equality holds? If the equality holds, this must be 0. If the equality holds, then, this must be 0; this must be 0, means, x equal to alpha y. So, x and y are linearly dependent. So, this.

That is it. Thank you.