Functional Analysis

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Module No. # 01

Lecture No. # 22

Further Properties of Inner Product Spaces.

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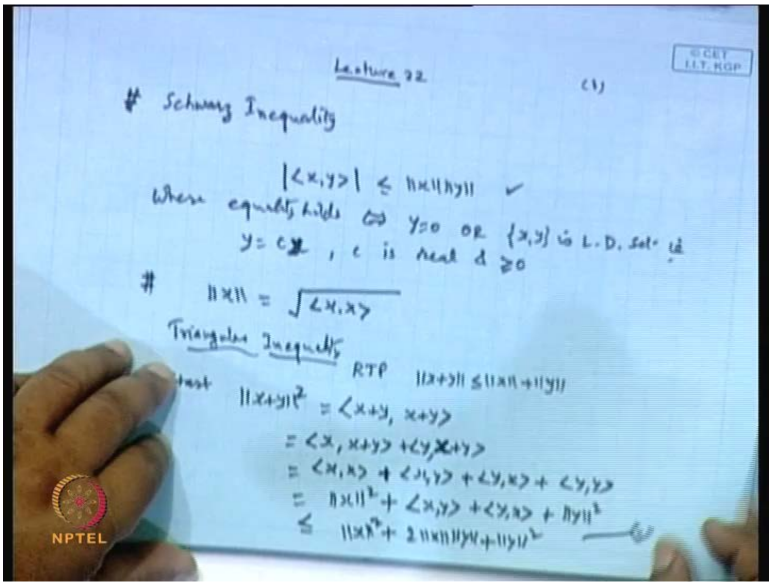

We have discussed this Schwarz inequality and this gives a relation between the inner product and norm; and we must have seen that, the modulus of the inner product x y is less then equal to norm of x into norm of y where, the equality holds, if and only if, either y is 0 or x is c y, or this set is linearly dependent set, is a linearly dependent set; that is, y is equal to c y, c y, where c is real and greater than 0.

Sir, c x.

c x, c x, where c is real and greater than 0, or equal to 0, at the most, clear. Now, this Schwarz inequality will be used to establish the relation or establish the triangular inequality, which will be helpful in justifying the norm. Can be written in terms of the inner product; and, in fact, when we have introduce the concept of the inner product, then, we also define this norm of x as the mod, inner product of x x under root of inner product of x x. Because this is a non negative quantity, greater than equal to 0; norm of x. And, we have seen that, this norm satisfy the, this definition satisfies all the conditions of the norm, except the one, which we have not verified, the triangular inequality.

So, now, once we have the Schwarz inequality, it will help you in establishing the triangular inequality also. So, let us see the triangular inequality, triangular inequality. We wanted to show that, norm of x plus y is less than equal to norm x plus norm y, is it not?. So, start with this. Norm of x plus y whole square and this can be put it as, x plus y x plus y under the inner product sign. Now, open it. So, we get from here is, x, x plus y plus y, y, x plus y, is it not?. This is and this further can be x written as x, x plus x, y inner product plus y, x plus y, y inner product; but this is norm of x square, by this definition, and this one, let it be as it is, x y plus y x plus, this is norm of y square, let it be 1, ok. Now, we know this. Schwarz inequality says that, modulus of the inner product x y is less than equal to norm x into norm y. So, this inner product x y will be less than equal to norm x norm y; this inner product y x will also be less than, because it is a conjugate of x y only, ok.

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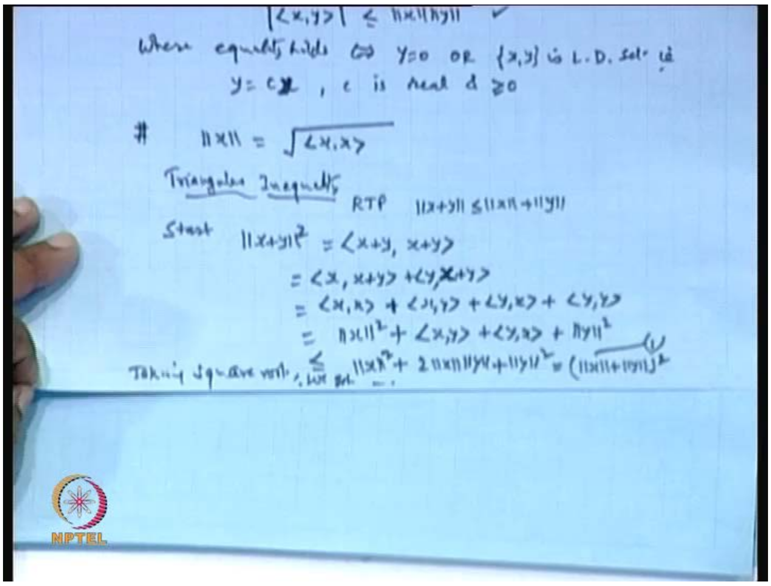

So, modulus of this, so, it will be further less than equal to norm of x whole square plus two times norm of x norm of y plus norm of y square and that will give… So, that will give you the norm of x plus norm of y whole square and taking the square root, we get the norm. So, taking square root, we get the triangular inequality estimates, we get the desired inequality. So, nothing to worry. Now, when this equality holds, what he say is that, result, if you put it this thing in the form, I am just writing as a result or lemma.

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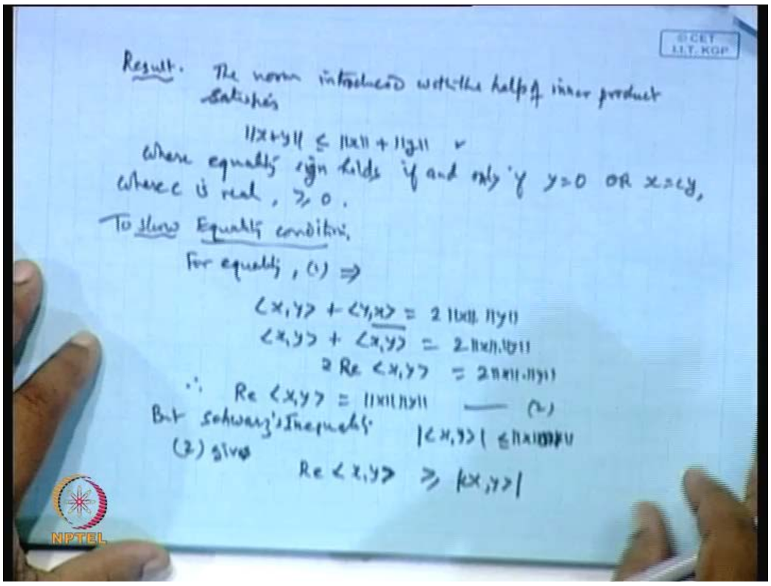

Result says, the norm, the norm introduced with the help of inner product, inner product satisfies this thing, norm of x is less than equal to norm x plus norm y, where the equality sign holds, sign holds, if and only if, either y is 0 or x equal to c y, where c is real and greater than equal to 0. So, we have established this result. Now, what we want is, to establish whether if the equality holds, these two conditions satisfy; obviously, from this side is, obviously, true; if y is 0, the equality is there; if x is equal to c y, then, we can write this 1 plus c x and here also, we can write 1 plus c, c is greater than 0 and the equality is satisfied. So, from here to here, is nothing to prove; only when this equality holds, you have to prove that, x and y are linearly dependent or x can be expressed as a c times, where c is real and greater than equal to 0.

So, to show the equality condition, that under what condition the equality sign shows, what we do, we have taking this 1, where the norm x square y square is equal to this; ((then, now)) from here, x by n by x, I have put it less than equal to this, is it not. So, from here only the problem is coming, with the sign less than equal to this, clear. So, let us see, ((they how do)), we have, for equality the 1 equation implies that, inner product of x y plus inner product of y x must be equal to two times norm of x into norm of y, in a state of less, is it not; but, is it not the same as x y plus conjugate of x y, which is two times norm x into norm y and this equal to two times real part of x y, which is equal to two times norm x into norm y. So, this shows, the real part of this inner product is the same as norm x into norm y, is it clear now. So, we get 2. But Schwarz's inequality says, Schwarz's inequality says that, modulus of the inner product x y is less than equal to norm of x into norm of y, is it not. Norm of y. So, we can put this thing in there.

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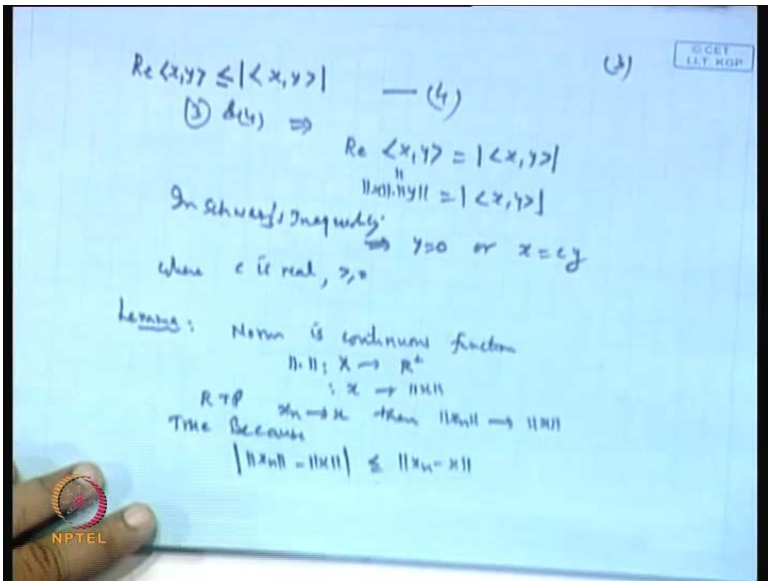

It means, therefore, 2 gives the real part of inner product x y is greater than or equal to the modulus of inner product x y, is it ok or not, modulus of inner the product x y. Because, this is, this norm of x plus y is greater than equal to modulus of inner product, clear. Now, in case of a complex inner product space, complex inner product, the modulus of the inner product x y, the modulus of the inner product x y, this will not be less than equal to the real part of this; rather than, the, this is always be greater than equal to real part of the inner product x y, clear.

So, and here, we are getting, in 3, we are getting the real part is greater than equal to this, while in this case, you are getting real part is less than equal to this. So, only third and fourth shows that, real part of x y must be equal to the modulus of inner product of x y, is it correct or not; only equality sign holds and when this equality sign holds, and what is this, this is nothing, but the norm of x into norm y, is it not; because, this will be real part of this, is nothing, but norm x norm y; so, which is the same as norm x into norm y equal to the modulus of inner product x y. So, in Schwarz's inequality, when the equality sign is there, then, either this y will be 0 or x is equal to c y, is it not, where c will be real and greater than 0. Even also, c can be proved and greater than 0; even c can be proved to be real and greater than 0, because, the once you are saying modulus of this inner product is real part of this, it means, this will be greater than non negative quantity, modulus.

So, it is greater than equal to 0. So, once it is greater than equal to 0, replace y x equal to c y. So, we can take c outside, and we get c to be greater than equal to 0, clear. So, this establish the triangular inequality, that this result…

Sir.

I think, sir, we can directly (( )) from the Schwarz's inequality, either less than equal to sign is coming (( )), from the Schwarz's inequality is (( )) norm is 0.

Yes.

And that…

Then you are getting equality sign…

But there, in the Schwarz inequality, x is equal to alpha y, where alpha may be a complex also; just a linear combination; x and y are linear combination, that is all; any scalar, clear. So, what here it is should be, real and greater than 0. So, in fact, this Schwarz's inequality, this will be equal to x y is linearly dependent, when you say x y is linearly dependent, this is so, and this part is not covered in the Schwarz, this portion.

(( ))

This portion, in fact, this we will take up in the, here; for equality we can say, because both side is nonnegative quantity, basically; but Schwarz, when, because it is a inner product, inner product, inner product scalar quantity; so, that is why, this x and y are linearly dependent, that is all; alpha may be any real number. So, we get, two. So, this completes your that.

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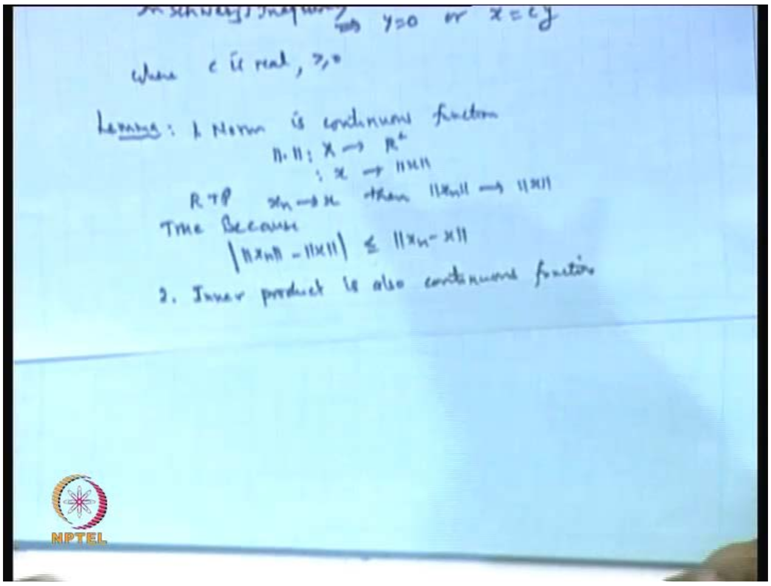

Now, there is another lemma, which shows, the inner product is a continuous function; just like a norm, we have seen that, norm is a continuous function, because we have seen that, this word, the norm, is it not, is a continuous function. How did you prove this thing? Remember, the norm is a mapping, is it not, from capital X to R plus, R greater than equal to 0, of course, such that, x goes to norm of x and this norm is a continuous means, if any sequence x n converges to x, then, norm of x n converges to norm of x n. So, what is required to prove is, when x n goes to x, then, corresponding norm of x n must go to the norm of x. And, this can be, then, this is true, because modulus of norm x n minus norm of x, this is less than equal to norm of x n minus x, is it not.

Now, if x n goes to 0, I can say this x 2 go to 0. So, it is coming. So, norm is a continuous function. Similarly, we can show that, inner product, inner product is also a continuous function; in fact, it is continuous with respect to both coordinates, jointly continuous sequences, whatever, continuous function.

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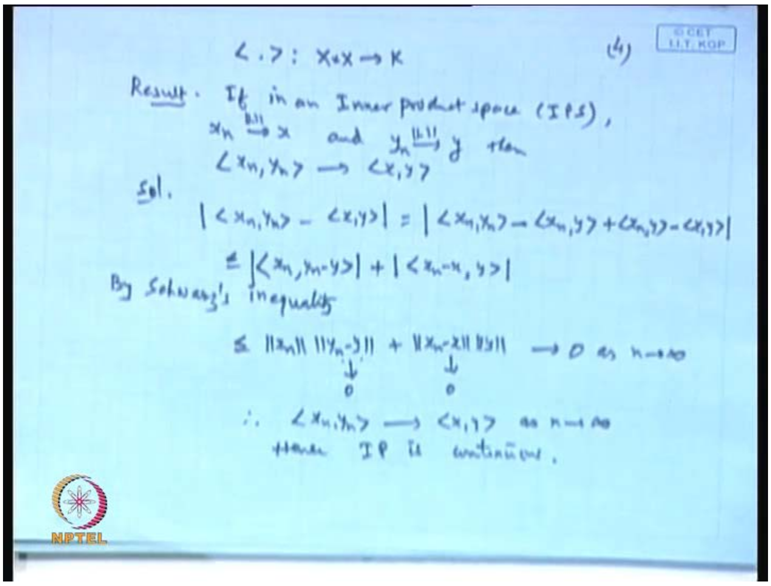

Why? The proof is like this. This we can say in the form of lemma. So, the result is like this. If an inner product space, if in an inner product space, inner product space, we write short IPS, if an inner product space, the sequence x n converges to x and y n converges to y, then, the inner product x n y n will go to the inner product x y. So, this shows, the inner product is a continuous function; because the inner product, the inner product, this is a mapping from x cross x to k, k is field of scalars of x, x is a vector space. So, picked up the two point x and x means, ordered pair x n y n and under the inner product, this gives the value of a scalar quantity k.

So, when this is a continuous function, we have to show the continuity means, pick up a sequence in the inner product x n y n type, and if it goes to x y, and under the modulus, because it is a scalar quantity, then, we say it is a continuous function. So, start with this, modulus x n y n minus inner product x y. We are giving the proof of this. Now, this can be written as modulus x n y n minus x n y plus x n y minus x y, is it not, clear. And, this will be equal to modulus x n y n minus y modulus plus, I am writing less than equal to, plus modulus x n minus x, y; combine these two, combine these two and we get…

Now, apply the Schwarz's inequality. By Schwarz's inequality, we get, this is less than equal to norm of x n into norm of y n minus y plus norm of x n minus x into norm of y. Now, this is given that, x n converges to x; y n converges to y. So, when we say, the x n converges to x, y n converges to y, it is automatically inherent that, it converges to a norm or to a metric, defined, introduced with the help of norm. So, this converges in the norm, which is, which gives or define in terms of the inner product. This converges in the norm, introduced with the help of the inner product. So, this goes to 0, y n minus y will go to 0; this x n minus x will go to 0, is it not.

Y n converges to y means, norm of y n minus y will go to 0; this will tends to 0; this will tends to 0. So, total it will go to 0, clear, as n tends to infinity. Therefore, this sequence x n y n converges to x y, as n tends to infinity; hence, inner product is continuous. Now, this gives an application. This lemma helps you in establishing that, every inner product space can be made complete; that is, if we take a Cauchy sequence in this inner product and if the Cauchy sequence will be convergent, then, we say it is complete space. So, with the help of this result, one can say that, every inner product space can be made complete; just like a completion; just like a normed space, we have seen, if x be a normed space, then, one can define the isomorphism and then, with the help of isomorphism and the isomorphism, the normed space can be made complete, is it not, and that will be unique. So, for the isometric fault is, means, with reference to the isometric only. So, similar results holds good, here.

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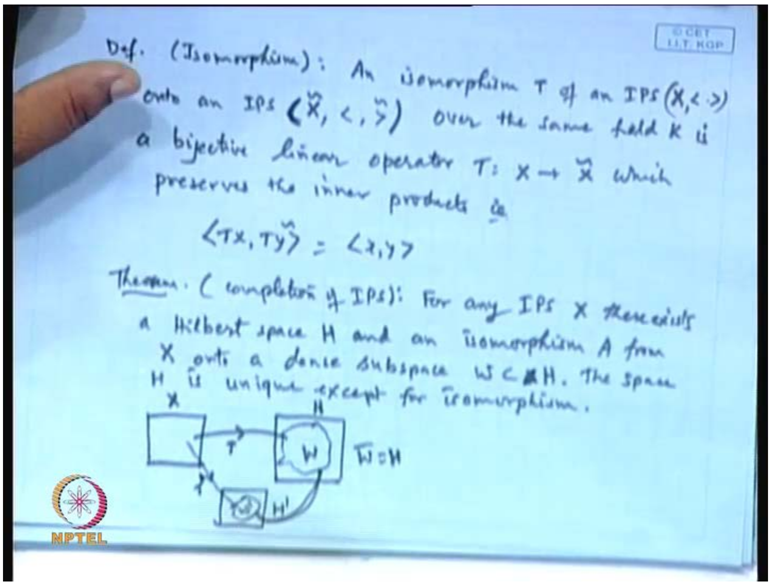

So, before going for the completion, we require the definition of isomorphism. So, isomorphism, we defined the isomorphism over inner product as follows; an isomorphism T of an inner product space capital X, onto an inner product space X delta;

I am putting the, taking the same inner product in both case; it may be different of course, but for the sake of simplicity, I am choosing same inner; otherwise, we can write this as inner product and this is as delta, both will be ok; or we can see this also and if you want, we can put this delta; this is our one inner product and this is another inner product, over the same, over the same field of the scalar, of scalar, over the same field K.

So, an isomorphism T of an inner product X onto inner product X delta, over the same field K, is a bijective, is a bijective linear operator T from X to X delta, which preserves the corresponding inner product, which preserves the inner products. That is, the meaning is, the inner product of T x T y under this inner product value will be the same as x y. Then, we say it is the isomorphism operator. So, one is the T must be 1 1, T must be 1 to 2, and it preserve the operation.

Now, when we say, it preserve the operator, this inner product, it automatically gives you the guarantee that, because of the linear part, bijective linear operator, I am taking T as a linear operator, it means that, addition and scalar multiplication which is defined on capital X is also preserved under T, ok. So, basically, the T which is an isomorphism from one inner product to another inner product, it preserves all the operations defined, algebraic structures, operations defined on X, whether it is addition, vector addition, scalar multiplication or inner product introduced on it. So, that way is, is it clear?

Now, this list gives you a theorem. Of course, I will not show the proof; it is a completion of the inner product space. The result is, for any inner product space, for any inner product space X, there exists a Hilbert space H and an isomorphism, A from capital X, on to a dense subspace W of X, W of H, sorry this is H and then, subspace W of H. The space H is unique, unique except for isomorphism. What is the meaning of this? What he says is, suppose, X is an inner product space; this is our X; this is not a complete inner product space; but what we can say is, we can always find a complete inner product space of X, say H, Hilbert space and an isomorphism T can be defined from X to a set X into W, such that, closure of W is H, is dense in H.

So, a one to one mapping can be introduced from X to W, which preserves all the operations of X and W. So, basically X and W are identical space; they behave as if they are carbon copy of each other; and once they have carbon copy, the closure of this is H. So, only the limiting points are the extra points, ok. So, that is why, we are, what we are saying is that, if X be a inner product of the space, one can identify a Hilbert space, corresponding to H, which has a subspace which is dense in H and has a one-one correspondence with, isomorphic with X; and this representation means, corresponding to X, we get a unique, H is unique, except for isomorphism; means, suppose, there is another, another H is, dash is obtained, which has the same property say W dash and then, isomorphism between this and this is also there; then, this W and W dash, they are isomorphic.

Sir, W and W dash are isomorphic?

They are isomorphic, ok.

So, that way, if there is no...

There will be separate T?

Yes, separate, separate T dash, clear, we get this. So, this is, except uniqueness, except this isomorphic, is it clear? So, this way.

Sir, is there any technique to find out (( )).

No, technique is there; we have to start with the sequence, Cauchy sequence and the same Cauchy sequence must behave here. So, that sequence is given and identify; that can be checked, ok. Now, this almost completes the concept. I do not know, whether last time I have given the concept of subspace or not; no. So, let us see the definition of the subspace, of the Hilbert space or inner product space.

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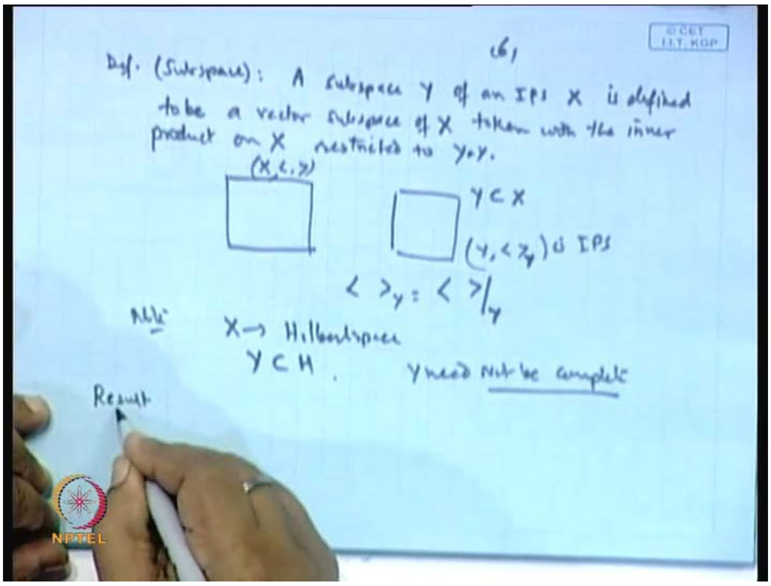

A subspace Y of an inner product space X, X is defined to be, to be a vector subspace, to be a vector subspace of X, taking with the, taking with the inner product, taking with the inner product on X restricted to Y cross Y, ok. What is the meaning is, suppose X, this is be a inner product space and Y be a subspace of this, this capital Y, which is a subset of X; we say Y is a subspace of this inner product space, if Y as a vector space, is a subspace of X, clear. And then, once it is a vector subspace, then, it should form the inner product also.

So, what the inner product is, that Y under this, say capital Y, is an inner product space, where, what is this, this Y is the restriction of this on of X on Y. This is the inner product on X; we are restricting on Y. So, only choosing the point of Y and same inner product, I am using. So, the restriction of this, gives you the corresponding inner product on Y. So, if Y is a vector subspace and it also forms a inner product space under the restriction on Y, the inner product of X on Y, then, we say by, with this new restricted inner product, is a vector, is a subspace of the inner product. Similarly, if suppose, X be a Hilbert space.

Sir, suppose X is a inner product space, and Y is a subspace, it must be a subset.

Yes, it is a subset.

So, that always the same type of inner product structure can be…

Yes, can be, yes, that is what, that is what this restriction is.

So, what are the requirement is that, it must be vector subspace?

Nahin, but sometimes, what it mean, we need not take the same inner product here. We can introduce some other inner product, but that must coincide with the restricted one; that must coincide with the restricted one. So, that is what. If X be a vector…In place of the inner product, if I replace X by H, if X is replaced by H, Hilbert space, then, we say Y is a subspace of Hilbert space, provided, all these conditions are satisfied; Y need not be a complete; here, Y need not be complete. So, for a subspace, the completeness of the Y is not required, just like a norm case we have. Then, we have some result and the proof runs on the same line, as we did in case of normed space. What is that further? Let Y be a subspace of a Hilbert space, Hilbert space H.

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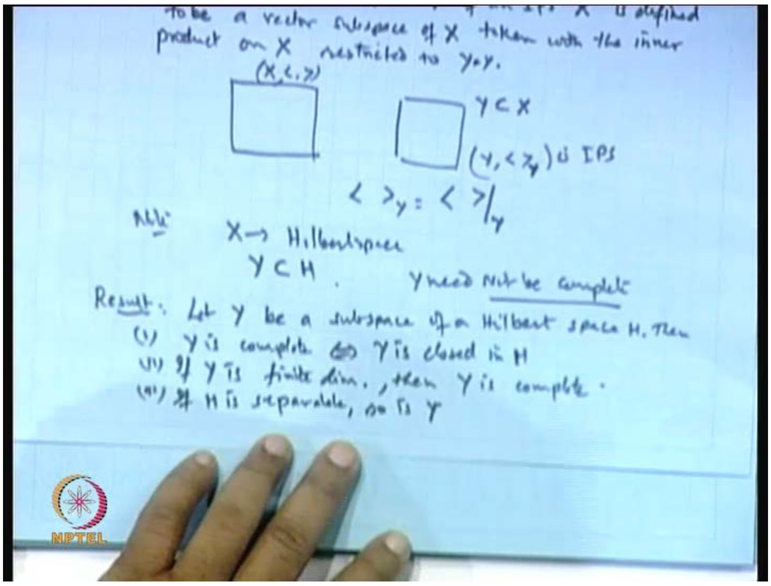

Then, the following results holds. Y is complete, if and only if, Y is closed in H. This was also result, in case of normed space, is it not. Y is closed in H and second, means, every closed subspace of a Banach space is Banach, is it not. Every closed subspace of a Banach space, is Banach. So, like this. Then, if Y is finite dimensional, finite dimensional, then also, Y is complete, is it not. And, third is, which can be proved directly, if H is separable space, so is Y; means, every subspace of a separable Hilbert space is separable. Separable means, yes, vector space is said to be separable, it has a countable subset, which is dense in itself. So, if H is given to be separable, it has a countable subset which is dense in. So, Y is a part of H. So, we can prove that, either Y is the same element or if it is not, then, we can identify the set, which is dense and also countable; that we did it in normed space, is it not. So, same result we continue here. So, is Y. So, these result follows, without any problem.

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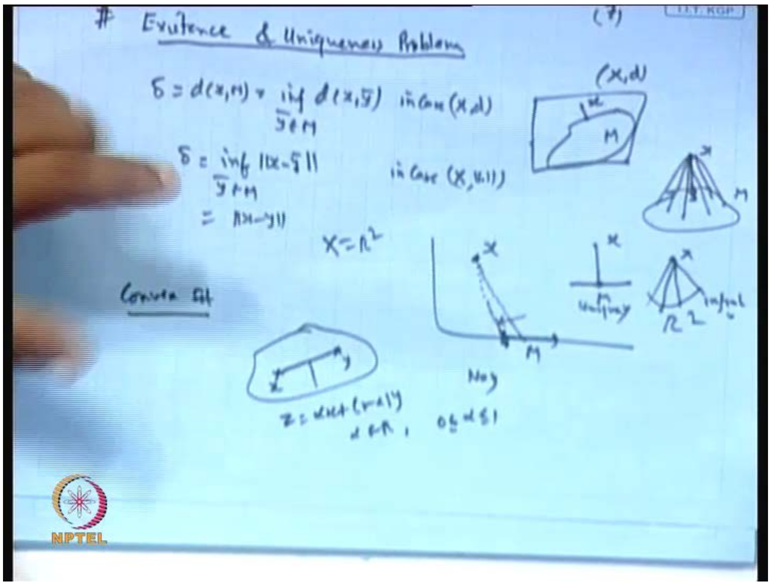

Now, next concept in this is, basically, the uniqueness, existence and uniqueness problem in the Hilbert space, existence and uniqueness problem. This is very important concept problem, particularly, used, when we go for the approximation part. Approximation theory, this is a very major thing; existence and uniqueness problem. What is the existence uniqueness problem? In case of, say normed space, in case of the metric space, suppose, x be a metric space; this is a metric space X d and let this be a M, which is a subset of X. If I take x here, and are interested to find the distance from x to M, then, how to find the distance d of x n, that is equal to infimum of d x y bar, where y bar belongs to M, is it not.

This distance, let it denoted by delta. So, infimum, if it is, in case of the normed space; in case of, sorry, metric space, x d. Now, in case of the normed space, the delta, we can say, it is infimum of the norm x minus y bar, where y bar belongs to M; just distance (( )) replaced by norm, because it is also length, ok. Now, the question arise, when you are taking the distances of x from various point of M, in fact, this is the point x and here is

M. What you are doing is, you are calculating the all sort of distances, is it not and then, among all distance, you are choosing the infimum value, clear. And then, you say, this is the distance, because that, smallest, among the greatest lower bound of all these distances will give the distance from x to M.

The question arise, whether this really infimum exist? Are you getting, some really a point here y, in the set M, where this exactly, this x to y distance, is the same as the infimum of this. If it is so, then, we say, this existence of the y is granted; then, whether you have a one y or more than one y, so, uniqueness y is also important, clear. So, when you are approximating the thing and when you are going for this thing, then, the uniqueness of the solution is important; when you are approximating something, uniqueness of the solution is important. Existence and uniqueness of the solution is important, when you are dealing with some physical problems; because if the solution is not unique, there is no use; and if the solution does not exist, there also we do not have any interest in such type of work.

So, that uniqueness in in in this, depends on the space which you are treating. In case of the metric space, this situation is very different; varies from case to case. For example, if I take x is equal to R 2 and suppose, I take this distance, point x here and this is our R 2 space. Let us picked up this M, say, this is our M; this line. Now, if I draw this, then, this will be the shortest distance. Now, if this is open interval, you cannot achieve this point; whatever the value you will get, it is only this way; that is all; but in order to go up to this point, the, we require the point at the end, but end point is not available. So, it means, no such y is available. Then, we are getting no y, is it not; because this is, or should I write like this further; if they are existed means, I will write x minus y; suppose, there exist y, which is equal to this. So, no y is available here. On the other hand, if picked up this thing; suppose, this is our M and here is x; then, draw the perpendicular. So, you get a unique y, is it not, and…

Sir, in earlier case, if you take the closed interval, then also…

Then, it is ok. But here, I am taking this one; different cases. And then, if I take this one, and then, point x is as center and then there are so many ys are available, is it not. So, infinite y. In fact, so, what we see here is that, in case of the metric space, the situation depends where the point is located, what is the set M, clear. We cannot give the guarantee the infimum will always exist; even if it exists, it may not give the guarantee, whether it is unique; same case will be the norm.

But in case of the inner product space, the situation is much more simpler and we will have a guarantee, if we choose M, a particular type of the sub space M, either it is a subspace or may be a convex space, which is complete; then, there will be guarantee that, always this infimum will be attained by a point inside the M; that is, the existence of y will be guaranteed, ok. So, that is why, this uniqueness and existing problem has importance and this space, inner product space or Hilbert space is widely used for this purpose, clear. So, let us see, using this, we require two concept, one is I think convex set.

Convex set, you know, a set is said to be convex, if we pick up any two point here, then, entire line segment must lie inside it; this is the line segment; the line segment z, will be of the form alpha x 1 minus alpha y, where x is this point, y is this point, alpha is a real number and lying between 0 and 1; 0 less than alpha less than equal to 1. So, if any two arbitrary point, if we picked up, then, the entire line segment joining these two point, if totally lies inside the set, then, we say, this set is a convex set. Subspace is always be a convex set. When we say vector subspace or subspace, then, this is a convex set. Why? Why it is convex? What is the property of vector subspace? How to prove the vector subspace? Any two point, linear combination must be available. So, I take the linear combination, alpha and beta; in place of beta, I take 1 minus alpha. So, that is why, every subspace is a convex, clear; like that. So, this will be used.

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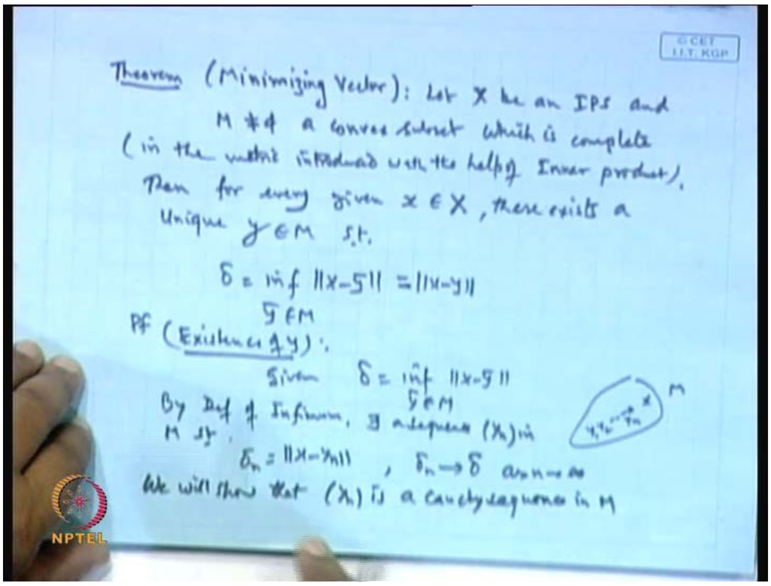

Now, we have (( )) results, which based on this uniqueness problem. The result is called the minimizing vector theorem, vector theorem. Let X be an inner product space, inner product space and M be a non-empty convex, non-empty, M be, and let M, a non-empty, convex subset, subset which is complete; complete in the metric, in the metric introduced or obtained with the help of inner product; with the help of inner product, you introduce the metric and then. Then, what this result say. Then, for every given x, belonging to capital X, there exists a unique y, there exist a unique y, belongs to capital M, such that, delta, which is the infimum of norm x minus y bar, y bar belongs to M, is equal to norm of x minus y.

It means, there is guarantee that, this infimum will be attained at some point, inside the M. So, uniqueness and also, unique is there. So, existence and uniqueness is guaranteed, if I start with a set M, which is a convex set and complete, in a inner product space. So, proof. So, first, we will show the existence of y, existence of y. Now, this is given; delta is the infimum of norm x minus y bar, y bar belongs to M. It means, there must be a sequence in M, y 1, y 2, y n, which goes to the infimum value, is it not. This is tending to, these are all sequence converging to x, so that, the infimum is coming to be y, is it correct or not.

So, we can say that, by the definition of infimum, there exist a sequence, infimum, there exist a sequence, there exist a sequence y n in M, such that, norm of x minus y n, say, is equal to, say, delta n, goes to delta n, goes to delta, as n tends to infinity; delta n which is goes to 0, as n tends to infinity, is it correct or not. Because, this infimum is delta. So, there must be a sequence y n, so that, the difference x minus y n, that is, the norm of this delta n, will go to delta, as n tends to infinity. Now, we want to show, we will show that, this sequence y n is a Cauchy sequence, is a Cauchy sequence in M. Once it is Cauchy in an M, M is already complete. So, it must be convergent. So, we get a point y, belongs to M and existence of y will be guaranteed. That is the, our procedure. So, how to show the y n is Cauchy?

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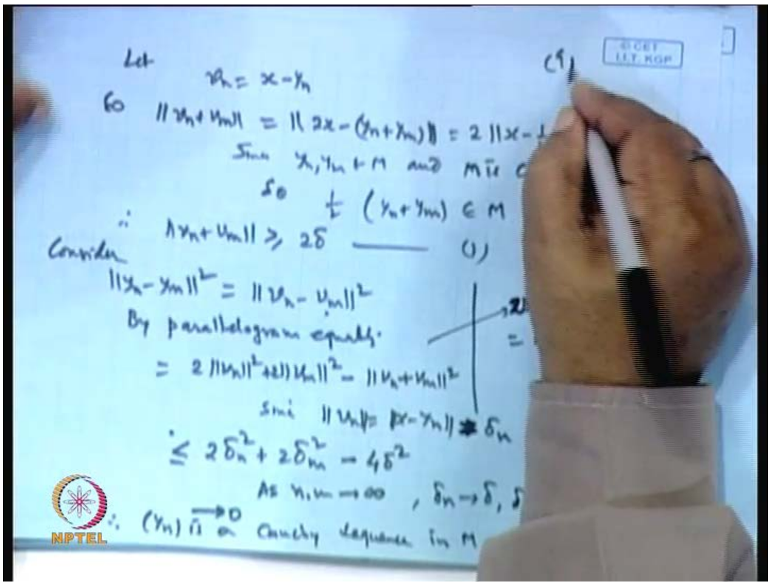

So, let us put it, let this term x n minus y n is v n, x minus y n, let it be v n. So, we get, norm of v n plus v n, what is this? This is equal to, this plus this, so, two times x minus y n plus y m, is it ok or not. So, take 2 outside, we get x minus half y n plus y m. Now, this will be delta, infimum of this thing is delta; it means, if I remove infimum, then, whatever the y belongs to M, the norm of x minus y, will be greater than equal to delta. Now, here, y n and y m, both are the points in M; and M is, since y n, y m belongs to M and M is convex, M is convex set.

So, half of y n plus y m is also point of M. Once it is point in M, therefore, norm of v n plus v m, this will be greater than equal to 2 delta, agreed or not. Is it correct? Clear. Let it be 1. Now, consider, norm of y n minus y m, because we want this to be Cauchy. So, norm of y n minus y m goes to 0, when n m is sufficiently large, then, it is Cauchy. So, let us consider this square, but y n minus y m, what is this? Is it not the same as v n minus v m whole square, v n minus v m. So, what you get, basically, you are getting y n minus y m. Now, every inner product and the corresponding norm, satisfy the parallelogram law, ok.

So, by parallelogram law, or parallelogram equality, what we get is, what is the parallelogram, do you remember? That is, that is parallelogram law is, norm of x, 2 times norm x square plus 2 times norm y square equal to norm of x plus y whole square norm of x minus y square, this is the parallelogram; write it. So, using this, this one is given, this is given. So, everything, you are writing here. So, this will be equal to two times v n square two times v m square and then, minus sum of this, square, is it ok or not.

Now, norm of v n, what is this norm of v n? Since v n is equal to x minus y n, so, norm of v n equal to norm of this; y n is point in what, M. So, this will be greater than equal to delta; similarly y m is also greater than equal to delta. So, we are getting, this is, this y n minus y, yes. So, this will be, no, sorry, this is delta n; I am taking delta n; sorry, let it be delta n, because, this one is (( )), x minus y n is delta n. So, here x minus y n, which is v n, is delta n. So, using this, we are taking ,this is 2 delta n square 2 delta m square and then, what is this, v n plus, v n plus v m, what is the v n plus v m? This, from 1, is greater than equal to 2 delta. So, minus of this, is less than equal to. So, we add the less than equal to sign, minus 2 delta, means, minus 4 delta square, is it ok. Now, as n m goes to infinity, this delta n goes to delta; delta m goes to delta. So, entire thing will go to 0, exactly, ok.

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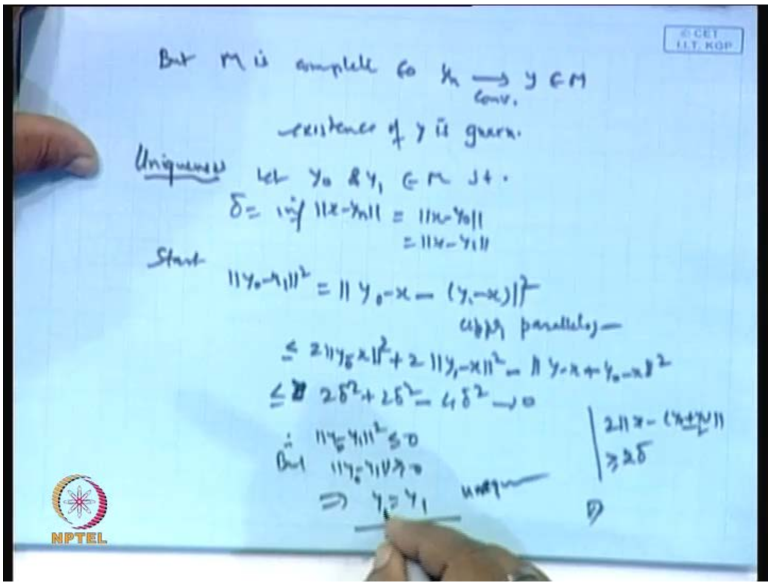

Once entire thing goes to 0, therefore, this sequence y n is a Cauchy sequence in M; but M is complete, what, M is complete. So, every Cauchy sequence is convergent. So, this sequence y n is convergent. So, y n converges to an element y, belongs to M. Therefore, existence of y is guarantee, guaranteed. Now, uniqueness of y. Suppose, there are two values. Let y naught and y 1 are the point in M, such that, delta, which is the infimum of norm x minus y n, is equal to norm of x minus y naught and this is also same as norm of x minus y 1.

Suppose, there are two points y 1 and y 2, where the unique limit exists; infimum exists. So, if it is unique, then, y naught should not be equal to y, y naught should be equal to y 1. So, start with norm of y naught minus y 1 whole square, and this can be written as y naught minus x minus y 1 minus x square and again, use, apply parallelogram law, ok.

Once you apply the parallelogram law, you will see that, this will come out to be, y y naught minus x whole square y 1 minus x whole square minus y minus x plus y naught minus x whole square, and this will be, this is delta. So, this will be 2 delta square; 2 delta square and this part, because y x can be taken outside, 2 can be taken outside; so, this greater than equal to delta. So, we get, this is less then equal to, sorry, this part greater than, yes, this will be belongs to this; so, it is greater than; so, this will be less than equal to, this will, now, what is this?

If you take 2 outside, you are getting x minus y naught plus y 1 by 2. Now, this is the point of y. So, it will be greater than equal to 2 delta. So, it is less than equal to minus 4 delta square and this will go to 0. Therefore, norm y minus y 1 whole square is less than 0, but norm cannot be negative. So, implies y equals to y 1; and, this shows the uniqueness; that is all. Thank you.

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y naught equal to y 1, yes, thank you.