Functional Analysis

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Module No. # 01

Lecture No. # 24

Representation of Functionals on a Hilbert Spaces.

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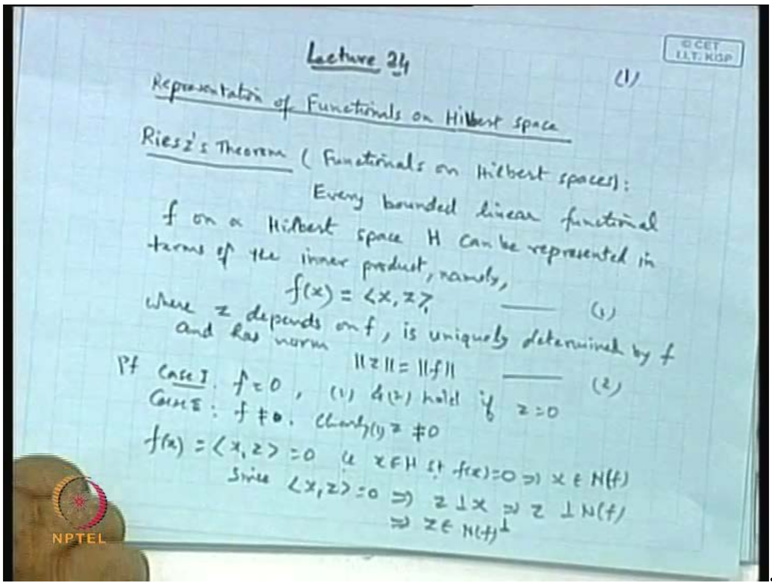

Representation of functional could be bounded linear functional, sesquilinear functional on the Hilbert space. So, normally the practice is, whenever we develop a structure, whether it is a vector space, normed space, Banach space or Hilbert space, we study what sort of the functionals are there on these spaces; their representation and the corresponding norm, if it is bounded and so on. In case of the Banach space, we have seen that, as soon as we change the space, the corresponding form or representation varies, changes; and also, the representation, you cannot say, it is unique, in case of the Banach space. Say, for example, c naught, the dual of this is l 1; l 1 dual is l infinity; representation is in the form of the series; but when you go for the c a b, set of all continuous function defined over a b, then, the representation not is in the form of the series, it is in the form of the integral. So, this is a very complicated case, when we deal with a Banach space; means, you have to identify, what is the representation, you have to identify the space, which behaves as a dual space for this unlike.

But in case of the Hilbert space, the situation is much more simpler. This, in fact, we will see here that, whatever the bounded linear functional be there, it will always have same representation, in the form of the inner product. Now, if the inner product changes, corresponding value will change; that is all. But the representation can be expressed in terms of the inner product and this result is given by the Riesz. So, we start with the Riesz theorem and this theorem says, which gives the functional, in fact, it is a theorem on bounded linear functionals on Hilbert space, on Hilbert spaces. What this result say, every bounded linear functional, functional f on a Hilbert space H, on a Hilbert space H, can be represented in terms of the inner product, inner product, namely f x, f is a bounded linear functional, x is a point in the Hilbert space; f x can be represented in the form of the inner product x z, where, where, z which depends on f, is uniquely determined by f and has norm equal to norm of f.

So, what this result says is that, if H be a Hilbert space and f be a bounded linear functional on the Hilbert spaces, then, we can always express this f x in the form of the inner product and this inner product is basically, the inner product define of H. This is the inner product taken over, of course, H. So, it will be and this z, it will depend on f, uniquely determined by f and the norm of z and norm of f will be identical. So, in fact, we wanted to prove three things; one is, every linear functional has a representation, 1; second one, the z which you are getting, is uniquely determined, depend on it, uniquely determined and third is norm of z equal to norm of f. So, let us see the proof. Case one.

Now, if our bounded linear functional f is a null operator or null functional, then, obviously, this 1 and 2 holds, hold if z is 0; because if f is 0, this f x will be 0, and here, z is 0, then, it will be inner product 0, and z is 0, so. So, basically, this holds if one, if z is 0. So, nothing to prove here.

So, if, let us take the case when f is non equal to 0, not 0, means, it is a non-zero bounded linear functional on H. Now, if f is non-zero, and representation f x we wanted into this one, so, let us see, what are the property of z, that can be enjoyed or that can be enjoyed by z, so that, the 1 and 2 retains; what property of z enjoys in order to get 1 and 2. So, clearly, z cannot be 0; why, because if suppose, z is 0, then, inner product will be 0.

Now, inner product is 0 means, this is true for any x. So, f of x 0 means, for every x, so, it is only possible, when f is 0; that will contradicts our assumption, f is non-zero. Therefore, z cannot be 0, clear. z will, sorry, z will not be a 0; z, clearly z will be always be non-zero; it cannot be, it cannot be 0, clear.

So, once z is not 0,then, second part is, if the inner product x z is 0; it means, inner product x z means, 0 means, f x is 0. So, those point, that is, the point x belongs to H such that, f x is 0 for that particular z. So, this x will be a point of the null space of f, ok. So, that z, which, where it is 0, we get the null space of f. Now, this null space, x is in null space. The inner product of x z is 0; if the inner product of the two nonzero vector is 0, then the possibility, the only possibility is, there must be orthogonal to x. So, this shows that, z is perpendicular to x. It means, z must be orthogonal to N f, is it not; or, z will be the point of n f perpendicular.

So, this shows that, z will be a point of N f perpendicular. I am using this symbol as N f perpendicular to be this, this orthogonal complement of N f. So, we get. So, what we, basically, is dealing is that, when we choose f to be nonzero, then, the z is such that, it cannot be 0; it, it can, it will always be takes as a non-zero term; it cannot be 0. Second one is that, for non-zero z, we are breaking the entire H into the two subclasses; one is N f; another one is the N f complement. So, both the things, null space as well as this orthogonal complement will be available for our use, is it correct.

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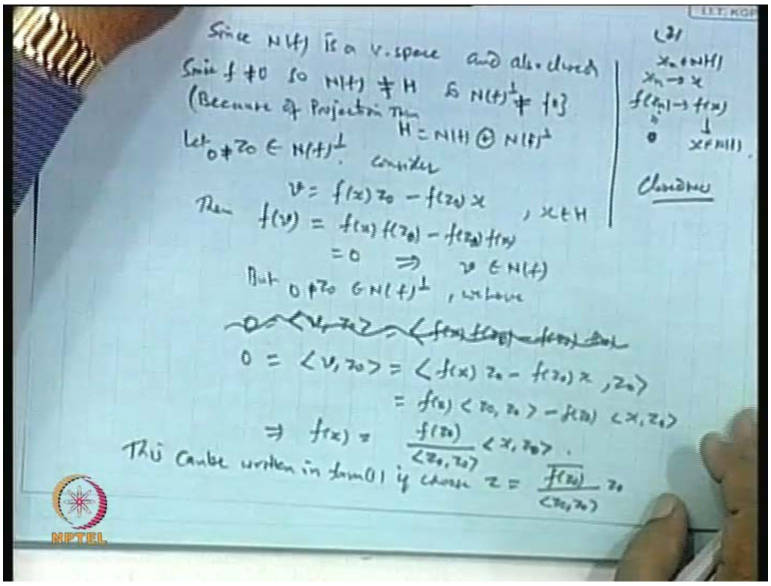

Now, one, another point, which we have, since N f, which is a null space is a vector space, is it not, and also closed, and also closed closed, because if x n converges to, closeness follows like this; if x n converges to x, and f is a linear functional or bounded linear functional, then, f of x n will go to f x; but x n is an element in the null space N f. So, this part will be 0. So, for each n 1 2 3 we are getting basically, sequence, null sequence, 0 0 0 0, and that converge to f x. So, f x must be a point in null space, clear. So, closeness follows like this. So, closeness, ok. So, N f is closed as well as null space and f is not equal to 0. So, the null space of this N f will be different from H; why, N f will not be equal to H; that is the entire Hilbert space will not be a null space; why, because…

Because it (( )).

Yes, because if N f is the entire Hilbert space, it means, the set of those point, where f x is 0. So, f x is 0 for all x. So, f must be 0, which contradicts. So, f, if f is nonzero, then, null space will be a subset of H. We have already got the possibility of getting the orthogonal complement of N f, clear. Therefore, by projection theorem, we can express Hilbert space H as a direct sum of the y and y orthogonal T. So, we get from here. So, what we get is that, N f orthogonal complement. So, N f orthogonal complement will be different from the singleton set 0. Why, because of, because of projection theorem; because of the projection theorem, is it not. The reason is, because if we take H a Hilbert space, this can be expressed as a direct sum of N f and N f orthogonal.

Now, N f is not equal to H, clear; that, it means, every element of x uniquely represented by the sum of this two. Now, if this is 0, if this is 0, then, H becomes N f, only without…These are intersection set and we get only singleton set 0. So, or no points common should be there. Therefore, which contradicts again, the thing. Therefore, this cannot be a null space; means, it cannot contain only the 0; null space, it will be a something called nonempty set, will be available. So, let z naught be a point in the orthogonal complement of this, which is not 0, agreed. Clear? Means, they are all the points, which are different from 0 in this. Now, let us consider, now, consider this term v as f x z naught minus f z naught x; just I am considering; because f is given; z naught, already we have got it; x is a point in H. So, everything is known. So, we can construct this v; then, what is the value of this v under f.

We see that, f of v becomes f x f z naught minus f z naught f x, is it not 0? Because, f x is a some scalar, f is a linear functional. So, f of x will be scalar. So, f is a linear functional, bounded linear. So, it will be operated on this, sum of or difference on this and we get f z naught; and f z naught, again a scalar, can be taken outside; f x and this comes out to be 0. This shows that, our v belongs to the null space of f, clear; but z naught, which is not 0, is a point of the orthogonal complement of null space. So, we get the inner product of v and z naught is 0; but v is f x f z naught minus f z naught, sorry, this is, I am sorry, this is, this is f of, is it not. So, 0 equal to v z naught; substitute the value of v; f x z naught minus f z naught x comma z naught, is it ok. Open it.

So, once you open this, then, what we get is, f x can be taken outside; z naught, z naught inner product; minus f z naught can be taken outside; inner product of x z naught. From here, we get the value of f x as, this comes here; f z naught divided by this into, f x divided by into inner product of x z naught, is it ok or not. So, we are getting this. Now, what, our aim was to represent the linear functional f into this point. I claim that, we have got this representation in this form; why?

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If we take, this gives, this can be written in the form, or this can be written, written in the form one 1, if we choose z to be f z naught conjugate over inner product z naught z naught into z naught. What he says is, for a given f, we want z. So, if f is given, if we choose z in this form, and substitute it here, is substituted here; then, this f x coincide with 1; you just see, clear. So, you have substituted this thing in 1; what you get, f x, in place of z you write this part; so, you are getting, this thing is, if I write this is f x, which is equal to x z. So, start with this, x comma z, this is equal to x, z, you write like this; f z naught f z naught conjugate over z naught z naught and z naught, is it ok or not.

So, this will be equal to, take this thing outside. So, this is taken from the second coordinate. So, conjugate of conjugate will becomes original, is it not; and then, inner product of x z naught; is it not this one, which coincide with our 3, 3. So, we have proved that, corresponding to each f, f is a bounded linear functional, we can find a z, depend on f and f x can be expressed into this form, the inner product of x z, is it ok.

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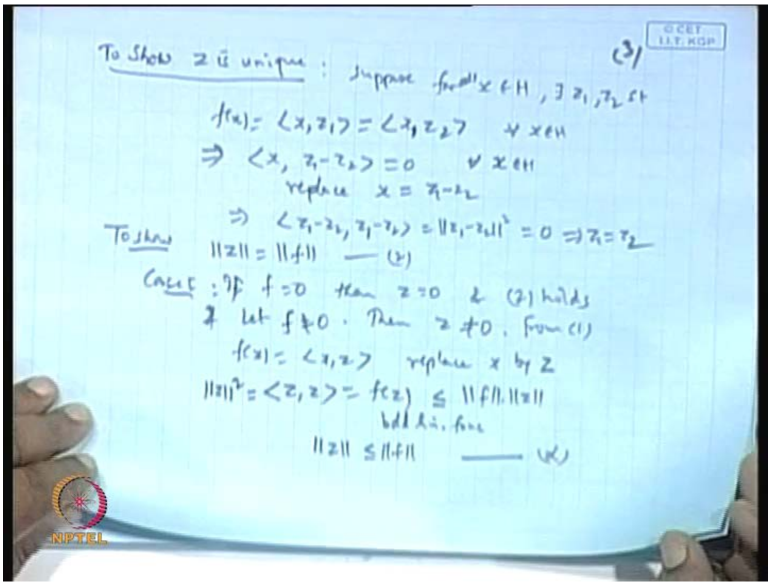

Now, second part which we want to show that, this z which you are getting, is unique. So, second is to show the uniqueness, to show z is unique. Suppose, we do not have the uniqueness; suppose, there are two z 1and z 2; suppose z 1and z 2 are two available, such that, for x belongs to H, there exists z 1 and z 2, such that, f x has the representation as inner product of x z 1, which is the same as the inner product of x z 2, agreed. Now, this can be written as x comma z 1 minus z 2 equal to 0 and this representation we want for all x, for all x. So, this is true for all x. So, let us write, replace x by z 1minus z 2, we can choose. So, this becomes, inner product of z 1minus z 2 z 1 minus z 2, that is equal to norm of z 1minus z 2 whole square, which is 0; but norm of anything is 0 means, that point must be 0. So, this implies that, z 1equal to z 2. So, uniqueness follows.

Now, third case, to show norm of z equal to norm of f. Yes, something wrong? Norm of z equal to norm of f. So, again, we will take two cases; if f is 0, then, z is also 0 and 2 holds, and 2 holds; this is 2, 2 holds. So, if f is not equal to 0, let f is not equal to 0; then, z is also not equal to 0; otherwise contradiction that, we think. Now, from this one, we get, from 1, that is, the f x is represented in the form of x z, replace x by z. So, what we get it, inner product z z, which is equal to f z, but f is bounded linear functional, this is a bounded linear functional, so, it will be less than equal to norm of f into norm of z, by definition. Now, divide by norm, but this is equal to norm z square. So, we get, this one is norm z square. So, we get norm of z is less than equal to norm of f. Let it be alpha, clear. Now, to get the converse inequality, consider mod of f x. This is equal to mod of inner product of x z; apply Schwarz inequality.

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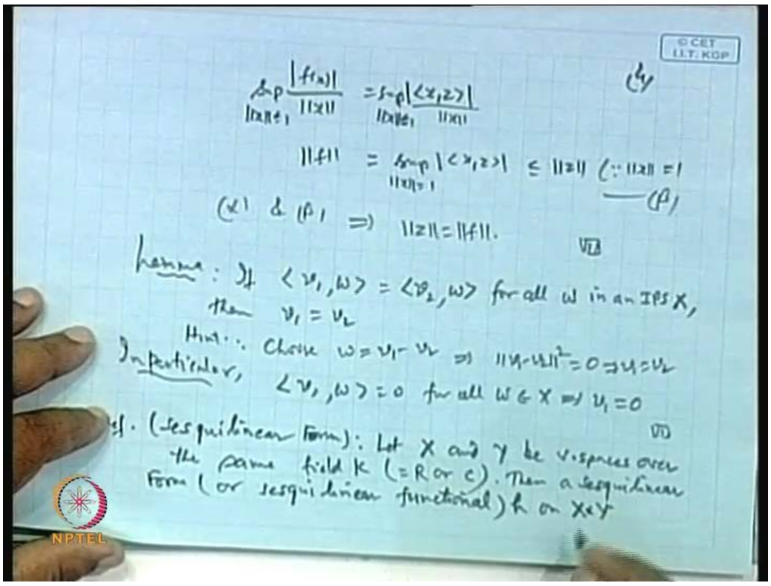

Schwarz inequality gives the relation between the inner product and the norm. So, it will be less than equal to norm x norm z; divide by norm x. So, we get, from here is that, if I divide by norm x, take the supremum over all x, such that, norm x is 1; then, this is equal to modulus of inner product x z; take the supremum over all x, where the norm x is 1. So, divide by x, that is why it is not norm x is 1; and then, in fact, let us take less than equal to 1, then basically, it is 1. So, this becomes norm of x. So, this is equal to norm x equal to 1. So, we are getting, this is supremum of this will give the norm f and this will give the supremum of inner product of x z, where the norm x equal to 1, is it ok.

But this part is less than equal to what; this is less than equal to norm x into norm z. So, norm x is 1. So, we get norm z; because norm x is 1. Modulus inner product is less than equal to norm x norm z; you are taking supremum over norm x equal to 1. So, this is … Therefore, norm x is less than this. So, let it be beta. So, alpha and beta gives the norm of z equal to norm of f; and that completes the result; it is proof. Now, as a corollary of this, or corollary of the result which you are using, we can say as a lemma; this lemma will be also used. If proof is, already we have done it, but just I want to state that lemma. If inner product of v 1 w is the same as the inner product of v 2 w, for all w, all w in an inner product space, inner product space X, then, v 1 equal to v 2. This is already done, earlier, is it not? Take the v 1 minus v 2 w and w replaced by v 1 minus v 2. So, norm v 1minus v 2 whole square, you are getting v 1, is it not. Clear? Hint is, replace, choose w to be v 1minus v 2, and then, this thing will, can be written as. So, entire thing can becomes v 1 minus v 2 whole square is 0, is it right, and that implies v 1 equal to v 2, clear.

So, this will, lemma, we will make use of that. And, in particular, in particular, we can say, if v 1, inner product of v 1 w is 0 for all w belongs to X, then, implies v 1 must be 0, clear; that is the same way we can prove. Now. Now, so far, this we have taken, but as you know, the inner product, it has the set, vector space X together with an inner product defined only is the inner product space; and inner product is a sesquilinear form, because it has a two coordinate; one, with respect to the first coordinate, it is linear; with respect to the second coordinate, it is conjugate linear. So, we say, it is a sesquilinear form.

Now, this sesquilinear form, that is a example, particular example we have taken as a inner product and we call it as a one and half times linear. But in general, the sesquilinear forms are also play, they also play a vital role over the Hilbert space; so, in particular, when we go for the Hilbert adjoint operators. Then, the representation of these things will be given in terms of the sesquilinear forms. So, in order to the study the Hilbert adjoint operator, we require the concept of the sesquilinear form and its representation of over the Hilbert space.

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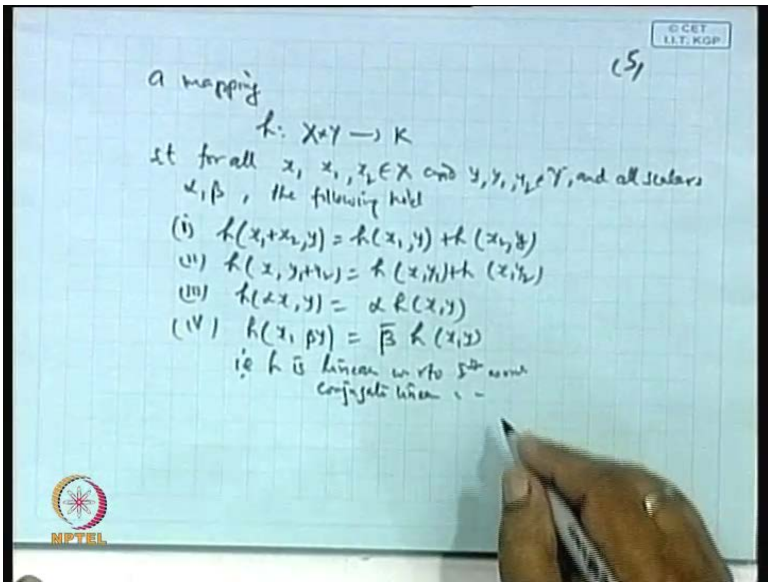

So, let us see, what is the sesquilinear form. Now, one more thing is, which I will explain, let X and Y be vector spaces over the same field, over the same field K, which is either R or C. Then, a sesquilinear form, then, a sesquilinear form or sesquilinear functional, functional h on X cross Y, on X cross Y is a mapping, is a mapping from X cross Y to K, such that, for all x, x 1, x 2 belongs to capital X and y, y 1, y 2 belongs to capital Y and all scalars alpha, beta, the following condition holds, the following hold.

First is, h of x 1 plus x 2, y equal to h of x 1, y plus h of x 2, y; that is, it is linear with respect to the first coordinate. Second, h of x y 1plus y 2 is h of x y 1 plus h of x y 2; there is no constant, constant is 1 1; but if the constant is alpha, say h alpha x, y, then, we say, alpha outside h of x y; and if there is a beta in multiply by y, then, we get beta conjugate h of x y, beta conjugate h. Now, this beta, which is complex, is here, then, we simply say, beta bar equal to beta and so on; in that case, h x y equal to h of y x or something, ok. So, this properties are satisfy, it means H is, that is, H is linear with respect to first coordinate, first argument and with respect to first coordinate, and conjugate linear with respect to the second coordinate; and it is called the sesquilinear form; that is why is it ok.

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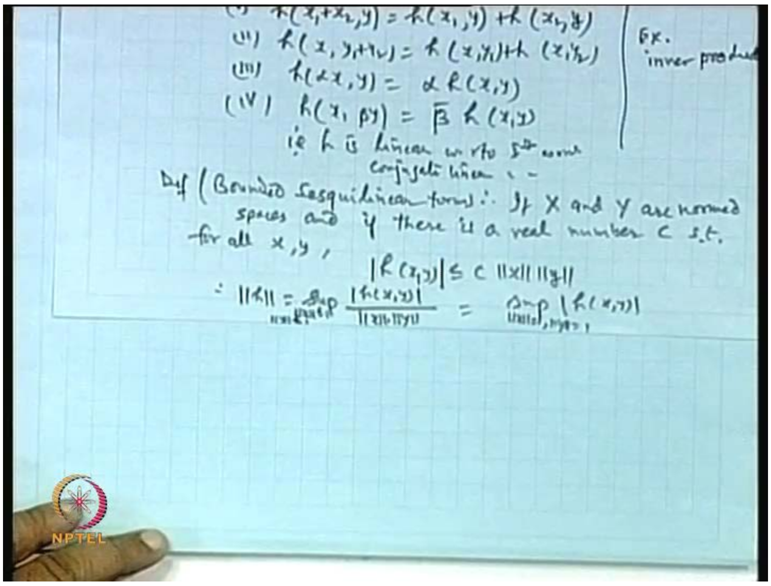

And then, norm of h, in a similar way, we define, a bounded sesquilinear form means, the examples of this is the inner product, example is the inner product . So, that will be (( )). Bounded sesquilinear form, if X and Y are replaced by a normed space, then, we can say, if X and Y are normed or a Banach space, normed spaces and if there is, there is a real number C, such that, for all x y, we have modulus of x y, absolute value of this is less than equal to C times norm of x into norm of y; then, we say, x is a bounded sesquilinear form and from here, we can write it. So, we can write, norm of h as the supremum value of modulus h x y over norm of x into norm of y, where the supremum is taken over all x, such that, norm x is 1 and norm y is 1, and norm y is 1 ok.

So, supremum is taken over norm x is 1and norm y is 1; that is the supremum. So, basically, we can side, less than equal to 1 x minus this, x minus this; so, it is equal to supremum modulus of h x y, when the norm x equal to 1, norm y is equal to 1; that is the norm of this, is called the…According to the previous, just it goes, as if it is the previous one. And, combining this, we can say, this is called the norm of h and combine this, so, we get, modulus of x y is less than equal to norm x norm x, norm h into norm x, y and this condition satisfied, say here is 3 or 4; what 6; something say gamma or theta, something number, you just write anything.

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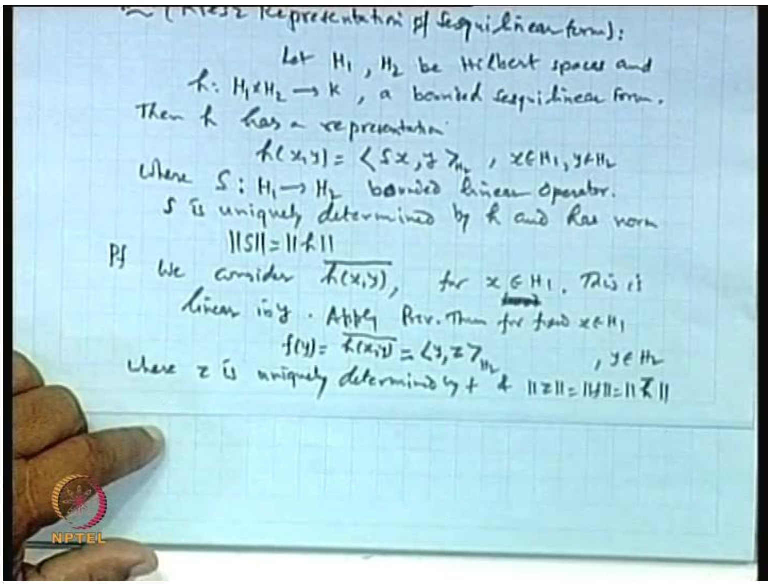

This, we say, H is a bounded linear sesquilinear form, if it condition is satisfy the mod of h is less than equal to C times norm, c becomes the minimum value, infimum value is c, is norm h. Now, since h is defined over H X cross Y, the dominant is X cross Y. So, earlier the domain was only a single variable, say H, where element U x belongs to H; it is a Cartesian product. So, instead of saying the functional, we use to prefer right form;

otherwise, we can also say, it is a bounded linear functional on a sesquilinear functional; that also true; but because, to differentiate between the bounded linear functional defined on a space, which has a single variable and the space which has a Cartesian product, double variable, we call it form from the second one and functional for the first one. So, this is our notations. ((let me)).

Now, the Riesz has also given the representation of this sesquilinear functional; just like he has given the linear functional, bounded linear functional representation in the form of inner product, similarly, he has also given another result, which gives how this bounded linear functional behaves over a Hilbert space. So, again, we get the theorem, which is given by the Riesz, representation of sesquilinear form, sesquilinear form, ok. So, what is this say. Let H 1 and H 2 be Hilbert spaces, let H 1, H 2 be Hilbert spaces, and h is a mapping from H 1 cross H 2 to K, a bounded sesquilinear form, a bounded sesquilinear form. So, H is given to be a bounded sesquilinear from H 1 cross H 2; instead of taking just inner product, here it is set in as a, both to be a Hilbert space; H 1 is also Hilbert space; H 2 is also a Hilbert space.

Then, what this result says is, then, h has a representation h of x y as in the form of the inner product, as s x comma y, s x comma y, where what is s? s is an bounded linear operator from H 1to H 2, a bounded linear operator from H 1 to H 2. Now, this s is uniquely determined by h and has the same norm as the norm of h. So, basically, this result is the parallel result, but only the difference is, here h is considered to be a sesquilinear form; there, h, h was the bounded linear functional f. So, what he says is that, every sesquilinear form on H 1cross H 2 will always be represented in the form of the inner product.

But inner product is on H 2; this, you remember that, you see, y is a point in H 2; here remember, x is in H 1; y is in H 2. So, this inner product is basically, the inner product of H 2. And, what is s? s should be a mapping from H 1 to H 2. So, if x is a point in H 1, the H of s of x will be a point of H 2. So, it is a well defined thing, is it correct. So, we have to prove two things; one is that, first, this representation s will be of this, this form; second one, s, which you are getting, should be a bounded linear operator from H1 to H 2 and third is, both are having the same norm. So, first is, in order to prove the h has a representation this, we consider…

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Uniqueness follows already, yes, uniqueness will, yes, it will be, but it will be follow from the previous, because we will make use of this; s is unique, yes; that is also, uniqueness is also used. So, another point s should be uniquely determined by H. So, we consider the h x y conjugate, let us consider this, ok. Now, here, if I fix x, fix x, or x belongs to H 1 fixed; suppose, I fix x. What happen this? This will behave as a function of y; but because of the conjugation, because of the conjugation, it is linear with respect to y, is it not; because x is a conjugate linearly with respect to second coordinate, but we are taking conjugate of it. So, that way, second, with respect to second coordinate, this entire thing becomes linear. So, we can say it is linear. This, consider the, this is linear, this is linear in y, is it ok or not? Ok.

Then, apply previous theorem, apply previous theorem for fixed x. So, for this fixed, you can write that, for fixed x belongs to H 1. So, once you are applying previous theorem, then, this will behave as a function of y, is it ok; because x will not play role. Now, any bounded linear functional f defined on H 2, f is a point in H 2. So, f y, sorry, y is a point in H 2. So, f will be bounded linear functional on H 2. So, any Hilbert space, if it is defined, f, bounded linear functional will be represented in the form of the inner product of that. So, this will be given in terms of the inner product of H 2 and this inner product will be in the form of y, z; because here y is variable. So, corresponding to f, we can find z, such that, f y can be represented into this form, where the z is uniquely determined, uniquely determined by f and both are having the same norm, as the norm of y, that is norm of h bar; but norm of h bar is h, is it clear or not; is uniquely determined and norm of this (( )), clear. Now, this y z, if we look this part and our representation this. So, what is our this, h y x? Hence, we can say, the h x y, the represented h x y is coming in the form of z y; let it be this equation, say double i, ok.

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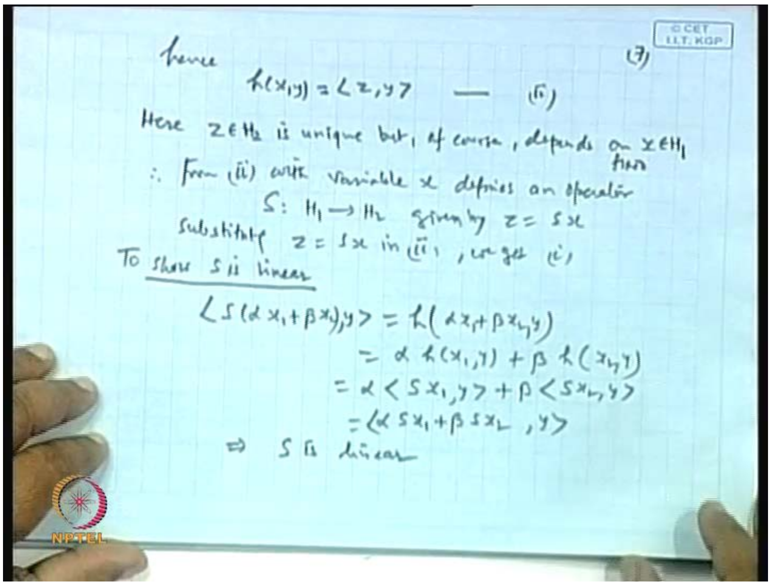

Now, here, the z belongs to H 2 is unique, because of this result, (( )) h is unique, but of course, it depends on, depends on x, which is a point in H 1; because x is fixed. Therefore, but z will always be unique, corresponding to each x y. So, it follows from 9, therefore, we can say from 2, with variable x, one can defines or defines an operator s, from H 1to H 2, given by z is equal to s x; comparing with that. Substituting z, substituting z is equal to s x in 2, we get our first one, i, that is it. So, the representation, that we have obtained the existence or we have proved the existence of s from H 1to H 2, which will give the corresponding (( )), ok.

Now, once existence of s established, then, we will prove s is linear and bounded. So, to show s is linear. Now, we start with this, s alpha x 1plus beta x 2, y. This is bracket alpha x one comma y. now, by definition of this, h of x z y, so, we can write h of alpha x 1 plus beta x 2 y; but this can be written as, alpha x 1, y plus beta h x 2, y, because h is linear; and then, this will be equal to alpha, again, inner product s x 1 y plus beta inner product of s x 2 y and this can be written, alpha x 1plus beta x 2 inner product comma y and this shows s is linear, ok. Now, to show the s is bounded, to show s is bounded. Now, if s is 0, then, nothing to show; it is obvious, is it not. So, let us take s is non-zero. So, let s is non-zero. Then, consider the norm of h. Norm of h, this h, we have taken as s z means, in this form. z equal to h x y.

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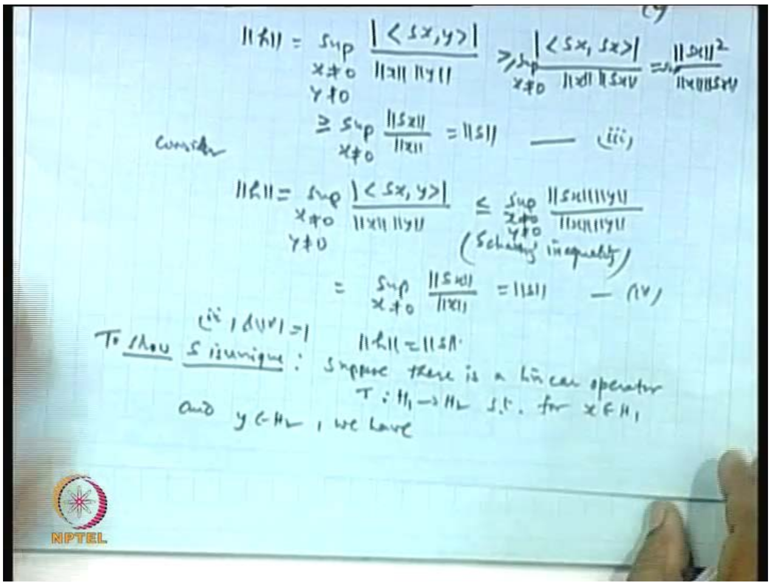

So, when you take the norm of s, this is by definition, supremum modulus of inner product s x y over norm of x norm of y and then, x is not equal to 0, y is not equal to 0. Now, this is true for all x and y, supremum is taken over. So, let it be replaced, y by a particular value as s x. So, it will be greater than equal to modulus s x s x divided by norm x into norm of s x; and then, supremum is taken over all x, which is not equal to 0; but this s x, this inner product of this s x is norm s x square, is it not. So, we get norm x norm s x; and then, supremum. So, this will be equal to, greater than or equal to supremum x is not equal to 0, norm of s x by norm x, and that is nothing, but norm of s.

So, we get, this norm of s is greater than equal to norm of x. To see the reverse inequality, consider the reverse inequality will be again, norm h, which is equal to supremum by definition, inner product of s x y over norm of x norm of y, x is not equal to 0, y is not equal to 0, ok. Now, apply the Schwarz inequality. So, we can say, it is less than equal to norm of s x norm of y and denominator remain as it is, norm of y; supremum is taken over all x not equal to 0, y is not, by using the Schwarz inequality; but this will be equal to what, norm of supremum norm of s x over norm x and x is not equal to 0. So, we get, this is the norm s. So, fourth, third and fourth implies the norm of h equal to norm of s, clear.

Now, what is left is, s is unique; it has bounded linear operator from H 1 to H 2. Now, only thing left is, s is unique and this. So, to show s is unique. Suppose, there is two representation; suppose, there exists, there is an bounded linear operator T, there is a linear operator T from H 1 to H 2, H 1 to H 2, such that, for all x belongs to H 1 and y belongs to H 2, we have norm of, inner product, sorry, this is, we have representation h x y as the inner product s x y as well as inner product T x y.

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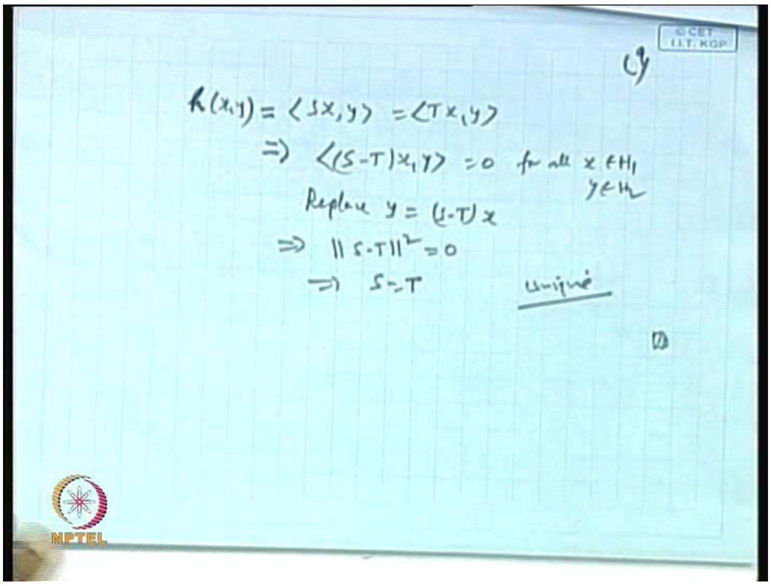

Suppose, we have the two representation of this h x y, in terms of the operator s and T; but this, is equal to inner product s minus T x comma y equal to 0; this is true for what, this is true for all x belongs to H 1 and y belongs to H 2, is it not. Now, once this is true for all x 1and s 2, then, what it can write, y is equal to this operator itself. So, replace y by s minus T x, where the s and T, both are the operator from H 1 to H 2; S minus T, bounded linear operator. So, s minus T will be an operator, operate, the element s minus T x is element of y. So, I am writing this one point y to be s minus T x. So, this gives you norm of s minus T whole square equal to 0 and this shows s is equal to T. So, uniqueness follow, is it clear. Hence, s equal to T. So, it completes the proof of this. Thank you.