Functional Analysis

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Lecture No. # 04
Separable Metrics Spaces with Examples

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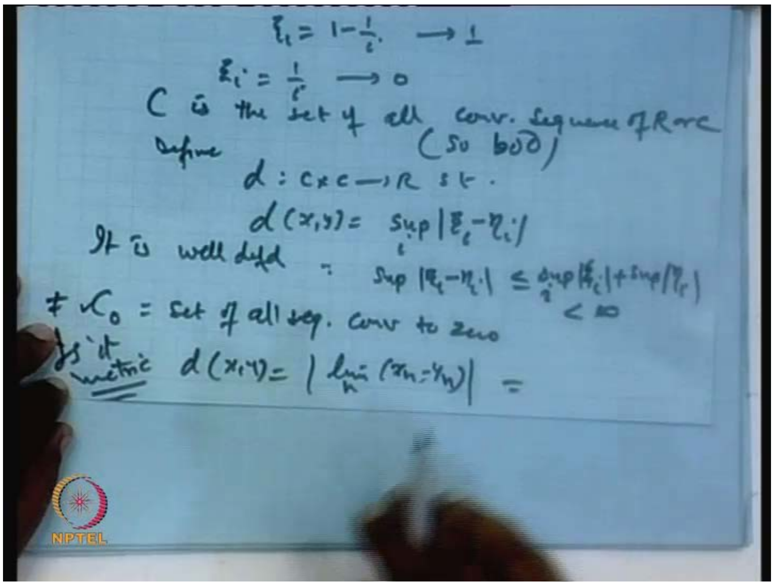

That is c 0, the set of all convergence consist converging to 0, and I asked whether this metric d x,y is mode of limit x n minus y n, whether this forms a metric on c 0 outcome. Now, if we look this is a well defined thing, because x n, y n both are convergence sequence. So, x n minus y n will be convergent and the limit will exist. But if we see the properties, then all the properties are satisfied except the first one. That is d x,y is 0 if and only if x equal to y.

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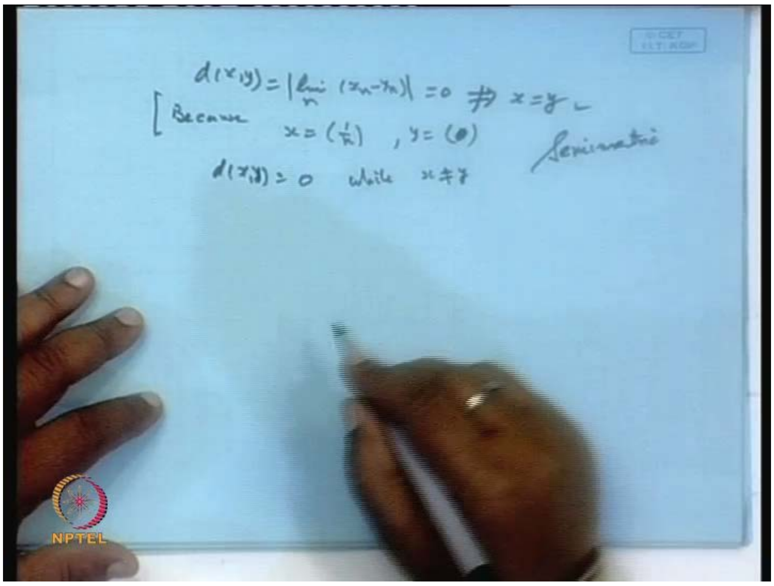

In that case, one way it is true. That a d x,y which is limit of this, x n minus y n n, and if I take it to be 0 then this does not give the guarantee x is equal to y. Because the reason is if we consider the sequence say x which is 1 by n and say y which is identically 0 sequence, all the terms are 0s, then in that case the difference x n minus y n that is comes out to be 0 while x is not equal to y. So, this shows that - though this is satisfy most of the property of the metric space, but the problem is coming only the this property that when d x,y is 0, x need not be equal to 0. So, such a mid notion of the distance which satisfy all the properties of the metric space except this one is known as the semi metric space - semi metric and the corresponding space we call it as a semi metric space.

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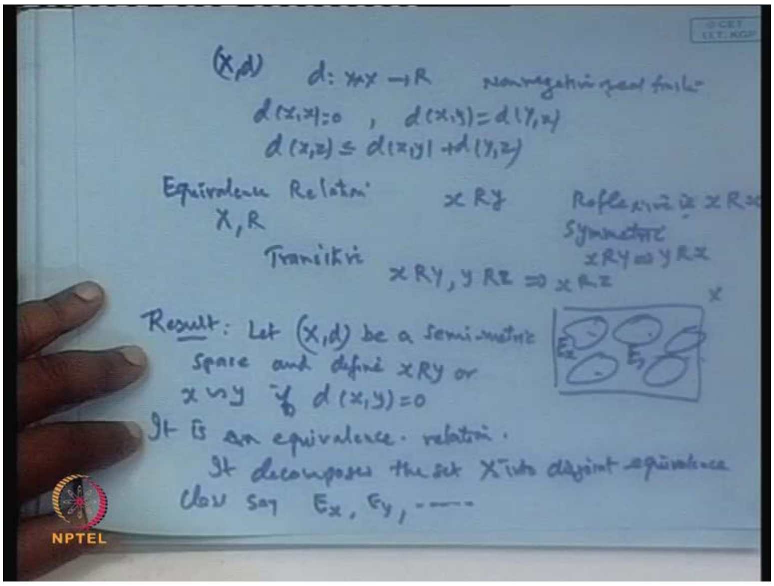

So, we formally define the semi metric space as a pair x,d, here x is a non empty set and d is a function defined form x cross x to say R, this satisfied the property that d x,x is 0, then d x,y equal to d of y,x , then d of x,z is less than equal to d of x,y plus d of y,z. So, if these properties are satisfied, these are non-negative real finite number is already there non-negative real finite. We satisfy this condition here except of property where d x,y equal to 0. So, a distinct point may also have a distance 0 in that.

Now, there is a standard procedure to convert the semi metric space into a metric space. So, because this normally we deal with the metric space. So, even if we come across about the semi metric space, then there is a way to converting into a metric space and the procedure is followed. This requires the concept of equivalence relation. So, the equivalence relation we mean… Suppose x and y are the two elements of the sets and x is related to y under certain relation, say suppose I take the relation x related to y, if x is the divisor of y or x plus y equal to 3 and like that way, so many time - so many ways one can relate.

So, if this relation is reflexive; that is x is related to x itself for every x belongs to the set. It is symmetric, symmetric means if x is related to y, then y must be related to x, and transitive transitive that is if x is related to y, y is related to z, then x must be related to z. So, if over a class x, if we define the relation R which satisfy these three property, then we say R is an equivalence relation on x. And every equivalence relation when we define this decompose the set into equivalence classes. That equivalence classes they are disjoint, paired by disjoint and union becomes the entire space x. That is if this is our set x, then if we define the relation R which is an equivalence relation, then it will decompose these set into disjoint equivalence classes. So, that the union total becomes x. So, this is the property.

Now, using this concept, we can have a method which helps you in transferring the semi metric space into a metric space, and the result is like this. Let (x,d) be a semi metric space semi metric space and define a relation x is related to y or we can say x is related to y, if d(x,y) equal to 0. Suppose x is a set and we are defining a relation between the two points of the set as if the distance of x and y is 0 then those two set elements are related. So, once we define this relation then it will satisfy these three conditions - reflexive, symmetric and transitive, because if it replace y by x, d(x,x) is 0. So, reflexive property satisfy, if d(x,y 0), d(y,x) will be 0, and similarly d(x,z), if it is less than equal to d(x,y) plus d(y,x). So, both are 0 and since it a distance function, it cannot be negative. So, d(x,z) must be 0. So, it is an equivalence relation define. So, it will decompose the set x into equivalence classes say E x, E y, E x, E y, etcetera, then it will show it is an equivalence relation. It decomposes the set set capital X into disjoint equivalence classes classes say E x, E y, etcetera, etcetera.

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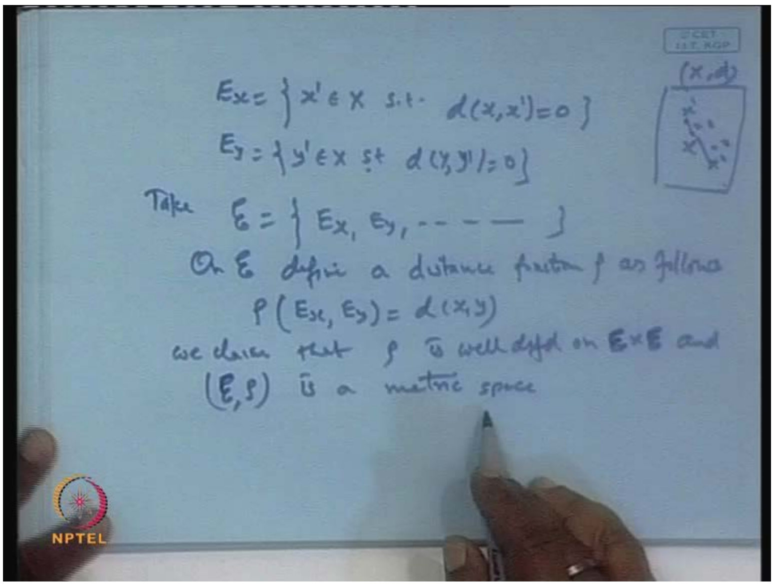

Where E x we mean is the set of those point x dash belonging to capital X such that the distance from x remains 0. So, out of this x, this is our x d and here we are taking say x. So, pick up those point x dash, x double dash and so on whose distance from this become 0, clear. This will form one class. Similarly, E y is the set of those point y dash in x whose distance from y is 0. So, this way if we get then it will decompose x into these classes which are disjoint and if we take the union of all such it becomes capital X. That is the one.

Now, take a collection E at the set of all these equivalence classes. Now, on this E define a distance function distance function rho as follows. Take rho(E x,E y) as the d(x,y). We claim that this rho is well defined on E cross E, and this E cross E means this capital E, and this capital E under the rho is a metric space. So, basically (x,d) is not a metric space, but if I define a equivalence relation on x. So, it decompose the equivalence classes and with the help of equivalence classes, you are now introducing the notion of the distance on E - capital E. So, basically the capital E, these are the subsets of x only and in the equation. How does it? First thing, whether it is well define. How it is well define? Second is whether it is metric or not. These two things we wanted to test.

It is a well define thing means (E x,E y) these are the two classes; one is E x and another one is E y both are distinct class disjoint classes. What we are doing is we are taking x from E x class, y from E y class and finding the distance. Now, this distance d if it is independent of the choice of the points then it is well define. Independent choice means instead of taking the x, I can take x dash, because this equivalent to x or I can take y dash, because it is equivalent to y, and if the distance between x dash and y dash is the same as the distance between x and y, then our problem is solved.

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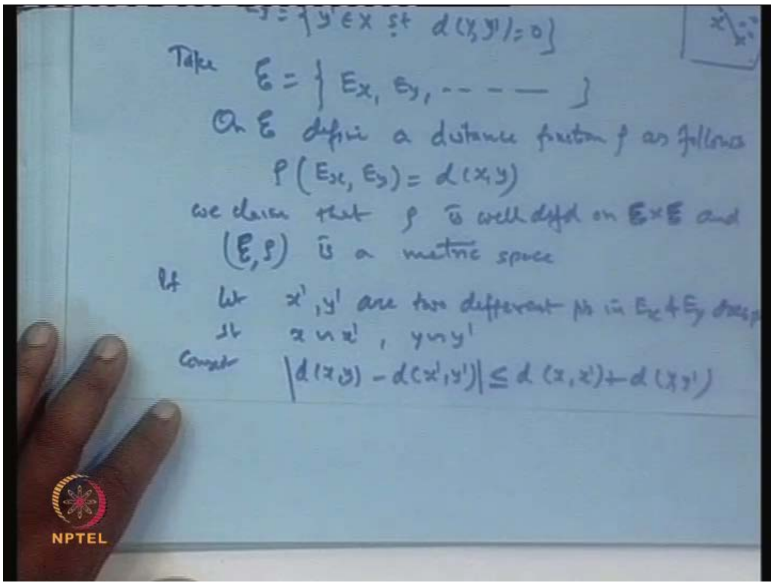

So, let x dash and y dash are the two different points in E x and E y respectively respectively; then such that x is related x dash and y is related to y dash. So, consider this thing, d(x,y) minus d(x dash,y dash), mod of this we know this is less than equal to d(x,x dash) plus d(y,y dash) clear. Why why this is so? Because the reason is if I start with say d(x,y), we can write this thing as d(x,x dash) less than equal to plus d(x dash,y dash) plus d(y dash,y) by triangular inequality. So, transfer this thing towards the left hand side and then take this this thing. So, it is less than equal to d(x,x dash) plus d(y,y dash).

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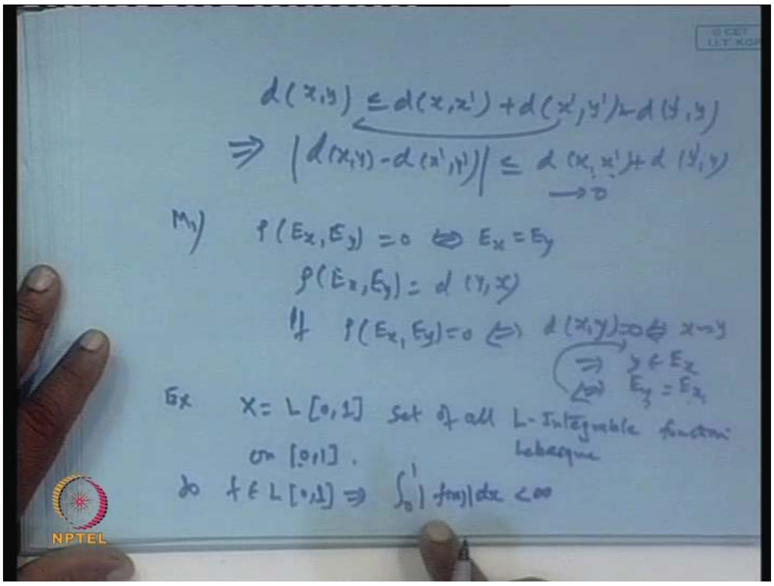

Now, if I interchange the position of x to x dash, y to y dash, then mod sign will come and we can see. Clear? So, this is that. Now, x and x dash they are the same point point in the same class E x. So, according to the construction d x,x dash must be 0. Similarly, d(y,y dash) must be 0. So, basically this tends to 0; so, d x,y equal to d y,dash x. It means the function which you are defining rho x y in terms of d x,y is a well define thing. Now, either it is a metric or not, let us see the property.

The first property is M 1 that is rho(E x,E y), it is 0 if and only if if and only if E x equal to E y. Is it not? This is… Now, what is the rho of E x equal to E y? So, let us take, if rho(E x,E y) is define as the d(y,x) or (x,y). Now, if it is 0, it means if rho(E x,E y) is 0. What it mean? It means d(x,y) is 0, d(x,y) is 0 does it not imply x is related to y. So, does it not imply that y belongs to E x or the or the class generated by y and the class generated by x both will be identical. Therefore, now, conversely if both are equal, then it means the class generated by x and generated by y are same.

So, if this is true then this will happen; and if this is happen it is true; when this is true it is true. So, it means it will form the first property is obviously satisfying and other property is obviously true; why, because if we write the (E x,E y) interchange the position then distance d(y,x) and d(x,y) makes no difference. Therefore, this property will also continue over to third property. Similarly, triangular inequality you can say that this is true. So, all the properties are satisfied, therefore it will form a metric on this. So, this is the way how to convert the semi metric space into a metric space. Clear?

Now, there is another example which we will take later on, but this is means example I am writing the detail when we go for the Lebesgue concepts this is. Suppose we have a class x which is L - L 0 1; L 0 1 is the set of all all Lebesgue integrable function – L-integrable function - Lebesgue lebesgue integrable function. On the interval 0 1 on the interval 0 1, then if we define… Since function f, so, if f belongs to this class L 0 1 then by definition of Lebesgue integration integral 0 to 1 modulus of f x d x is finite. This is by definition, Lebesgue integrable means those functions which are integrable absolute value of this is integrable over 0 to 1. When we say L p then we has to raise the power p; so, p eth integrable functions. This is different the concept is a higher concept than the Riemann integrals. Here, we have function here, which are Lebesgue measurable function - Lebesgue integrable functions.

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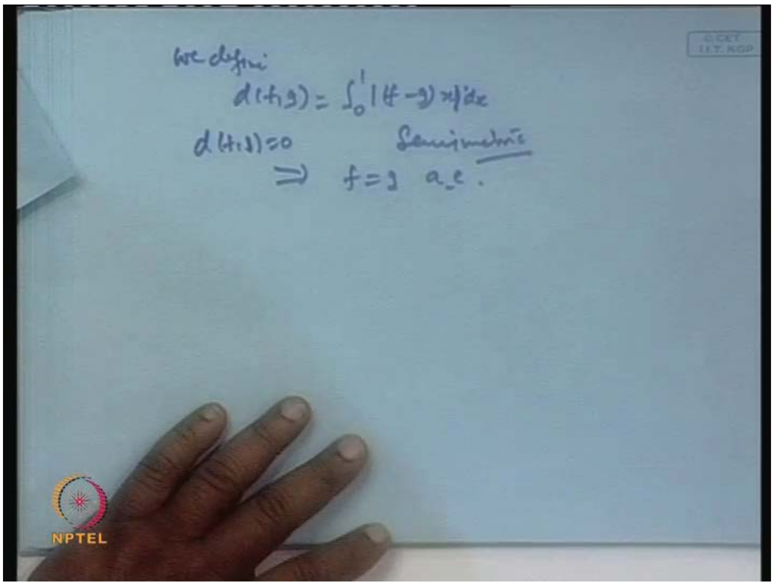

Then on this class if we define a metric d(f,g) as 0 to 1 mod f minus g x d x, then you will see that this satisfy all the property except of 1 that is d(f,g) equal to 0, does not imply f equal to g, it imply f equal to g almost if and this shows it is a semi metric. That is it.

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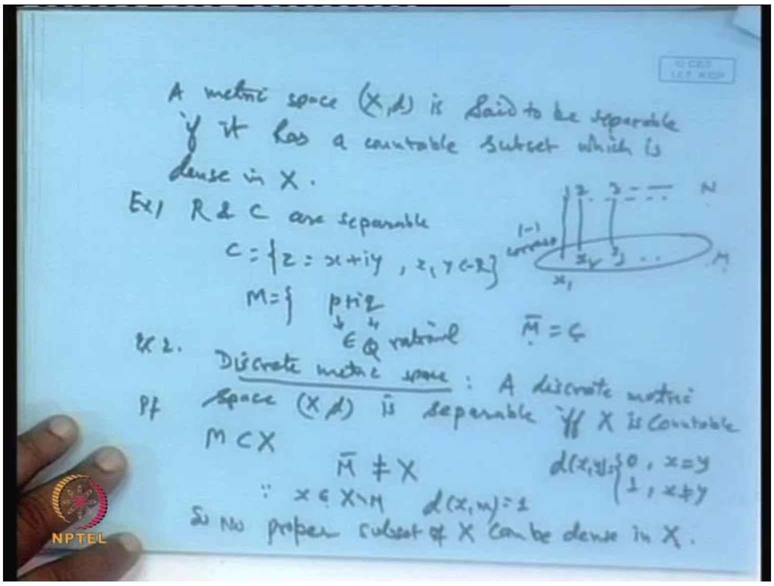

A metric space (x,d) is said to be separable separable, if it has a countable subset countable subset which is dense in which is dense in x countable subset which is dense in x. What is the meaning of countable? A set if if there is a 1 to 1 correspondence between the set and the set of positive integer that is a natural number. Suppose 1, 2, 3, etcetera these are the set of natural number and this is the set M, if corresponding to each x, we get 1 1, we get 2 another number, 3 we get another number and so on. So, if there is a 1-1 correspondence between the set of the points and the set of natural number of positive integers, then this set is said to be a countable set. So, a set which is countable and dense if such a set exist in a metric spaces, then we say metric space is a separable.

Now, for example is set of real number, it is a separable metric space. So, R and C are separable R and C are separable, because in case of R you can identify the set M which is a set of rational number. So, rational numbers are countable and it is dense. So, it will be a separable. In case of the C, because the C this is the set of all complex number of the form x plus i y where x and y both are real. So, if I pick up a set M if I pick up a set M whose elements are say p plus i q where p and q both are rational numbers both are rationals number. Then this set closure of this will be C and it will be a countable set. So, we can say that C has a countable subset which is dense in set C countable set which is dense is C. So, C is separable.

An another example let us see, if I choose a discrete metric space a discrete metric space discrete metric space then we have a discrete metric space metric space (x,d) is separable is separable if and only if if and only if x is x is countable. Why, if we look that proof, a discrete metric we means a metric space which satisfy this condition 0 and 1, when x is equal to y the distance between x and y will be 0 and when it is not equal to y then distance is 1. So, suppose I take any subset M of X - any subset. Now, this M closure of M will it be X or not. It cannot be equal to X. Why, because if I take that any point x, because if we take any point x belonging to capital X, but not in m, then distance from x to m, where m is an elements of capital M. They are distinct. So, the distance will be 1.

It means if we draw an open ball around the point x with a radius less than 1, you would not get any point of m. You would not get, because as soon as you take that x other than the points of m, distance will already 1. So, if we take a point less than 1, this cannot contain any point of m, it will be the point x itself. Therefore, this cannot be dense set. It means no proper subset of this can be dense. So, only the dense set will be the x itself. So, no proper subset proper subset no proper subset of X can be dense in X. So, what is the proper what is the dense set left out now only the X space X itself.

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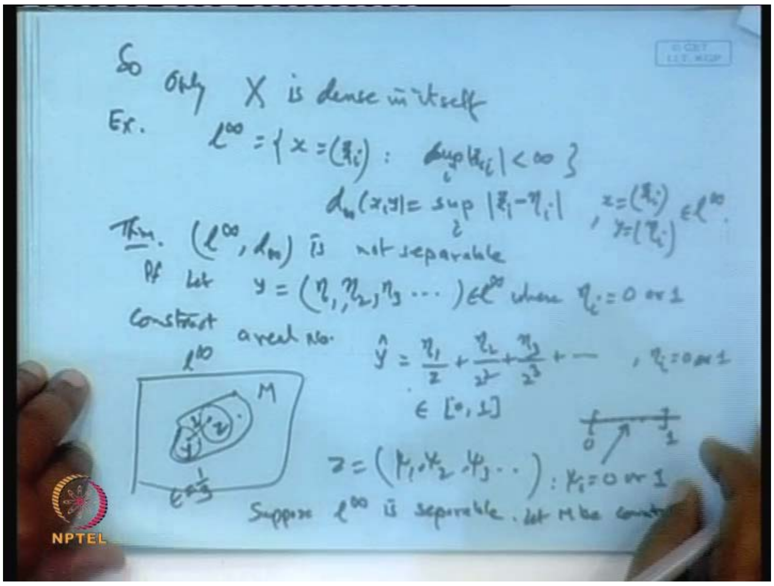

So, only X is itself is dense in itself. And condition is giving the X is also countable, because this condition I obtain if X is countable and X is the only set which is dense, it means X must be separable. So, X discrete metric is if a metric space X which is countable, and then obviously it will be separable by definition, and if (x,d) is a discrete metric, then according to the definition of the metric only set which is dense in x if the x itself, and therefore it will be separable only when x is countable. So, that is. I think it is clear.

Then another nontrivial example is our say l infinity. l infinity is the set of those point x which is an infinites sequence of real-complex numbers such that the supermom of mod psi i is finite, set of all bounded sequences. Is it not? And notion of the distance d infinity, we have defined in terms of the supermom mode psi i minus eta i where x is equal to psi i y is equal to eta i both are in l infinity. We claim that l infinity under this metric is not separable is not separable. So, it means either we are unable to get a set which it dense and countable both. Is it not? Then only we can say this is not separable. So, if a set is dense it may not be countable and if a set is countable it may not be dense. So, let us see how this proof goes.

Let us pick up a point y whose element are suppose eta 1, eta 2, eta 3 and so on where eta i is either 0 or 1. It means we are taking a sequence whose elements are either 0 1 or 0 1 like that; may be all 0, may be all 1, all may be some 0, some 1s. So, what will be the supermom of this? It will be maximum 1. So, it is finite, bounded. Therefore, this will be a point in l infinity this is the point in l infinity. Now, if we construct a real number a real number say y hat as for eta 1 by 2, eta 2 by 2 square, eta 3 by 2 cube and so on. Now, eta i's are 0 or 1. So, is it not a binary representation of a real number y hat. If y hat is a real number is binary representation will have this type of expansion; eta 1 by 2, eta 2 by where, eta 1, eta 2, eta n, make assume either 0 or 1, clear.

So, this is the binary representation of y hat and this will definitely leave point belonging to interval [0,1]. Why [0,1], because all etas are 0s, then y will be y hat will be 0; if all are 1 then it is half plus half square plus 1 by 2 cube and so on an infinite geometric series. So, a over 1 minus r 1 by 2 divided by 1 minus 1 by 2, that is your 1. So, maximum it can take the value 1 and minimum is 0. So, it can take all the possible, you know, all the real numbers in between 0 and 1 clear. Giving the different choice of eta 1, eta 2, eta n, you can fill up the gaps, fill up these close interval 0 point, clear.

Now, let us take a other values. Suppose I picked up a point in this interval; say any point in this interval can be express into this form. So, take the numerators only. So, you can construct a sequence y whose coordinates are eta 1, eta 2, eta 3. So, it will be a point in l to infinity and having the coordinate either 0 and 1. It means there is a 1 to 1 correspondence between the collection of this these type of point and the point in the [0,1] clear. So, if this is our l infinity and here this is a set m where the elements are of this type y. Similarly, another element say z; similarly, another element like this; and each one will have the coordinate either 0 and 1. So, what is the distance between these two points? Maximum distance will be 1. Is it not?

If y and z both are identical if the distance will be 0, but if y and z - z is any point say psi 1, psi 2, psi 3 and so on where the psi's are 0 or 1, then what happens, the distance between these two point, d(x,y) means difference between the corresponding coordinate, eta 1 minus psi 1, eta 2 minus psi 2 and so on take the supermom. So, if all eta i is equal to psi i the value will be 0; if at least one of place it differs the value will be infinity. So, it means the distance between y and z will be 1, if they different they are different otherwise 0.

Now, let us draw the ball around the point y and around the point z with a radius epsilon 1 by 3. If we draw the ball around the point y with a radius 1 by 3, draw a ball around the point z with a radius 1 by 3, then they are disjoint balls. Now, this y and z these are the points these are the points in l infinity, corresponding point in 0,1 and vice versa. So, it means the how many sequence point you have in l infinity, whether it is countable or uncountable. Corresponding to each point on 0,1, we get 1 point in l infinity, but 0,1 is an uncountable set, 0,1 is an uncountable set clear. So, it means there are uncountable number of points are there y hat. Therefore, correspondingly we had the uncountable point in l infinity that is in the set m. So, this m is an uncountable set, means these points are uncountable. Now, we are drawing the ball around each point which are disjoint clear. Now, suppose M is suppose l infinity is separable; let us consider assume this part. So, it means if it is separable it must have a countable dense subset. So, let M be a countable dense subset of l infinity countable dense subset of l infinity such that closure of this equal to l infinity clear.

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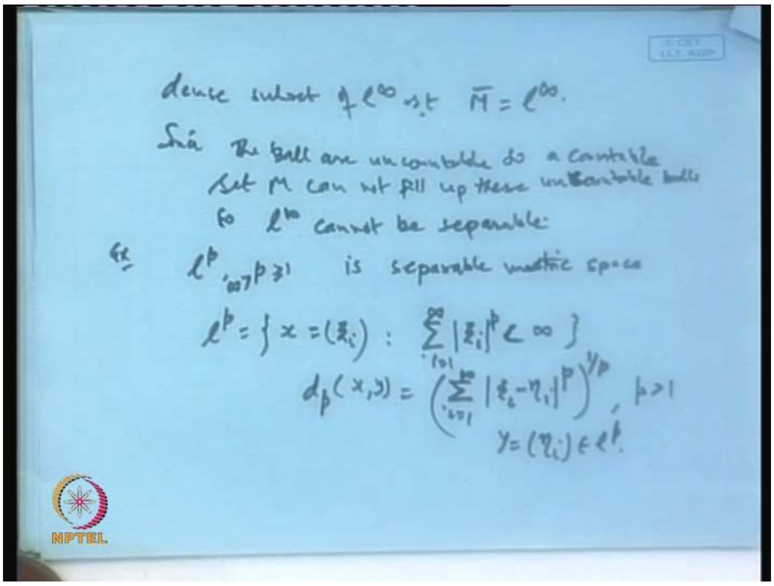

Now, if closure of this is l infinity and countable. So, if we pick up any arbitrary point of l infinity then and draw the ball, then this ball must include the points of M. So, there is a one point M here, another point M here, but how many such ys are there, they are countable; it means you are getting a countable number of the sorry uncountable and uncountable number of these open balls are available, and in each ball we want some point of M. So, M has to be uncountable, because a countable set cannot fill up the uncountable ones. So, M has to be uncountable. It means if a set M is dense in l infinity, it cannot be a countable set. Therefore, l infinity is not separable clear. So, that is what it is. I think it is clear. So, l infinity is not a separable, dense in l infinity (( )).

Now, since since the balls are uncountable. So, a countable set countable set M cannot fill up these uncountable balls uncountable balls clear. So, l infinity cannot be separable. So, that is it. Now, let us take another example where it say separable. l p when p is a greater than 1 equal to 1 at the most is an, but infinity less than infinity, but p infinity is k, because l infinity is not separable. So, this is a separable metric space separable metric space. What is l p? The l p is the set of those points x who which is in the form of the sequence psi i. Such that its p eth power is absolutely separable; this is that. And that is notion of the distance d p on (x,y) is defined as 1 to infinity mod psi i minus eta i power p power say 1 by p where p is greater than 1 and y is equal to eta i is a point in l p. So, this is our… Now, we want this to be separable. So, we have to show the existence of a set which is countable and dense. So, this can be done like.

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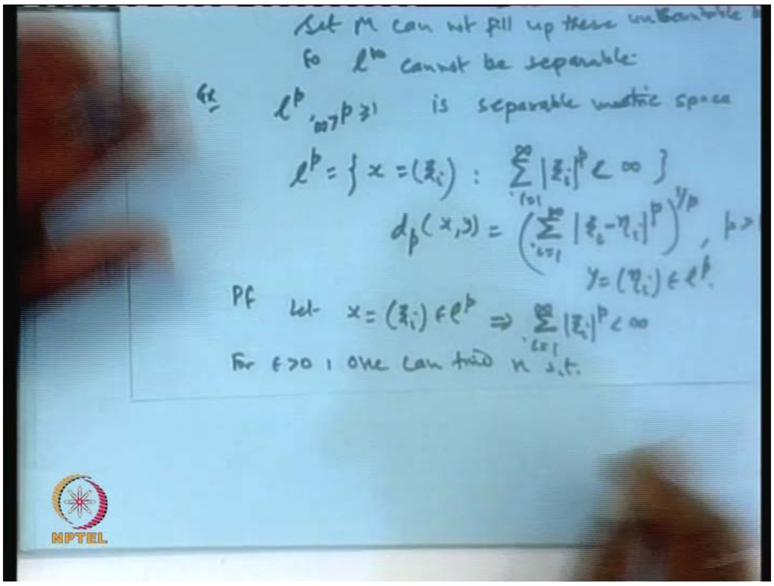

Let x which is psi i belongs to l p. So, once it belongs to l p, it means this condition will satisfy, power p is finite. Now, this is a series an infinite series which is convergent. So, if I truncate the series after a certain stage the remainder can be made as a small as we please, depending on how many numbers of terms after that you are discarding or truncating.

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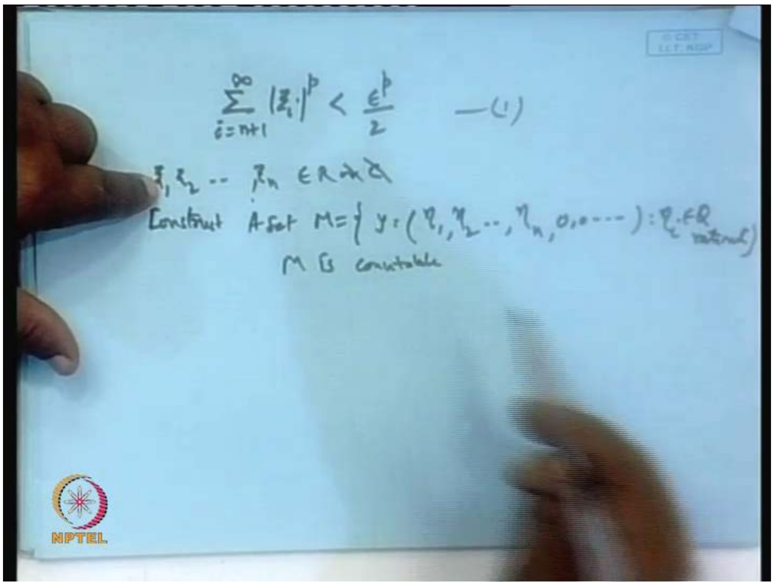

So, for the given epsilon greater than 0 one can find n such that this series i is equal to n plus 1 to infinity mod psi i to the power p is less than say epsilon to p by 2 for this epsilon. So, epsilon to the power p is also another epsilon 1. So, this is say epsilon 1. So, with this epsilon 1, we can identify the number of points starting from psi 1, psi 2, psi n. So, if I truncate after all any term the remaining term which is the remainder of the convergent series can be made as small as this number.

Now, the points which are left now psi 1, psi 2, psi n, these are the points left out. So, now we construct a set M which is a set of all the points y, which is of the form eta 1, eta 2, eta n with respect this n; 0, 0, 0; where eta i are rational numbers; they are rational clear. So, we are taking those points whose first n elements are rational and rest are 0s. Now, this set M is obviously countable. Why, because these are rational points. So, you keep on changing eta 1, eta 2, eta n, you get another points y 1, y 2, y n, but eta 1, eta 2 are rational number and rational numbers set is countable. So, you can have a countable number of collections there, we can replace eta 1, eta 2, eta n from the countable set. Therefore y 1, y 2, y n can be made constructed and this will form a countable set. So, this is a countable set. So, M is countable. Now, only thing left out if it is also dense in l p then we have a at least this set which is countable and dense. So, now psi 1, psi 2, psi n which is either a real number or complex number depending on the set, if l p is a real space then this coordinates will be real, if l p is a complex then this will be a complex.

So, let us take with this real case first. So, corresponding eta 1, eta 2, eta n is a rational points. If you start with complex then this will be the sum of the a complex form will each real element part will be rational part. So, like that. So, psi 1, psi 2 are real, every real number can be approximated by means of a rational number; every real number we can always find a sequence of the rational point which converts to the real number. That is the difference between the real and rational can be made as a small as you please.

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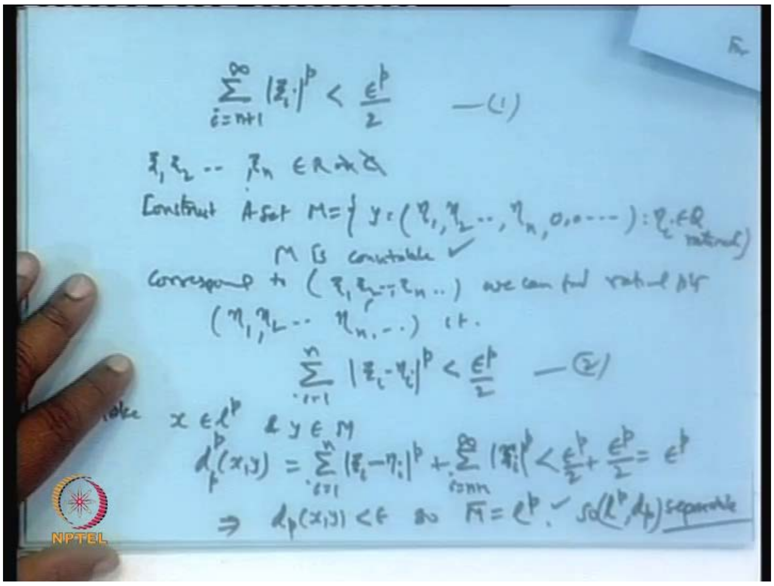

So, corresponding to the psi 1, psi 2 and psi n, we can find the rational points eta 1, eta 2, eta n such that the series i equal to 1 to n mode of psi i minus eta i power p is less than epsilon to the power p by 2; psi 1 can be approximated by eta 1, psi 2 can be approximated by eta 2, psi i n can be approximated eta n and in such that the sum of this their p eth power is less than this number. Now, take any point x; now choose any point x belongs to l then corresponding to this find the y and y belongs to M. So, what is that d? d p (x,y) by definition and power p; this is i is equal to 1 to n mod of psi i minus eta i power p plus i equal to n plus 1 to infinity mod of mod of psi i power. Is it not? Because this series is there; this is our series; this one is the series, this is. So, when i is equal to 1 to n the eta 1, eta 2, eta n are there, but when i is equal to n plus 1 eta n plus 1 is 0 and so on.

So, basically i equal to 1 to n you are getting this term and after that psi i minus 0 is that. Now, this is less than epsilon to the power p by 2; this is less than epsilon p by 2, total becomes epsilon p. So, take the power 1 by p both side we get therefore d p (x,y) is less than epsilon. So, what does show? This shows… So, M is dense in l p. M is countable, M is dense, therefore l p is countable, M is countable, M is dense so, l p under d p with d p is separable. Let us see, right so.

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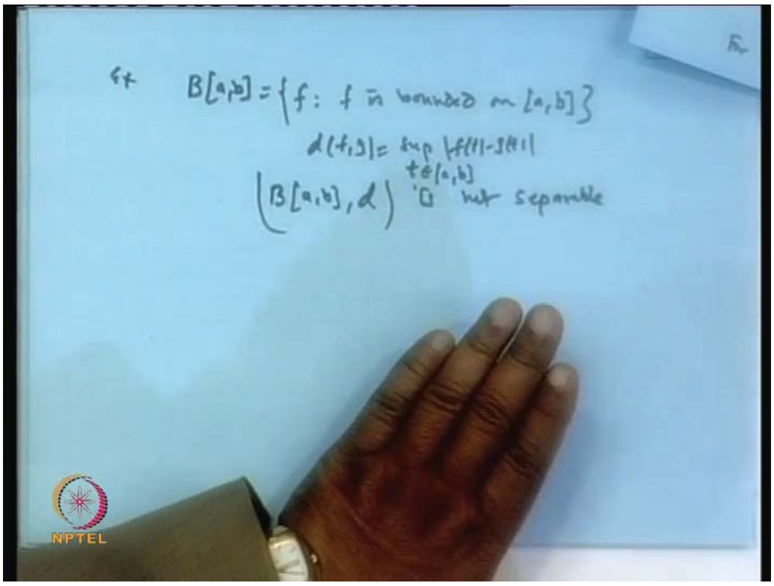

We can in fact, find out the any space like this whether it is separable or not. The B[a,b] for example, the set B[a,b]; what is the B[a,b]? Set of all functions f which are bounded on the close interval and the metric is defined in term of the supermom f(t) minus g(t) over t belonging to the interval, because it is a bounded function supermom will attain and this is the metric. Now, this B[a,b] under d is not a separable space. The hint I will give that. Why it is not? Because this basically if this metric B[a,b] set is nothing but a federal set in our l infinity; l infinity is what, l infinity is the collection of the sequences which are bounded. l in state of the sequence l getting the functions which are bounded and the same similar type of metric is defined supermom psi i minus eta i here the supermom of mod f(t) minus g(t).

So in fact, the proof that l infinity is not separable; on a similar line, we can show that this is also not separable. So, what you have to do is? You have to first identify the points real it has a value functions real it as a value 0 and 1. Find out the collections and then take this distances between the two function where it is 1, those points with the help of just like l infinity find out as point which are having distance 1 apart, draw the ball then if if you assume set is countable and dense, then it cannot fill this uncountable part let us. So, in a similar way… So, this you can try.

Now, there are certain more definitions which is required like the concept of the convergence, Cauchy you can say in a general metric space; just like in a real or complex number, we define the convergence part, a sequence of the real number is set to converge to a point x, if the distance mod of x and minus x is tending to 0 when n tends to infinity.

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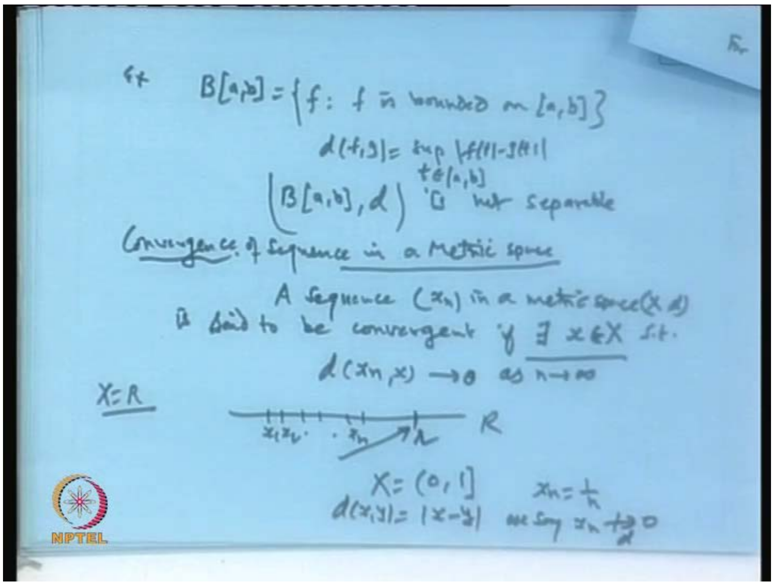

So, basically this concept has been generalized to an arbitrary metric space. So, convergence of a sequence in a metric space; a sequence x n a sequence x n in a metric space (x,d) is said to be convergent convergent is said to be convergent or converging, if there exist a point x belongs to capital X. Please remember, here this is very important part, this part is important, there exist a point a X belongs to capital a X such that the distance between x n and x tends to 0 as n tends to infinity. Why I am insisting this point, because in case of the real, if x be R take a sequence of real numbers x 1, x 2, x n, then this sequence of the real number will converge to a some real point which is available in the R. So, then we say that this mod of x n minus R tends to 0 or x n minus x tends to 0. But here is not granted the point x be available in the set x, because x need need not be as entire line.

If I choose x to be only this open interval 0 and 1, then what happens, if I pick up the sequence x n to be 1 by n and define the notion of the distance as our usual distance as x minus y, then this sequence x n will converge to 0, but 0 is not available in x, because x is only the semi close interval [0,1]. So, though the x n goes to 0, but because the limiting point is not available, therefore we say we say x n does not converge to 0 in this metric. Because the limited point is not available, but if suppose I remove this thing simply I take x n equal to 1 by n and entire real line x to be R; obviously, the point R will be available in 0 will be available in R.

So, this is sequence is convergent, but when you go for the metric space then what type of the set you are choosing, what type of the metric you are choosing, that takes a important that takes plays a important role in deciding whether the sequence is convergent or is not converge. So, that is the main idea we have.

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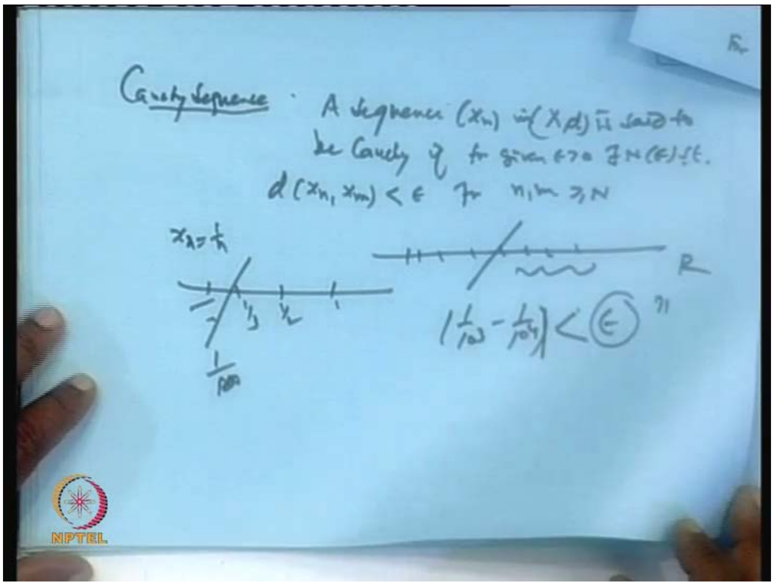

In a similar way, we define a sequence to be Cauchy sequence - Cauchy sequence or fundamental sequence; we say a sequence x n a sequence x n in a metric space (x,d) is said to be Cauchy, if the distance between x n and x m can be made less than epsilon for (n,m) greater than equal to capital N, that is if for a given epsilon greater than 0 there exist an n which depends on epsilon such that the distance between x n and x m can be made less than epsilon for all n greater than. Just like in a real line also if this is a sequence x 1, x 2, x n; what we do is, pick up the epsilon and then truncate it, after this the difference between any two terms of the sequence has to be less than epsilon. For example, 1 by n, x n is equal to 1 by n, if I take this interval 1, 1 by 2, 1 by 3 and so on.

Suppose I truncate after 1 by 100, then if you pick up any two term of the sequence 1 by 10 cube minus 1 by 104 it will remain less than the desired number epsilon whatever you choose. So, this epsilon will depend on how many terms you are choosing here. So, that is clear. So, this is done.

Let us we will see that; again in a similar way, if convergent in Cauchy that we will we see the completeness later on, because in case of the real or complex number every Cauchy sequence is a convergent one. So, that is why the real and complex number is a complete metric space complete, but in a general metric space not true that we will take later. Thank you.