Probability and Statistics
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Module No. #01
Lecture No. #12
Special Distributions - III

Let us look at some applications of the Poisson process.

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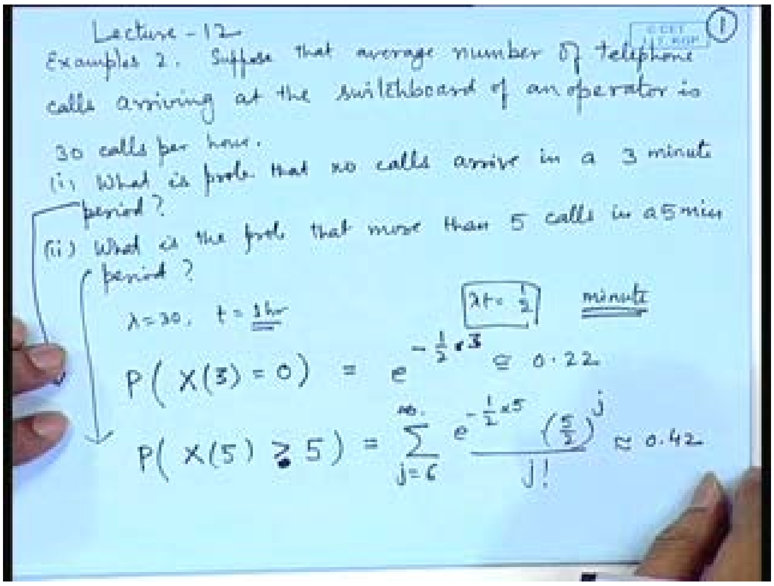

Suppose that average number of telephone calls arriving at the switchboard of an operator is 30 calls per hour, what is the probability that no calls arrive in a 3 minute period? What is the probability that more than 5 calls in a 5 minute period?

So, here if we see lambda is equal to 30 and t is equal to 1 hour. So, if we consider the unit as minute then in 1 minute there will be lambda t is equal to half where, if we are considering the unit of time as minute. So, if we say probability of no calls in a 3 minute period, this can be considered as probability of X 3 is equal to 0. So, it is equal to e to the power minus 1 by 2 into 3. This 1 by 2 is the rate for 1 minute. So, in 3 minutes it will be 3 by 2. So, here I am taking lambda to be half and t to be 3. So, it is e to the power minus 3 by 2 it is approximately point 2 2.

In the second one for example, what is the probability of more than 5 calls in a 5 minute period? That means, in a 5 minute period X 5 greater than or equal to 5. So, here the Poisson distribution will have half into 5. So, e to the power minus half into 5 then 5 by 2 to the power j by j factorial and summation from j is equal to more than 5 call. So, we will put strictly greater than here in place of greater than or equal to. So, it is j is equal to 6 to infinity.

So, from the tables of binomial distribution or by calculation we can check it is approximately 0.42. So, the probability that more than 5 calls will be received in a 5 minute period is 0.42 and probability of no call being received in a 3 minute period is 0. 22. So, here you can see that when we want to apply the Poisson process then the parameter lambda is dependent upon the unit of time for which we are considering. So, here initially it is given 30 calls per hour. So, if we consider the unit as hour then lambda is 30, but if we consider unit as minute then lambda will become 1 by 2 because 30 by 60. So, this is the way of evaluation in a Poisson process.

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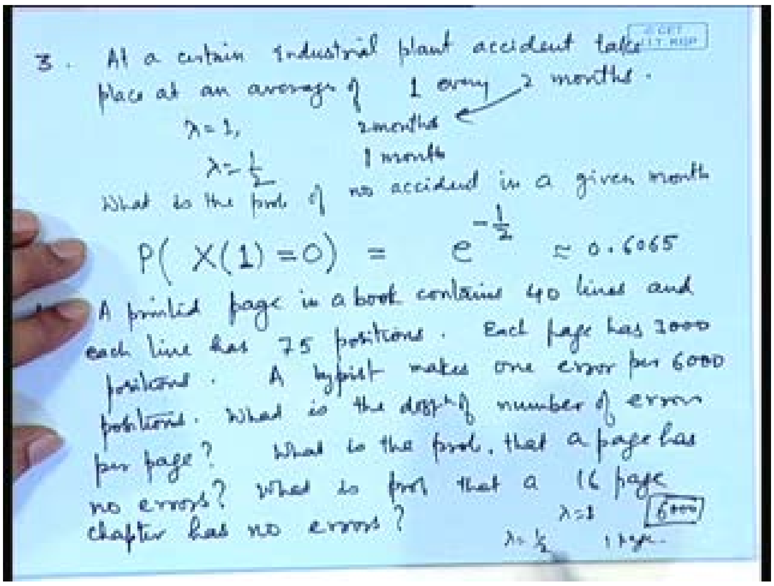

Let me take one more example here; at a certain industrial plant accidents take place at an average of 1 every 2 months. So, the rate is 1 accident in 2 months. So, if I consider lambda is 1, but time of unit of time is 2 months. So, if we consider unit of time as 1 month then lambda will become half. So, what is the probability of say no accident in a given month? Now, this means probability now month is the unit. So, probability that X

1 is equal to 0; so, in a 1 month lambda will be half. So, it is e to the power minus lambda t that is e to the power minus half that is 0. 6065. So, there is a 60 percent chance that there will be no accident in a given month.

If we look at the conditions of the theorem, conditions of the problem here, it is given an average of 1 every 2 months. So, you will feel that probability will be 50 percent of no accident or probability of 50 percent of 1 accident in a month, but it is not. So, actually the probability of no accident is more than that it is point 60.

Let me take one more application of Poisson distribution; here a printed page in a book contains say 40 lines and each line has 75 positions it is like 1 2 3 4 5 6 7 8 9 10. So, blanks are also counted here. So, each page has 3000 positions a typist makes 1 error per 6000 positions, what is the distribution of number of errors per page? What is the probability that a page has no errors? What is the probability that a 16 page chapter has no errors?

Now, here you see lambda is equal to 1 for 6000 positions; if the unit of area or space is 6 is the position then it is 6000 positions; then, lambda is equal to 1. If we consider the unit as 1 page then in a page there are 3000 positions then lambda will become equal to half if the unit is 1 page. So, in order to answer the questions here, the unit is 1 page therefore, lambda will be half what is the distribution of number of errors per page. So, this will be probability of X 1. So, 1 means 1 page X 1 is equal to n and here lambda is becoming half. So, it is e to the power minus half half to the power n by n factorial n is equal to 0 1 2 and so on.

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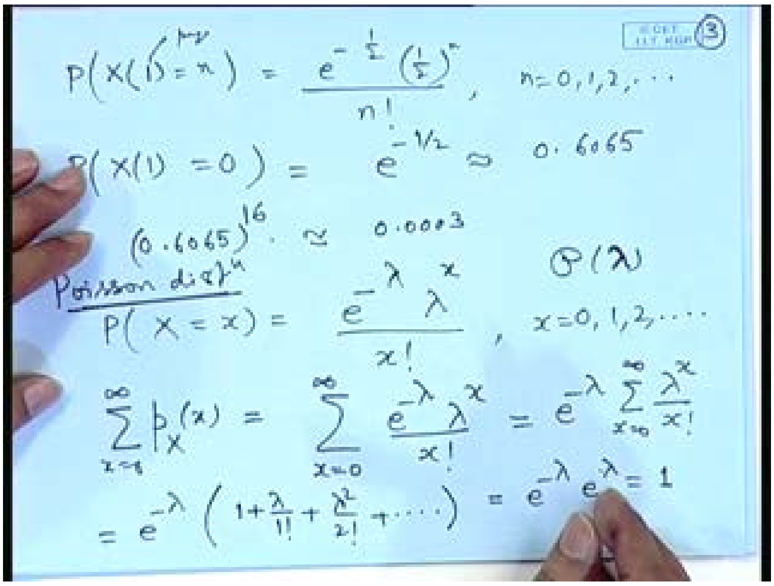

What is the probability that a page has no errors? So, that means, we want probability of X 1 is equal to 0. So, according to this formula it will be e to the power minus 1 by 2 which is approximately point 6065. What is the probability that a 16 page chapter has no errors now here each page containing an error or not can be considered as a Bernoullian trial now for 1 page not having any error is point 6065. So, 16 page chapter having no errors can be considered as point 6065 to the power 16; that is P to the power n which is approximately of course, point 0003 which is quite small; which is obvious that since in 1 page the probability of an error is not an error is point 6. So, in a 16 page chapter there will be no error; the probability is naturally going to be very very small.

Let us look at the characteristics of the Poisson distribution. So, for convenience we will denote probability X equal to x as e to the power minus lambda lambda to the power x by x factorial X is equal to 0 1 2 and so on. So, here that lambda t be replaced by lambda because what happens, that lambda is the rate of occurrence of the event when the unit of time area or space is taken as something which is denoted by t. So, that lambda t we can merge into 1 and we can write it as lambda again. So, this is a convenient way of expressing a Poisson distribution. So, this form of the probability mass function is known as a Poisson distribution and we will use a notation Poisson lambda. So, this means that rate is lambda here.

If we look at sigma of P X x x is equal to 0 to infinity that is equal to sigma e to the power minus lambda lambda to the power x by x factorial X is equal to 0 to infinity. So, this is equal to e to the power minus lambda summation lambda to the power x by x factorial X is equal to 0 to infinity. If we look at the series sigma lambda to the power x by x factorial, this series is 1 plus lambda by 1 factorial plus lambda square by 2 factorial and so on, which is nothing but the expansion of e to the power lambda. So, this is equal to e to the power minus lambda e to the power lambda that is equal to 1. Therefore, this is a valid probability mass function.

Now, this summation also suggests that how the moments of the Poisson distribution will be evaluated; that means, we will have to interpret or represent the infinite series as expansion of e to the power lambda terms.

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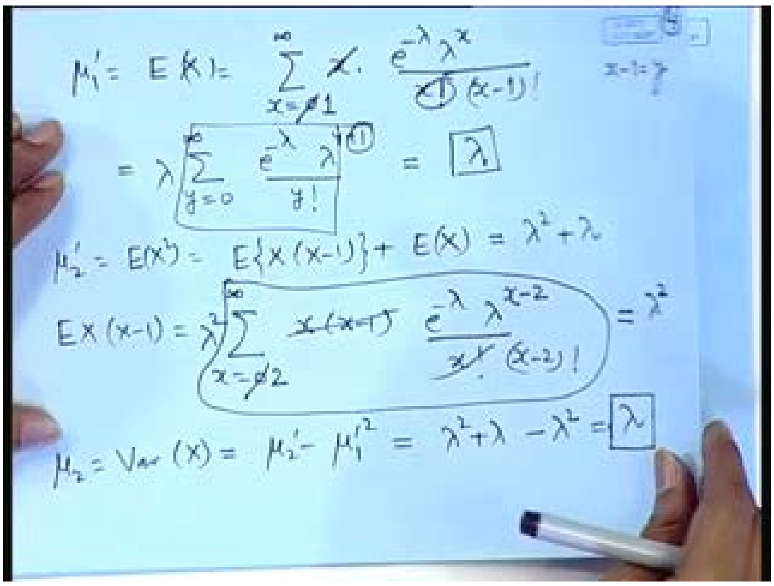

Let us look at say mu 1 prime that is expectation of X. So, that is equal to X e to the power minus lambda lambda to the power x by x factorial summation, from x equal to 0 to infinity. As we have seen in the binomial distribution or hyper geometric distribution when the factorial term is involved then, we have to adjust that. So, notice here that corresponding to x equal to 0 this term is vanishing. So, in effect this is actually summation from x equal to 1 to infinity. Now, this term can be adjusted and we can write it as x minus 1 factorial.

Naturally, we can substitute x minus 1 is equal to y then this becomes summation from y is equal to 0 to infinity e to the power minus lambda lambda to the power y plus 1 divided by y factorial. So, we can keep lambda outside and then this sum becomes 1. Therefore, the mean of a Poisson distribution is lambda which is obvious because, when we are saying lambda is the rate of occurrence, so, in a particular unit of time t the number of arrivals will be lambda t. So, when we replace lambda t by lambda here the mean must be lambda.

Now, this suggests that in a way to calculate say second moment we will need to calculate the second factorial moment. So, if we apply the same argument expectation of X into X minus 1 will be equal to X into X minus 1 e to the power minus lambda lambda to the power x by x factorial x equal to 0 to infinity. Notice here that corresponding to x equal to 0 and x equal to 1 this term vanishes. So, in effect this summation is from 2 to infinity and therefore, this x and 2 x minus 1 term can be adjusted with this term and we get x minus 2 factorial. Therefore, this we can write as lambda to the power x minus 2 lambda square and now this is nothing, but expansion of e to the power lambda.

So, the second factorial moment becomes lambda square now if we substitute this value in the expression for mu 2 prime this is lambda square plus expectation x that is lambda therefore, mu 2 that is variance of the Poisson distribution becomes that is mu 2 prime minus mu 1 prime square that is equal to lambda square plus lambda minus lambda square which is equal to lambda. So, we come to a surprising looking result here; mean was lambda and now the variance is also lambda. So, in a Poisson distribution the rate is mean as well as it denotes the variance of the distribution. Now, the way the calculations have been d 1 here. Since, the factorial is or involved it will be easier to calculate factorial moments in the case of Poisson distribution.

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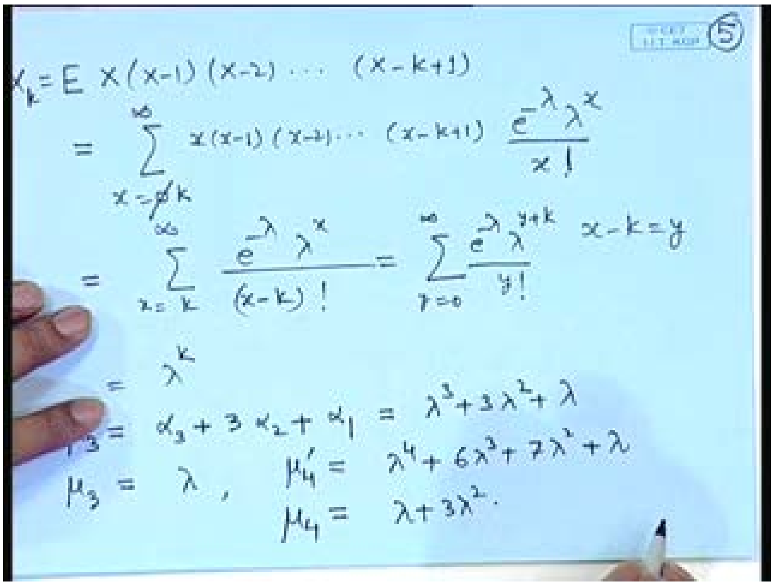

We may consider k th factorial moment that is expectation X into X minus 1 X minus 2 up to X minus K plus 1. So, this becomes summation x into x minus 1 into x minus 2 up to x minus k plus 1 e to the power minus lambda lambda to the power x by x factorial x is equal to 0 to infinity. So, noticing that for x is equal to 0 1 2 up to k minus 1 this term vanishes this is basically an expansion from x is equal to k to infinity. So, naturally then we can cancel this term from expansion of x factorial and we will get e to the power minus lambda lambda to the power x divided by x minus k factorial x equal to k to infinity.

So, if we substitute x minus k is equal to y then this becomes summation from y is equal to 0 to infinity e to the power minus lambda lambda to the power y plus k divided by y factorial. So, this is lambda to the power k into this term which will become actually 1. So, the k th factorial moment becomes lambda to the power k. So, we can use this and get the expressions for third 4th non-central moments and consequently the third and 4th central moments of the Poisson distribution. So, for example, mu 3 prime [ ] mu 3 prime will then become equal to the third let me denote it by say alpha. So, alpha 3 plus 3 alpha 2 plus alpha 1 alpha 1 is expectation X. So, this is, we can write alpha k. So, alpha 3 is equal to lambda cube plus 3 lambda square plus lambda, the third non-central moment.

Once again, if we make use of the relationship between the central and non-central moments then mu 3 is equal to lambda that is lambda cube plus 3 lambda square plus lambda minus lambda cube minus 3 lambda square that term is coming there. So, it is actually becoming lambda. So, this is again surprising because here the third central moment the second central moment and the mean they are all the same. So, in a Poisson distribution all the 3 are same.

In a similar way, we can look at mu 4 prime mu 4 prime is lambda to the power 4 plus 6 lambda cube plus 7 lambda square plus lambda and using that, mu 4 that is a 4th central moment is calculated to be lambda plus 3 lambda square.

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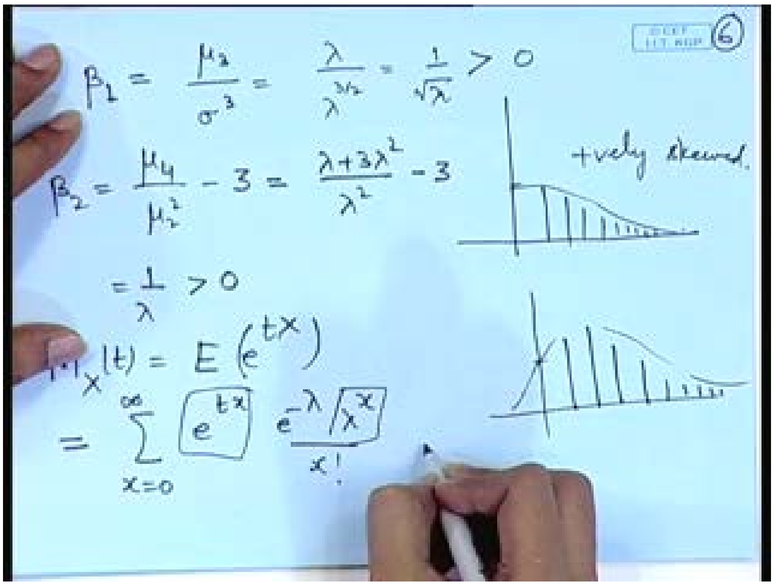

We can calculate the measures of skewness and kurtosis beta 1 is equal to mu 3 divided by sigma cube. So, here if we see sigma square is equal to lambda that is sigma is equal to lambda to the power half. So, if we substitute it here we get lambda divided by lambda to the power 3 by 2 that is equal to 1 by root lambda naturally seems lambda is rate it is a positive parameter; so, this is greater than 0.

However, you can observe that as lambda increases this will converge to 0. So, Poisson distribution is a positively skewed distribution which is obvious also because, the terms of the Poisson distribution are given by e to the power minus lambda lambda to the power x by x factorial. So, in the beginning if you see the first term is e to the power minus lambda then lambda into e to the power minus lambda then lambda square by 2 factorial into e to the power minus lambda. So, as x increases the denominator will be dominating x factorial term. So, the probabilities will rapidly decrease. It may increase (

) little bit in the beginning if lambda is greater than 1; if lambda is less than 1 then from the first step itself the probability will start decreasing. Therefore, it is a. So, if lambda is less than 1 it will decrease quite rapidly. So, it is a positively skewed distribution if lambda is bigger than 1. Then in the beginning may be it will increase, but there after it will start decreasing rapidly.

So, the shape of the curve is positively skewed. Let us also look at the measure of kurtosis; that is beta 2 mu by 4 mu 2 square minus 3. So, that is equal to lambda plus 3 lambda square by lambda square minus 3. So, that is equal to 1 by lambda and once again it is positive. The peak is a little higher than the peak of a normal distribution. However, we can observe that as lambda increases 1 by lambda is approximately 0 and therefore, the peak will converge to a normal peak as lambda increases in a Poisson distribution.

We can also look at the moment generating function of the Poisson distribution m x t that is expectation of E to the power t x this is equal to sigma e to the power t x e to the power minus lambda lambda to the power x by x factorial summation x is equal to 0 to infinity. We have already shown how the evaluations of the moments are d 1 that is by looking at the expansion of e to the power lambda term. Therefore, this term should be combined with this and we get e to the power minus lambda lambda e to the power t to the power x by x factorial x equal to 0 to infinity.

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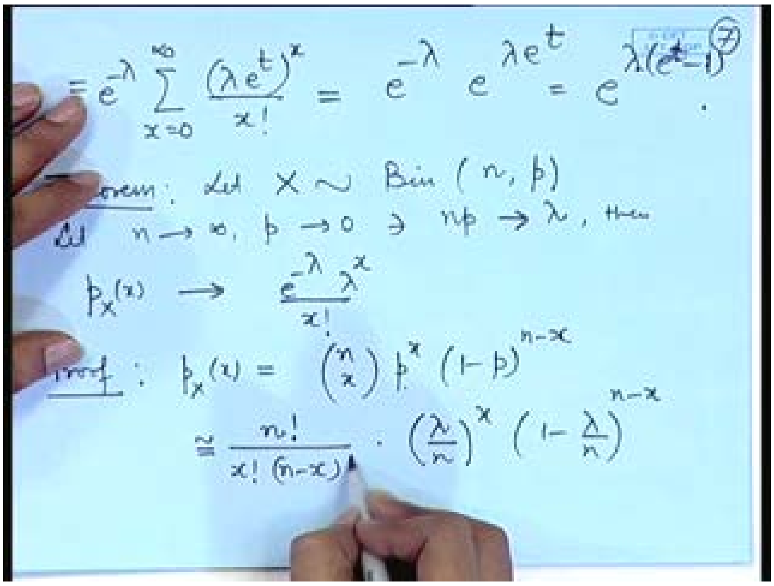

So, this is equal to e to the power minus lambda e to the power lambda e to the power t. This term is an expansion of e to the power lambda e to the power t. Therefore, the moment generating function of a Poisson distribution is e to the power lambda e to the power t minus 1 which is existing for all values of t.

You can observe here that at t equal to 0; this is equal to 0; therefore, it becomes e to the power 0, that is 1. If t is positive then e to the power t is greater than 1. Therefore, this will be positive and therefore, this term is greater than 1. If I am considering t to be negative than e to the power t will be less than 1 therefore, this term will become negative and since lambda is positive, this term will become less than 1; actually it will be between 0 and 1. So, this is the way the moment generating function of the Poisson distribution behaves.

Now, we have already given you the way a Poisson distribution arises in natural process. So, if we are looking at number of arrivals or number of occurrences during a process which satisfy certain conditions then the distribution or the number of occurrences during a specified time interval or during a specified area or during a specified portion of space follows a Poisson distribution.

However, it has also connection with the distributions which arise out of Bernoullian trials. Let us consider say x following a binomial distribution with parameter n and p; let n tend to infinity p tend to 0 such that n p tends to some number say lambda; then the distribution of x converges to e to the power minus lambda lambda to the power x by x factorial. That means, the binomial distributions probability mass function converges to the mass function of the Poisson distribution under the condition that n tends to infinity p tends to 0 such that n p tends to lambda a physical interpretation of this is that in a sequence of Bernoullian trials. If n becomes very large then it means that that the probability of a single occurrence becomes very small.

So, we were considering events such as probability of making a mistake in typing a certain section of a chapter probability of an occurrence of some accident at a particular traffic crossing. Here, you can consider this as Bernoullian trial in the sense that happening of an accident or not happening of an accident. So, there are 1000 of vehicle passing and one of them may be involved in the accident. Therefore, what will happen is that the number of trials… suppose, 1001 vehicles are there and 1 accident takes place.

So, may be 1 or 2 of the vehicles are involved in the accident. So, the probability of 1 vehicle meeting with an accident that is p is very small as compared to n.

However, there will be a fixed proportion of the number of occurrences which we call the rate. So, this seems logical that the binomial distribution should converge to Poisson distribution. let us look at a proof of this fact we may consider the p x and it is equal to n c x p to the power x 1 minus p to the power n minus x now here the limit process involves n and p; however, since n p itself converges to lambda; that means, in the limit there is a relation between n and p. So, we can write it as n factorial divided by x factorial n minus x factorial and this p we can write as lambda by n 1 minus lambda by n to the power. So, this is already I have taken the approximation of p as lambda by n because n p converges to lambda. So, p can be replaced by lambda by n and in the long run. Once again, in order to take the limit we have to look at since n tending to infinity. So, here factorials are involved. So, we have to simplify these terms.

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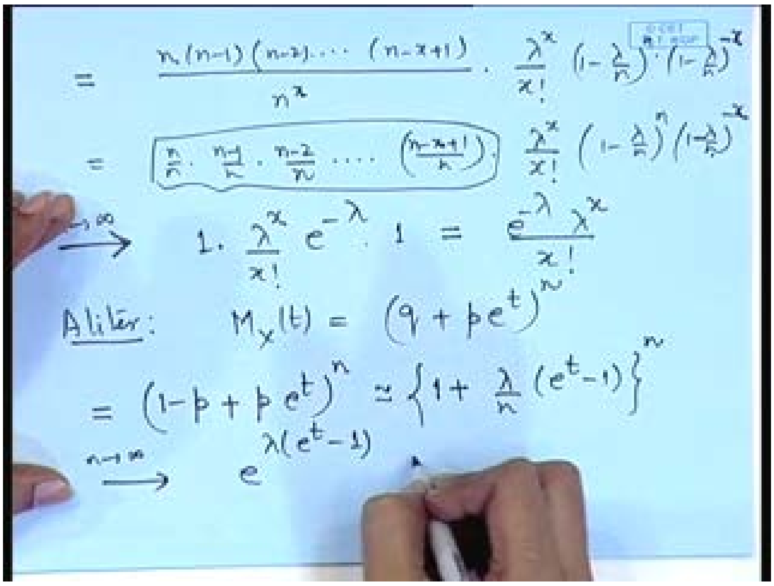

We can write it as n into n minus 1, n minus 2, up to n minus x plus 1. So, these terms has mean at this step in the denominator we have n to the power x. So, this term we put here. So, we are left with lambda to the x by x factorial; then, we have 1 minus lambda by n to the power n and 1minus lambda by n to the power minus x.

If we look at this ratio these are x terms each of the terms are divided by n. So, here you look at the first term that is n by n second term is n minus 1 by n which goes to 1 as n tends to infinity. The next term is n minus 2 by n as n tends to infinity this also goes to 1 and so on. And, n minus x plus 1 by n as n tends to infinity; this also goes to 1 because x is a fixed number between 0 1 2 and therefore, for a fixed x as n tends to infinity n minus x plus 1 by n will go to 1.

Then, we have lambda to the power x by x factorial 1 minus lambda by n to the power n 1 minus lambda by n to the power minus x. So, when we take the limit as n tends to infinity, this entire block if this converges to 1 then, lambda to the power x by x factorial then 1 minus lambda by n to the power n converges to e to the power minus lambda and here x is fixed. Therefore, lambda by n goes to 0 and this term goes to 1. So, this is nothing, but the probability mass function of a Poisson distribution with parameter lambda.

This proof can also be given using moment generating function. If we look at the moment generating function of the binomial distribution, that is, q plus p e to the power t to the power n. This we can consider as 1 minus p plus p e to the power t to the power n. Now, here you notice that the p and n both are involved here and we have to take the limit. Therefore, we can replace p as approximately lambda by n. So, this becomes 1 plus lambda by n e to the power t minus 1. So, when we take the limit as n tends to infinity this goes to e to the power lambda e to the power t minus 1. So, by the uniqueness of the moment generating function, we can say that the probability mass function of the binomial distribution converges to the probability mass function of the Poisson distribution.

So, if we are having a certain problem to be solved for binomial distribution where n is large, p is small such that, n p converges to a fixed number lambda then, we may make use of the Poisson approximation; let us consider one application here.

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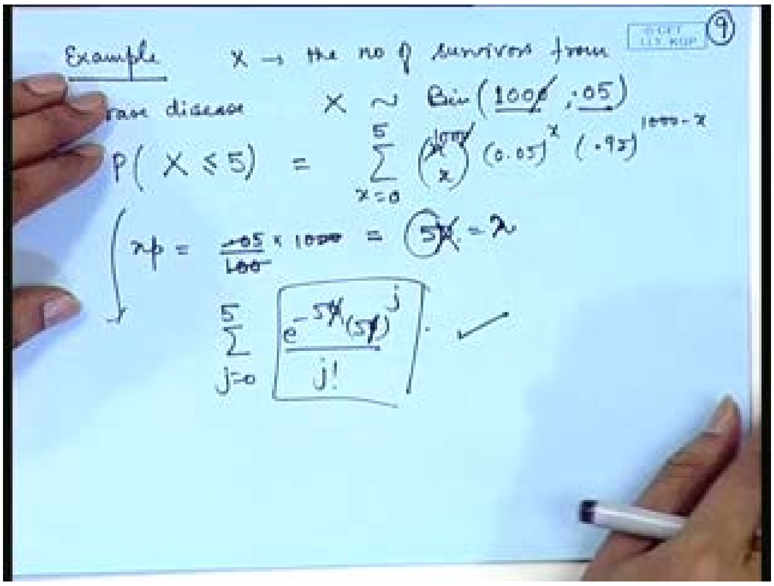

Let x denote the number of survivors from a rare disease the probability of survival is say point 05 and out of 1000 patients. What is the probability that say, x is less than or equal to 5? Now, if we directly want to calculate this 1 then it is equal to n c x point 05 to the power x point 95 to the power 1 1000 minus x here n is 1000 x is equal to 0 to 5. So, if you observe this term it can be calculated using certain extensive calculations, but this involves having approximations because point 95 to the power say 1000 point 95 to the power 993, etcetera. So, this will lead to lot of computational errors.

However, here if we observe n is large and p is small. So, here n p that is point 05 into1000. So, this can be considered as that is lambda. So, we may make use of e to the power minus 50, 50 to the power j by j factorial j is equal to 0 to 5. Suppose this n was 100 in place of 1000 then, this will become slightly simpler this will be 5 here and this will be 5 and this can be easily evaluated.

In fact, we will talk about this also that when lambda is large in a Poisson distribution then what happens when we discuss normal distribution. So, far we have discussed various distributions and many of them we gave direct origins that what kind of experiments lead to those distributions. Apart from that, there are certain distributions such as the distributions which can be represented as a power series because you observed say, geometric distribution where you are getting a term q to the power j minus 1 into p where j is from 1 to infinity. So, this is something like the power series in negative binomial distribution also and then, you have a finite polynomial sums like in binomial distribution or in Poisson distribution, you have certain term e to the power n then you have power series. So, there is a general family of power series distributions; that we will talk little later.

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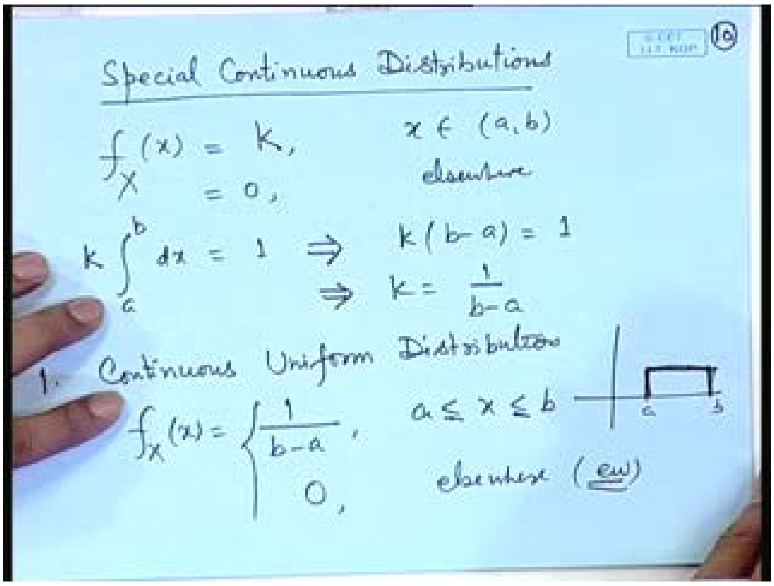

Firstly, let us discuss some special continuous distributions as in the case of discrete distribution. The simplest example of a continuous distribution could be where the density is uniform or density is a constant. For example, you consider a rod with a uniform weight; for example, you consider this sheet and here the sheet, it is the thickness or say weight of this sheet at every point will be same. So, if we consider say this portion of the pen then here in this portion the density the weight the thickness or the width is constant.

So, if the density is a constant over a certain interval say a to b and 0 elsewhere, we call it uniform distribution. Now, what should be the value of the constant? That can be determined by the condition that the density must be non-negative and the integral of the density must give you 1. So, k must be non-negative and integral of k from a to b must be 1 this means k into b minus a is equal to 1; that means, k is equal to 1 by b minus a.

A continuous uniform distribution has the probability density function given by 1 by b minus a; a is less than x less than b 0 for x lying outside this. we will use the notation e w for writing elsewhere ;of course, here it is immaterial whether we use strict inequalities or we may use equalities at some points less than or equal to because, the probability of a point is 0. So, in a continuous distribution inclusion or exclusion of a point does not make any difference.

So, if you look at the shape of these distributions suppose a and b is here then, 1 by b minus a is this. So, it is a, you can say a plateau kind of thing the continuous uniform distribution we may look at some of the properties here.

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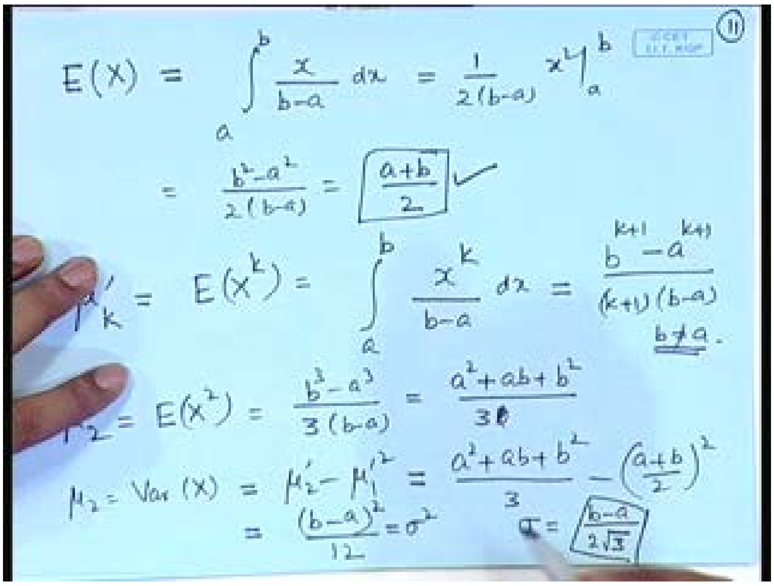

What is the first moment that is expectation of X? So, it is equal to integral x divided by b minus a d x from a to b. So, that is half b minus a x square from a to b. So, that gives us b square minus a square divided by twice b minus a that is equal to b plus a by 2. You can easily see that it is the mid-point of the distribution which is understandable because, the density is constant and therefore, the average value must be the mid-point.

Now, since it is a finite interval we can look at the moment of any order here. The moment of any order will exist x to the power k divided by b minus a d x. So, this is equal to b to the power k plus 1 minus a to the power k plus 1 divided by k plus 1 into b minus a if you have b is equal to a then it will not be a continuous distribution. So, in particular mu 2 is equal to expectation of mu 2 prime that is expectation of X square that will be equal to b cube minus a cube divided by 3 into b minus a that is a square plus a b plus b square divided by 3 into divided by 3. So, the variance of the uniform distribution that is mu 2 prime minus mu 1 prime square that is, a square plus a b plus b square by 3 minus a plus b whole square by 4. So, after some simplification a plus b by 2 whole square, you can make simplification here. This turns out to be b minus a whole square by 12 that is sigma square.

So, the standard deviation of this distribution is b minus a divided by root 12 that is 2 root 3. So, you can see here the range of the distribution is b minus a and the variability is b minus a divided by 2 root 3 which is slightly you can say much less because 2 into root 3 is root 3 is 1 point 71. So, this becomes b minus a divided by 3 point 4 2 kind of thing. So, this is much smaller than the range and it is because of the uniformity that the variability is much smaller.

However, if b and a are far apart; that means, if b minus a is a large number then even though after division the variability will be high. So, for example, you have too much of flatness say, a to b and the value of this is like this 1 by b minus a. Basically, it will become much smaller here if you plot 1 by b minus a here. So, that shows that the variability increases if the range of the distribution increases.

One may also look at mu 3 and mu 4 and calculate the measures of skewness and kurtosis. Obviously, it is a symmetric distribution; therefore, measure of skewnesss will be 0. However, measure of kurtosis will depend upon the difference between a and b as you have seen here. If the difference between a and b is smaller than the density will be plotted at a higher this one if the difference between a and b is large then it will be quite flat. So, mu 4 will be dependent upon the value of b and a.

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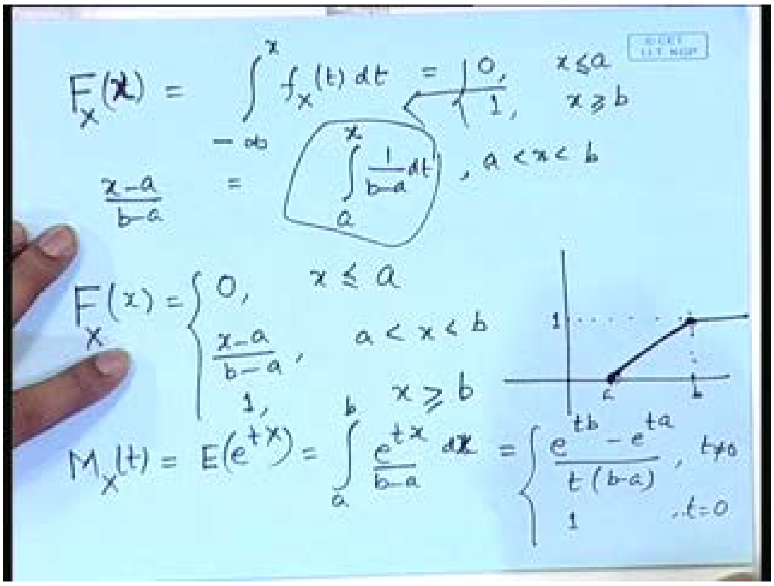

We also look at the plotting of the cumulative distribution function that is integral minus infinity to x f x t d t. Since, the distribution is 0 up to a and beyond b that means, this is going to be 0 for x less than a and it is going to be 1 for x greater than b you may put equality here also it does not make any difference. However, if I am considering x to be between a and b then it is integral from a to x of 1 by b minus a d t and this value is equal to x minus a by b minus a. Therefore, the c d f can be written as 0 for x less than or equal to a x minus a by b minus a for a less than x less than b and 1 for x greater than or equal to b.

You can easily observe that the function is absolutely continuous. The derivative will give you the density and the end points a and b of the intervals the function is continuous. If we plot this suppose a is here and b is here. So, upto a it is 0 from a to b. So, suppose this is the value 1 here. So, it is a line here and there after it becomes 1. So, it is a straight line joining the point 0 to 1 here.

We may also look at the moment generating function e to the power t x d x divided by b minus a from a to b. So, it becomes e to the power t b minus e to the power t a divided by t b minus a; obviously, at t is equal to 0 this is not defined, but at t is equal to 0 this is 1. So, this is t not equal to 0 and t equal to 0.

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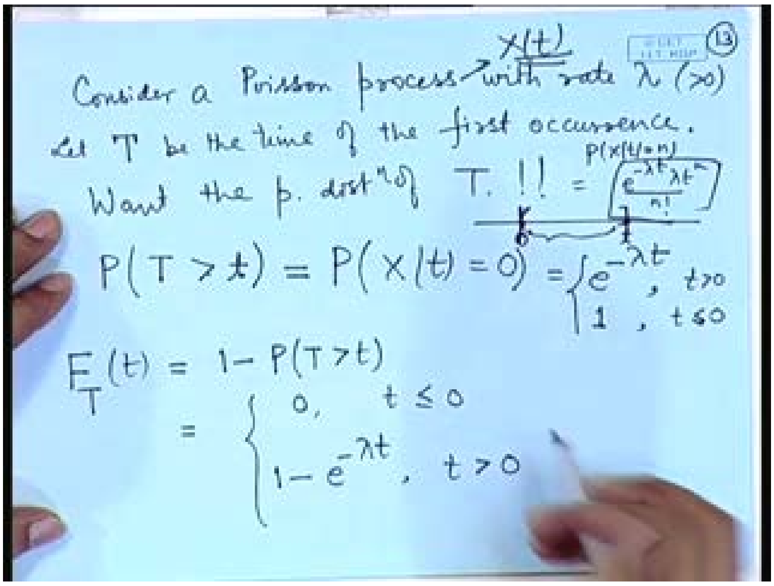

Let us consider a Poisson process. Consider a Poisson process with rate lambda; of course, lambda is greater than 0. Let us denote by say t be the time of the first occurrence. So, we know that it is a Poisson process the number of occurrences will follow a Poisson distribution. So, at some point of time we start observing the process. For example, we go and stand at a ticket counter of certain cinema and then we want to see when the first customer arrives.

We are, suppose I am a traffic police person and I go to the designated traffic crossing. Now, when I am standing there, I start observing when is the first accident taking place? So, if we consider t as the time of the first occurrence from the time when we start observing then t is a continuous random variable because it is the time. So, we want the probability distribution of t what is the distribution of t. So, we can look at an event say probability t greater than say small t what does it mean; that means, if we are observing the process from certain time t then up to the time small t if we are starting from 0 up to time small t that event has not taken place; that means, in the interval 0 to t the number f occurrences is 0. So, if we are considering this Poisson process as X t then, the event capital t greater than small t is equivalent to probability that X t is equal to 0 because if there is no occurrence in the interval 0 to t then capital T is definitely going to be greater than small t and vice versa. So, these 2 events are same.

However, the distribution of X t is assumed as a Poisson distribution because, we have made the assumption here that it is a Poisson process with rate lambda. That means, probability X t is equal to n is e to the power minus lambda t lambda to the power n by n factorial. So, here if I put n is equal to 0 then this is giving me e to the power minus lambda t. So, of course, this statement is true if t is greater than 0 because, we are observing from certain time onwards. So, this is for t greater than 0 this probability will be 1 if t is less than or equal to 0.

So, if we consider the cumulative distribution function of t that is capital F of t it is 1 minus probability of T greater than t. So, from this derivation it is equal to 0 for t less than or equal to 0 and it is 1 minus e to the power minus lambda t for t greater than 0. Now, you can observe here we are able to derive the p d f of this continuous distribution and it is an absolutely continuous function. So, if we differentiate this we can get the density function of the time for the first occurrence during the Poisson process.

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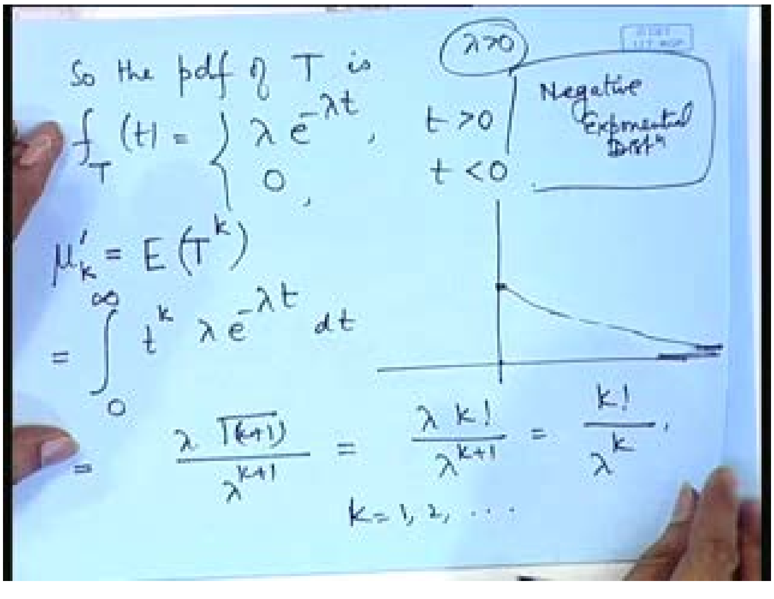

The probability density function of t is f t equal to lambda e to the power minus lambda t for t greater than 0 and it is equal to 0 for t less than or equal to 0; this is known as negative exponential distribution. So, this is a continuous distribution and this distribution arises as the distribution of the waiting time in a Poisson process for the first occurrence of the event.

Whatever occurrence we are trying to observe, naturally we would like to look at the properties of these distributions. So, for example, the shape of the distribution at t equal to 0 it is 0 up to 0 and at t is equal to 0 the value is equal to lambda and then, e to the power minus lambda t because lambda is positive. So, this will be less than 1 and this will be a decreasing function because and you will have decreasing to 0 as t tends to infinity. Let us look at its moments, etcetera.

So, if we consider a general moment of the k th order, it is ( ) of t to the power k that is equal to t to the power k lambda e to the power minus lambda t d t 0 to infinity now this is nothing, but a gamma function. So, this becomes lambda gamma k plus 1 divided by lambda to the power k plus 1. This is equal to lambda k factorial divided by lambda to the power k plus 1 that is k factorial divided by lambda to the power k for k equal to 1 2 and so on.

Of course, here you can see that if k was 0 then actually it was nothing but the integral of the density which would have been 1. So, the moments of all positive order exist here; in particular we can calculate mean, variance, etcetera.

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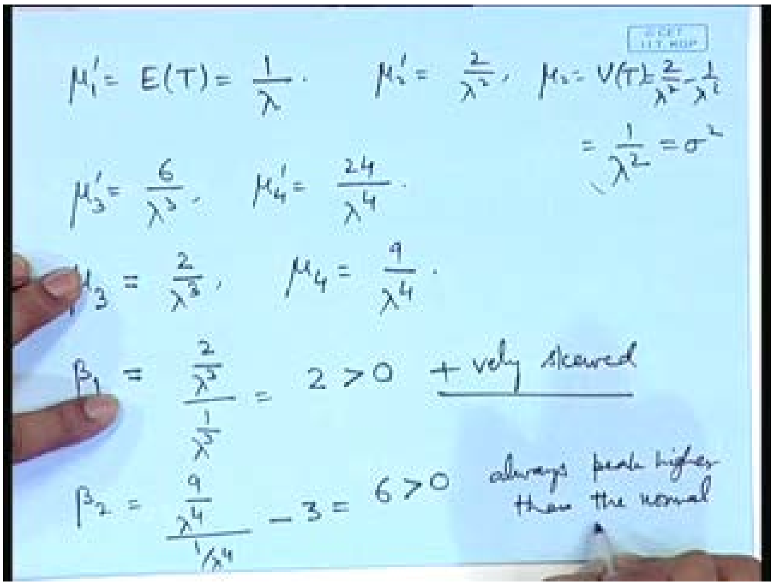

So, for example, mu 1 prime that is the mean of this distribution is equal to, if I put k equal to 1, here I will get 1 by lambda. So, if the rate of occurrence is lambda upper unit of time then the waiting time the average waiting time for the first occurrence is 1 by lambda, which is very natural to understand.

Suppose I say that, in 1 hour 2 events will occur. So, roughly average waiting time for first occurrence will be thirty minutes we may also look at mu 2 prime that is equal to 2 by lambda square and therefore, mu 2 that is the variance will become 2 by lambda square minus 1 by lambda square that is equal to 1 by lambda square. So, you can observe here the variance of the exponential distribution is square of the mean.

We may calculate mu 3 prime that is equal to 6 by lambda cube mu 4 prime will become equal to 24 by lambda to the power 4, using the relationship between the non-central and central moments mu 3 is equal to 2 by lambda cube and mu 4 is equal to 9 by lambda to the power 4. From here we can calculate the measures of skewness and kurtosis. So, beta 1 is equal to 2 by lambda cube that is mu 3 divided by sigma cube. So, this is sigma square. So, this becomes 1 by lambda cube that is equal to 2 which is always positive; that means, no matter what the value of lambda is the exp 1ntial distribution is always positively skewed. So, this you can see from the shape of the distribution also because here it is a constant.

Similarly, if we look at say, beta 2; beta 2 is equal to nine by lambda to the power 4 that is mu 4 divided by mu 2 square that is divided by 1 by lambda to the power 4 minus 3 that is equal to 6. So, no matter what the value of lambda is it is always having peak higher than the normal peak. So, this is always positively skewed and always peak high that is higher than the normal peak now here you observe this is slightly different from the earlier distributions in binomial distribution in Poisson distribution in geometric distribution, etcetera. The condition for the skewness and the kurtosis was dependent upon the parameter. That means, like in the binomial distribution if p was half the distribution was symmetric if p is less than half it was positively skewed if p was greater than half it was negatively skewed. Similarly, if p q was less than 1 by 6 it was leptokurtic if p q was greater than 1 by 6 it was platykurtic, etcetera.

Unlike those distributions, same thing we observe in the Poisson also. In the Poisson distribution the measure of skewness was 1 by lambda. So, it was positively skewed; but as lambda becomes large it approaches the symmetry. Similarly, the measure of kurtosis was also having lambda in the denominator. But, here these measures beta 1 and beta 2 are constant and they will always have the same behavior; that is, positively skewed and the peak higher than the normal peak. In the next lectures we will be considering extension of this concept of the time because, here the negative exponential distribution is obtained as distribution of time for certain occurrence. So, we will consider extension of these concepts in the next lecture; thank you.