Regression Analysis
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Lecture - 29
Polynomial Regression Models (Contd.).

Hi, this is my 3rd lecture, on polynomial regression and here, is the content of this topic

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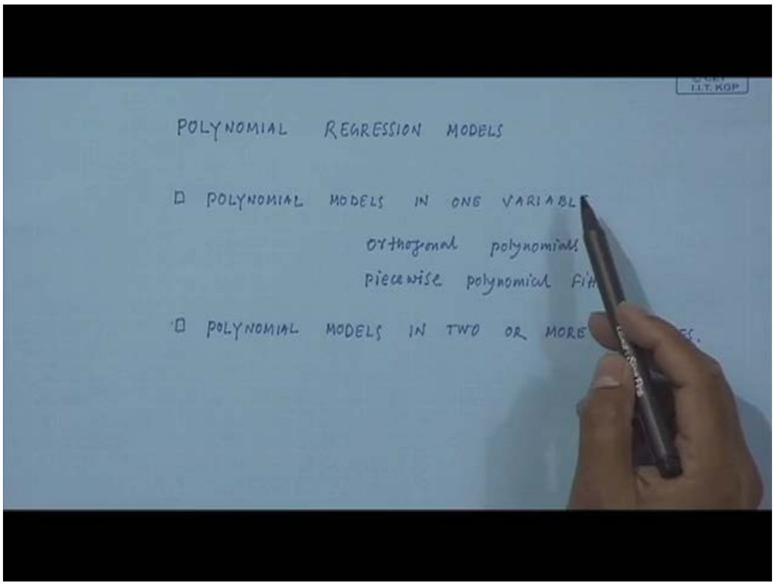

So, polynomial models in one variable orthogonal polynomials, piecewise polynomial fitting and polynomial models in two or more variables. So we will be talking, about this polynomial in two or more variable today. So, in the previous classes, we have studied polynomial in one variable and we know that, we know polynomial are used in situation when the response of variable is non-linear and we have studied, how to fit a k-th order polynomial using orthogonal polynomials and also we have studied piecewise polynomial fitting. So, piecewise polynomial fitting is used in situation, when a lower order polynomial does not fit the given data properly. But, increasing the order of the polynomial does not improve the situation substantially. So, this indicates that, the response function it behaves you know differently in different part of the range of x.

So, what we do in a common approach to deal with such situation is that, we divide the range of x into several segments and we fit an appropriate of each segments. So, we talked about all this things know in the previous classes, today will be talking about polynomial models in two variables, ok.

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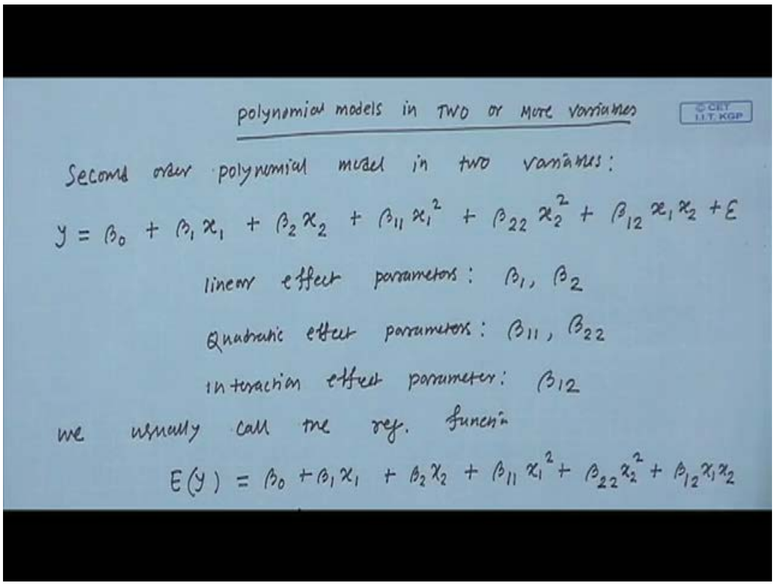

So, polynomial model in two or more variable, so second order polynomial model in two variables, is a y equal to beta naught plus beta 1 x 1 plus beta 2 x 2 plus beta 1 1 x 1 square plus beta 2 2 x 2 square plus beta 1 2 x 1 x 2 plus epsilon. So, this is 2nd order polynomial model in two variables ok. So, here the linear effect the parameters are beta 1 and beta 2 and then, quadratic effect parameters, are beta 1 1 and beta 2 2 beta 1 1 beta 2 2 and then, the interaction effect parameter is beta 1 2.

So, here we usually call the regression function, expectation of y is equal to beta naught plus beta 1 x 1 plus beta 2 x 2 plus beta 1 1 x 1 square plus beta 2 2 x 2 square plus beta 1 2 x 1 x 2. So, this is called response surface and so, this response surface is used in industry to for modeling the response variable, in terms of controlled variable like in a regression variable. So, they have in huge application in the industry. So, we will be talking now, how to fit 2nd order polynomial model in two variables.

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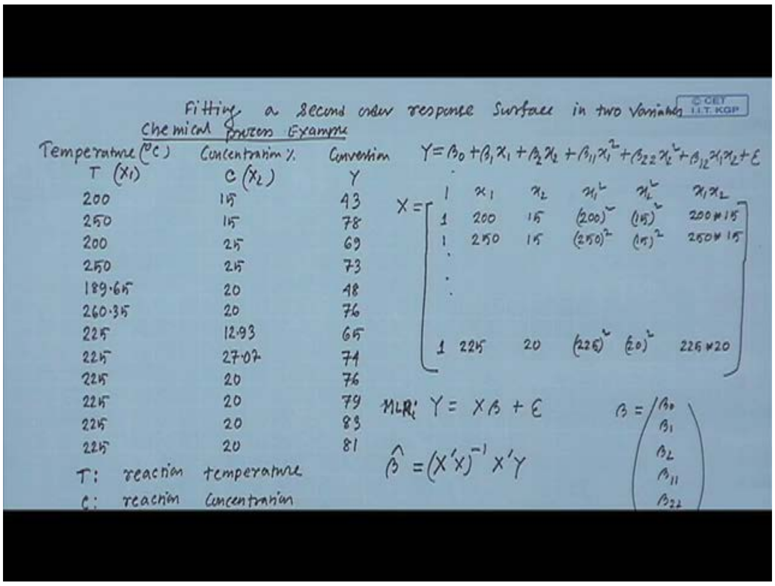

So, for that I will give an example, I will talk about, the fitting of 2nd order polynomial model in two variable using an examples here. So, this is you know chemical process example and here expand is regressor variable, x 1 and x 2 are regressor variable. So, x 1 stands for the temperature regression temperature and x 2 a stands for concentration and the response variable Y, it stands for percent conversion of a chemical process, ok. So, we have 2 regressor variable and we have 1 response variable and we have given some data, we have to fit a model like this, ok. This is the 2nd polynomial model in two variable x 1 and x 2. So, anyway I mean this one is nothing but, multiple linear regression model Y equal to X beta plus epsilon.

So, here X is the co efficient matrix or design matrix something we say. So, here the 1st column is correspond to beta naught I mean or say x naught here and then, the 2nd column is correspond to x 1 and then, x 2 then x 1 square x 2 square and then x 1 x 2 then how do you get this matrix, we are given x 1 here so, the 1st column is corresponds to this x 1 values your given the x 2 values also. So, 2nd column or the column associated with x 2 is correspond to this, columns and then, you can compute x 1 square you can compute x 2 square and you can compute x 1 into x 2 so, 200 into 15 for the 1st observation. So, this is how you get the co efficient matrix x and then, this one is same as you know multiple linear regression Y equal to X beta plus epsilon, where beta is beta naught beta 1, beta 2, beta 1 1, beta 2 2 and then beta 1 2.

So, we have to estimate this co efficient and you know how to do that, so beta hat is nothing but, X prime X inverse X prime Y. So, you know X matrix you know Y so, you can compute beta hat right.

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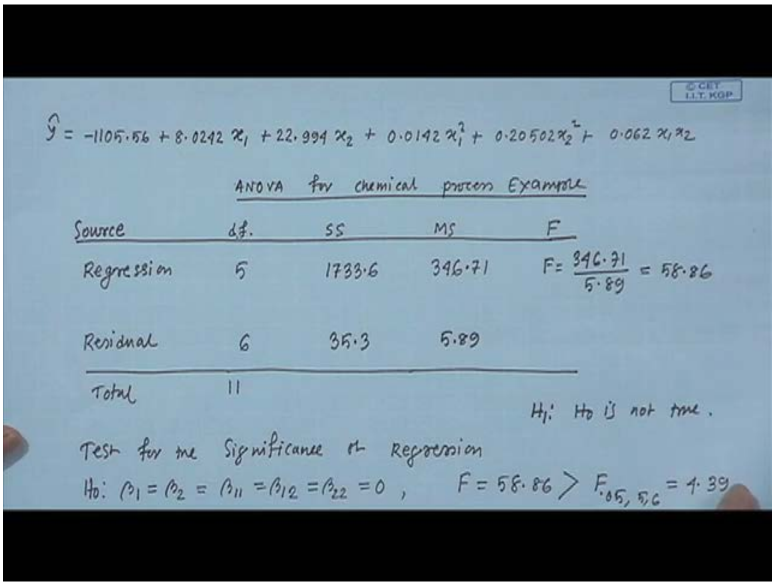

So, here is the fitted model now. So, these are the, this is beta naught hat and this is this is the fitted model for this chemical process example and then, here is the Anova table this one. So, we had their 12 observations. So, that is why S S total has degree of freedom 11 and as you can see here, that there are 6 parameters 1, 2, 3, 4, 5 and 6 and that is why you will have 6 restriction on residuals. So, there are total 12 residuals, because there are 12 observation and on this 12 residuals e i you have 6 restriction. So, you have the freedom of the using 6 residuals independently and then, the remaining have to be chosen in such way that they satisfies those restrictions.

So, that is why the residual degree of freedom is 12 minus 6 which is 6 again and the regression degree of freedom is 5, ok. So, you know how to compute this residual, you know how to compute this regression, S S regression ok. So, this S S regression it involve sort of S S regression due to beta 1 plus S S regression due to beta 2 plus S S regression due to beta 1 1, beta 2 2 plus beta 1 2. So, this is the total S S regression and then, you have the M S residual here and here is the F statistic what does is that it, this F statistic is used to test the significance of this model whether, this model is significant that means the parameters are significant.

So, test for the significant of regression, what about you have fitted which is same as testing the hypothesis that, beta 1 equal to beta 2 equal to 1 1 equal to beta 2 2 equal to beta 1 2 is equal to 0. So, all of them 0 means the null hypothesis says that, the regression fit is not significant and the alternative hypothesis h naught here h sorry hypotheses h 1 that is no h naught is not true, that means the fit is significant, ok. So, you have to test null hypothesis, you have the F statistic and which has value 58. 86. And now, this follows F with degree of freedom 5 6 and you get the tabulated value from the F table that is 4.39. So, the observed value is greater than, the tabulated value that means the h naught is rejected, which says that the regression fit is significant, ok.

So, overall the whatever, model is fitted 2nd order polynomial involving the two variables. That model is significant now, what we are going to do is that, we are we will be testing whether what you have contributions of the linear terms, what is the contribution of beta 1 and beta 2. So, that is what we will test the significance of the linear terms, in terms of beta 1 and beta 2 and then, will be testing the significance of the quadratic terms, ok.

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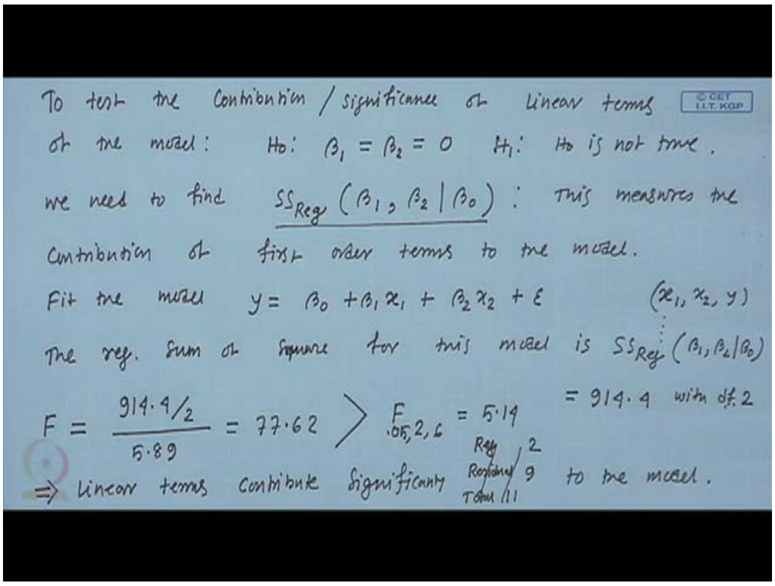

So, to test the contribution or significance of linear terms of the model. What you have to test? To test null hypothesis H naught that, beta 1 equal to beta 2 equal to 0. Because beta 1 is the coefficient of x 1 and beta 2 is the coefficient of x 2. So, if this is true that means if the null hypotheses, is true then the contribution of the linear term is not significant against the alternative hypothesis H 1 that, H naught is not true ok. So, to test this one, we need to find S S regression due to beta 1 and beta 2. So, this is the contribution of beta 1 and beta 2 in total S S regression. So, S S regression due to beta 1 and beta 2, in the presence of beta naught.

So, this measures the contribution of 1st order terms to the model, ok. So, how to get this one is that, you fit a model y equal to beta naught plus beta 1 x 1 plus beta 2 x 2 plus epsilon. So, you fit this model to your given data x 1, x 2, y so, you are given x 1, x 2, y for you have several observation x 1, x 2 and y here specifically you have 12 observations on x 1, x 2 and y .So, you. fit this model on the given observations and then, the regression sum of square for this model is basically this quantity, S S regression beta 1, beta 2 given beta naught. So, how to get S S regression due to beta 1 and beta 2 given that is beta naught is in the model. So, basically to get this S S regression you have to fit this model and then, you find the S S regressions for this model, ok. That S S regression for this model, you know how to do that right?

So, S S regression for this model, is same as S S regression due to beta 1, beta 2 in the presence of beta naught, for the model we considered like you know 2nd order model involving two variables, ok. So, this can be found that, this equal to 914.4 with degree of freedom 2, maybe I will explain why it is 2. So, the F statistics is so, why it is 2 you can construct Anova table for this one. So, the total degree of freedom is 11 that is total and then, the residual has a degree of freedom 12 minus 3 that is 9. That is why the regression degree of freedom is to 2 for this model, ok. So, the F statistic is 914.4 by 2 by 5.89, ok. So but, this test is for the full model and the M S residual for the full model, is 5.89.

So, this part is just to explain how to get S S regression due to this, in the presence of beta naught. So, this is the M S residual and this is equal to 77.62 and you now, this follows F distribution with degree of freedom 2 and 6 right. So, find the tabulated value 0.0526 from the table that is equal to 5.14. So, you see that, the observed value is greater than, the tabulated value. So, which implies that, H naught is. Rejected and H 1 is accepted. So, H naught is rejected means, beta 1 and beta 2, are not equal to 0 that means a linear term distribution is significant. So, it implies linear terms contribute significantly to the model, ok. So, we observed that, the contribution of linear terms is significant to the model.

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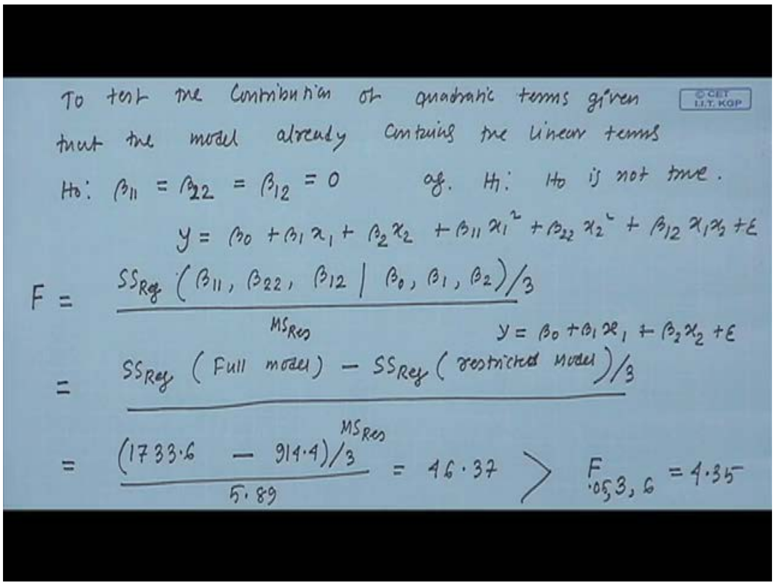

Now, we test for the significance of or what the contribution of the quadric term, to test the contribution of quadratic terms given that, the model already contains the linear term. To test this thing, the contribution of quadric terms given that, the model already contains the linear term, we have to test the hypothesis that, beta 1 1 is equal to beta 2 2 is equal to beta 1 2 is 0 against the alternative hypotheses h 1, that h naught is not true, ok. So, I hope that you can recall the model so, the model you are considering is y equal to beta naught plus beta 1 x plus beta 2 sorry beta 1 x 1 plus beta 2 x 2 plus beta.1 1 x 1 square plus beta 2 2 x 2 square plus beta 1 to x 1 x 2 plus epsilon so, this is the full model. Now, you know how to test this hypothesis using the technique of extra some of squares right.

So, the F statistic for testing this hypotheses is equal to, I will use the notation that S S regression due to beta 1 1, beta 2 2, beta 1 2. So, the S S regression due to beta 1 1, beta 2 2 and beta 1 2 in the presence of the linear model. Linear terms in the model that is beta naught, beta 1 and beta 2. This is, what we want to test I mean this is the notation for s s regression due to this, quadratic term in the presence of linear term by M S residual. And of course, I need to divide this by degree of freedom that is 3. I will explain why it is 3. Now, this one is this regression, S S regression due to this quadratic term in the presence of linear terms; this can be computed using the extra some of square techniques.

So, what we have to do it that you compute S S regression for the full model right. So, you compute the S S regression for the full model. This is the full model and you already have that, you have the Anova table for this one ok. So, this is the S S regression for the full model right. Now, what I will do is that now, this minus S S regression for the restricted model. What is my restricted model? My restricted model is, the model under H naught. That is y equal to beta naught plus beta 1 x 1 plus beta 2 x 2 plus epsilon. So, S S regression due to this model is nothing but, S S regression under the restricted model, ok by M S residual.

Now, you can just now, we have both the things. We have S S regression for the full model, from the Anova table. That is 1733.6 and for the restricted model also, we have it that is just now, you computed that is 914.4. Now, this has degree of freedom 5 and this has degree of freedom 2. So, the difference is 3 that is why you have to divide it by 3. So, this by 3 by M S residual is 5.89 which is equal to 46.37. Now, you check the tabulated value, this F follows, F distribution with degree of freedom 3, 6, 0.05 level of significance. So, this one is nothing, this is you get this value table from the F table that is 4.35. So, you can see the observed value is greater than the tabulated value, which implies that the null hypothesis is rejected that means, this coefficients are significance.

So, the final conclusion is that so, this implies that, the quadratic terms contribute significantly to the model, ok. So, we have observed that, you know while so, what we have done is that. We have studied it, how to fit a 2nd order polynomial model is involving two variables and then, 1st we computed you will fitted this model and then, we computed the Anova table for full model. And then, we observed that the model the significant by the F test and then, once the model is significant that means all the regression coefficients are not equal to 0, some of them are non-zero and then, what we did is that we tested the significance of the linear term separately and we also tested the significance of quadratic term.

And we, found that for this particular example, both the linear terms and the quadratic terms are significant. So, you cannot remove any term from the model so, the model is quite a significant. So, that is all about the 2nd order polynomial fitting involving two variables and now we will solve some problems on orthogonal polynomials so, where is the problem.

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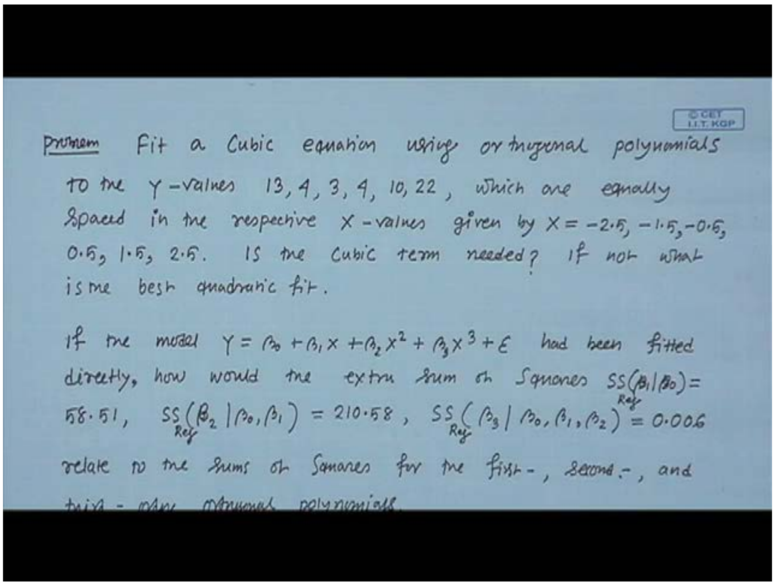

So, fit a cubic equation using the orthogonal polynomials to the Y values the values are 13 4 3 4 10 and 22. So, we have 6 observations for Y corresponds to the X value minus 2. 5 minus 1.5 so on and you can see that, the x values are equally space. So, the question is we are asking to fit cubic equation so, is the cubic term needed? If not, what is the best quadratic, ok? So, to solve this problem 1st what we do is that we have to fit a 3 degree polynomial, involving I mean using orthogonal polynomial, because orthogonal polynomial has some advantage and then, will test the significance of the quadratic significant of the cubic term ok.

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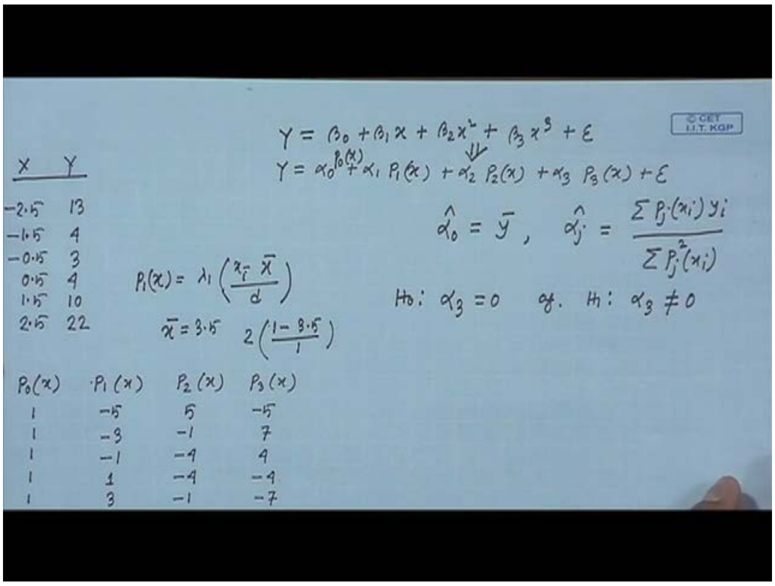

So, I wrote the observations here again. So, this is my X and Y and I have 6 observations and the question is to fit a cubic model. So, basically you have to fit Y equal to beta naught plus beta 1 x plus beta 2 x square plus beta 3 x cube plus epsilon. You have to fit this model and then, we know that you know instead of fitting, I mean we know how to fit orthogonal polynomial to fit, 3rd order polynomial. So, instead of fitting this model, we fit this one so, this one is this is also 3rd order but, involving orthogonal polynomial. So, this is orthogonal polynomial of order 3, order 2, order 1 and we have 6 observations. So, you know how to compute this orthogonal polynomial for 6 observations so, p naught here, you have p naught x.

So, you know the p naught x is always equal to 1 for all x and p 1 x is a minus 5 minus 3 minus 1 1 3 5 right? I hope that you can recall that p 1 x is equal to lambda 1 into x minus x i minus x bar by d. So, here I can check that x bar is equal to 3.5, I mean no need to consider this value, we can just replace them by 1, 2, 3, 4, 5, 6 because, they all equally spaced. So, you can code them by 1, 2, 3, 4, 5, 6. So, using those values my x bar is equal to 3.5 and then, for 1 it is 1 minus 3 .5 d is equal to 1 and I have to take lambda equal to 2 to make it integer so, this is minus 2.5 and I have multiply 2 to get minus 5 so, this is how you know to compute p 1 x and then, you see the formula p 2 x and p 3 x. Generally during the exam, you know this table is given so, you no need to memorize all these things.

So, what I want to do is that. Suppose well so, it says that fit a 3rd cubic equation and then, you know what is this alpha, how to estimate this alpha? We know that alpha naught hat is equal to y bar and alpha j hat is equal to P j x i into y i by P j x i square. So, you know everything so, if you want to compute alpha 1 hat if you have p 1 x you know y so, you can compute alpha 1 hat and similarly, alpha 2 hat and alpha 3 hat. So, that is not a problem now, the problem says that you know you just the significant of the problem says that, is the cubic term needed so, that means we have to test the hypothesis that, H naught alpha 3 equal to 0 against the H 1 that alpha 3 is not equal to 0, ok. So, how to do that, I think you 1st compute the Anova table for this one and then come back to testing this, because anyway you have to estimate the S S residuals.

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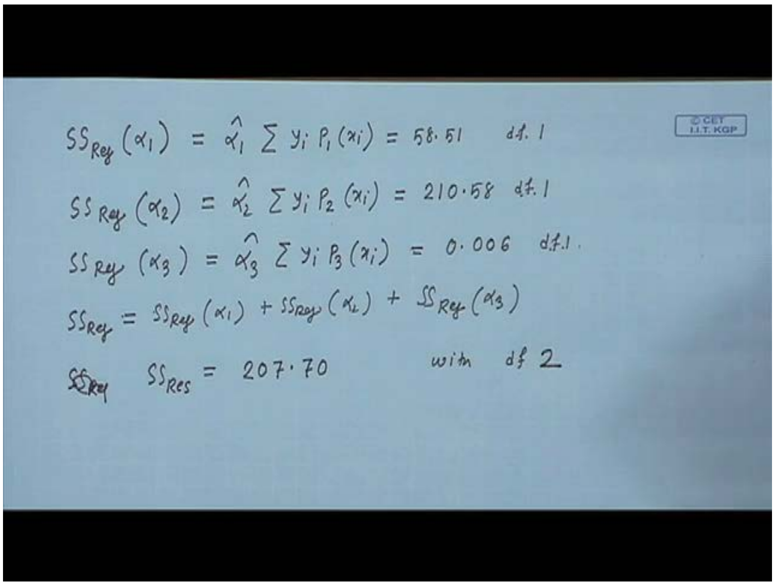

So, let me construct the Anova table 1st for that, I need S S regression due to alpha 1. So that one is nothing but, alpha 1 hat summation y i p 1 x i so, you can compute you know alpha 1 hat and you know y i s you know p 1 x i so, you can check that, this 1 is 58.51 with degree of freedom of course, 1. Similarly, you can compute S S regression due to alpha 2. That means the contribution of the quadratic term alpha 2 S S square right in the regression model. So, that is alpha 2 hat into y i p 2 x i this one is nothing but, 210 that you can check with degree of freedom 1 and then, s s regression due to alpha 3. This we need, because we need to test the hypothesis that alpha 3 is equal to 0 against alpha 0 not equal to 0 that is, alpha 3 hat summation y i p 3 x i right.

So, this one is very small. So, this clearly of course, will test it formally but, it clearly says that significance of alpha 3 is negligible with degree of freedom 1. So, if you add this 3 S S regression that will be, the S S regression. Total S S regression right and I mean, what I mean is that S S regression for the cubic model, involving orthogonal polynomial is nothing but, S S regression, is S S regression due to alpha 1 plus S S regression due to alpha 2 plus S S regression due to alpha 3 right. And then, you compute S S total and then, S S residual sorry S S residual can be obtained from the S S total minus S S regression that, you can check that this residual is 207.70 with degree of freedom 2. Why it is 2? Because we have 4 parameters in the model. So, 4 parameters mean 4 restrictions on the residual and there are total 6 residuals and you have 4 restrictions.

So, that means only 2 you can choose independently, you have the freedom of 2 and the other 4 have to be chosen in, such a way that 4 restrictions are satisfied. So, that is why the S S residual has degree of freedom 2. Now, I have the S S residuals so, I can to compute the F statistic to test this hypothesis is alpha 3 is equal to 0.

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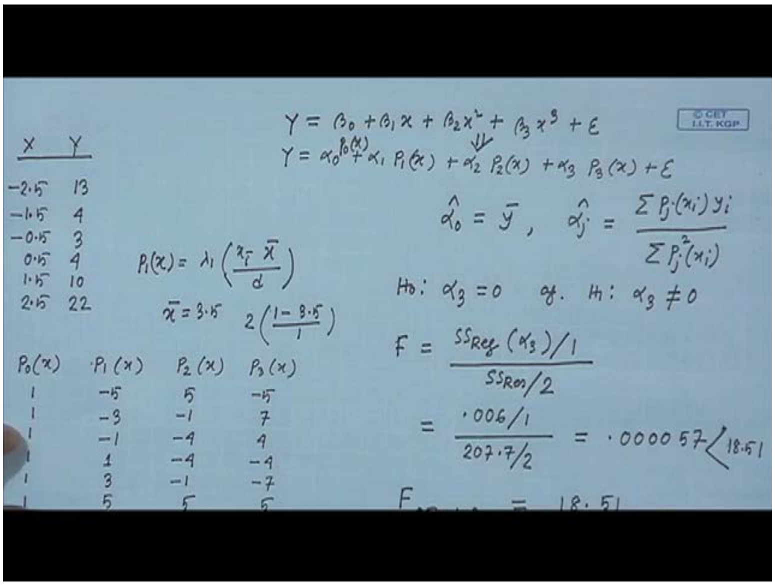

So, my F statistic is F is S S regression due to alpha 3 that, means the contribution of cubic term in total S S regression by its degree of freedom is equal 1 by S S residual divided by s degree of freedom that, is M S residual basically. So, this one is nothing but, 0.006 by 1 and S S residual you computed that, is 207.7 by 2, which is equal to which is very small 0.000057 and I mean this clearly says that, alpha 3 is not significant. So, still let me, find the value of tabulated F that, tabulated F 0.05 with degree of freedom 1 and 2 that is 18.51. So, this one is clearly very small very smaller than, 18.51. So, this F test implies that, you know H naught is accepted. That means alpha 3 can be 0 in the model. So, which in other words says that, alpha 3 is not significant.

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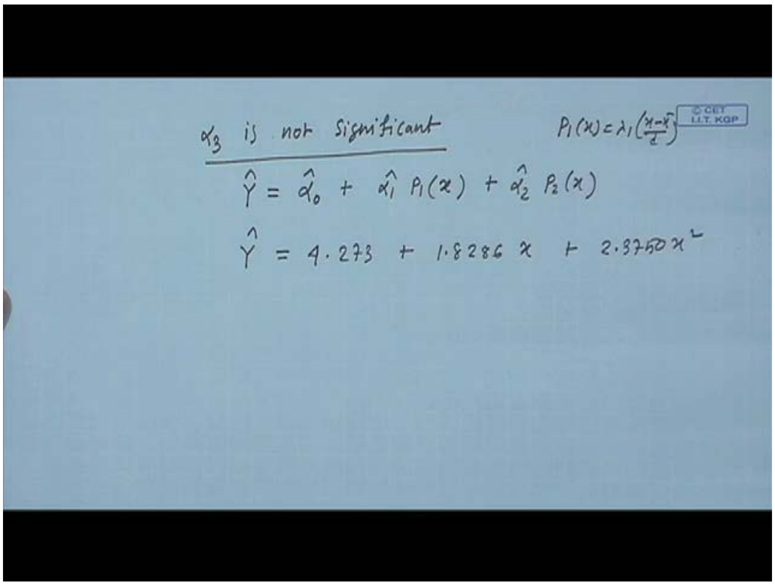

So, I can say here that alpha 3 is not significant so, not significant means I can go for the model y equal to alpha naught hat plus alpha 1 hat p 1 x plus alpha 2 hat p 2 x, I can ignore the cubic term. And finally, you can check that, this is y hat is equal to 4.273 plus 1.8286 x plus 2.3750 x squares see you know, this is not alpha 1 hat. What we have done here is that, you know you find alpha naught, alpha 1 hat and alpha 2 hat and then also you replace this, you write this equation in terms of x. Here is in terms of orthogonal polynomial so, you replace that p 1 x by that p 1 x is equal to lambda 1 x minus x bar by d. So, you can e lambda is equal to 2 d equal to 1 and so, finally, you have to get this equation in terms of x well.

So, the next problem is it says that so, this is the original problem. So, we tested the 3rd order term and we found that, the alpha 3 is not significant and we find the best quadratic fit. Now, if the model say this is the 3rd order polynomial had been fitted directly. How would the extra sum of squares S S regression due to beta 1, given beta naught which is equal to 58.51 and S S regression due to beta 2 in the presence of beta naught and beta 1 which is equal to 210.58 and S S regression due to beta 3 in the presence of beta naught, beta 1 and beta 2. How this things are related to the sum of squares for the 1st, 2nd and 3rd order orthogonal polynomial? So, you understand the problem.

So, what you done is that, you have fitted the 3rd order polynomial in terms of using orthogonal polynomial. Now, you can without using the orthogonal polynomial, you can use you can fit this model also. And then, the S S regression, you have the S S regression due to beta 1, in the presence of beta naught which is equal to 58.51. How this one is related to the S S regression of the 1st order polynomial? Involving orthogonal polynomials ok so, I hope you understood the problem. Now, if you see that this one is 58.51 and whatever, you computed before that S S regression due to alpha 1 is also 58.51 ok.

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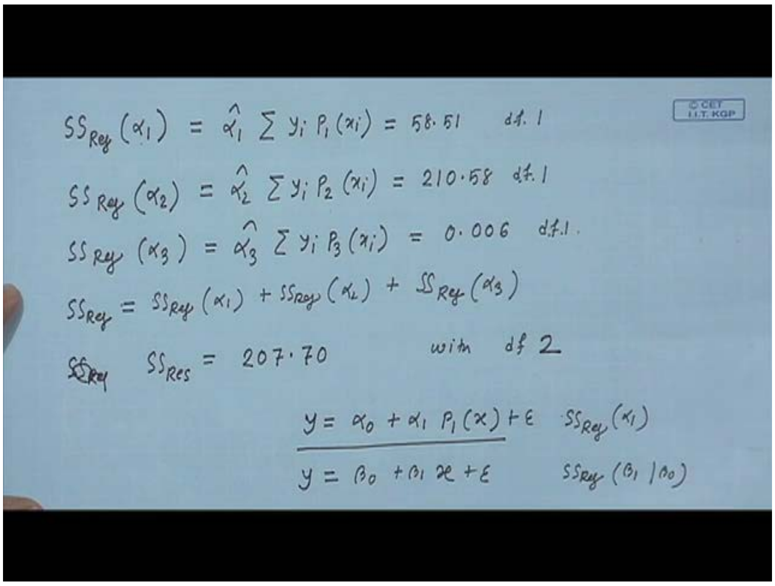

So, the S S regression for the model Y equal to alpha naught plus alpha 1, p 1 x the S S regression due to this model is same as S S regression due to the model beta naught plus beta 1 x plus epsilon plus epsilon. Because S S regression due to this model is nothing but, S S regression due to alpha 1 ok and S S regression due to this model is nothing but, S S regression due to beta 1 in the presence of beta naught. Now, see the question says how now, this one is related how S S regression due to beta 2 in the presence of beta naught and beta 1 that is 210.58. How this is related to a sum of square for 2nd order orthogonal polynomials? ok.

Now, we need to check so, this 2 quantity are same. This is same as basically S S regression due to beta 2, in the presence of beta naught and beta 1 which is same as S S regression due to alpha 2. So, what I want to, the message I want to give here is that now, I am talking about model of order 2.

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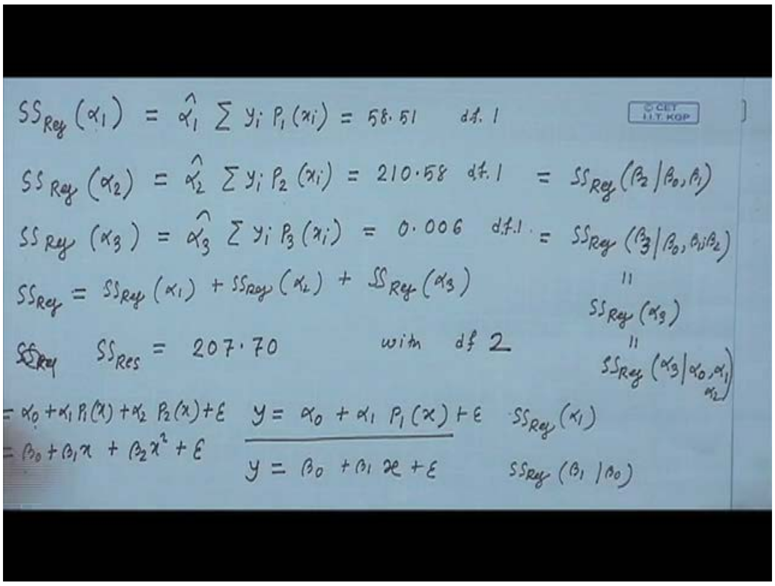

Let me, write down that y equal to alpha naught plus alpha 1, p 1 x plus alpha2, p 2 x. So, this is 2nd order model involving orthogonal polynomials plus epsilon. And let me write the polynomial 2nd order polynomial that is, beta naught plus beta 1 x plus beta 2 x square plus epsilon. Now, the contribution of what is this quantity? The contribution of beta 2 in S S regression, in the presence of beta naught and beta 1 is same as the contribution of alpha 2 in this model, in the presence of alpha naught plus and alpha 1. But, you are aware of the fact that you know, in case of orthogonal polynomial fitting. The S S regression are orthogonal, I mean s s regression due to alpha 2 does not depend on S S regression due to alpha 1.

So, the S S regression due to alpha 2 is same as S S regression due to alpha 2, in the presence of alpha naught and alpha 1. So, here you understand, I think you have to think about it so, that is why you know S S regression due to alpha 2 is same as S S regression due to beta 2, in the presence of beta naught and beta 1. And similarly, you can check that S S regression due to alpha 3. does not depend on the other term side. So, this is the term as S S regression due to beta 3, in the presence of beta naught, beta 1 beta 2 ok. This is same as this one and this is same as S S regression, due to alpha 3 right for, polynomial fitting using orthogonal polynomials. Because S S regression in polynomial fitting there are orthogonal. Also, this is same as S S regression due to alpha 3 in the presence of alpha naught, alpha 1 and alpha 2. It does not matter you know, because the

S S regression in polynomial fitting using orthogonal polynomial in there are independent, ok. So, that is all for today.

Thank you.