Statistical Inference

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Lecture No. # 27
UMP Unbiased Tests (Contd.)

Yesterday, we have discussed the case that when we have two sided alternative hypothesis, then the UMP test does not exist, we have also shown through an example, and then we said that if we have a condition called similarity, similar test, level alpha similar test. Then for distributions in the exponential family we have UMP unbiased test when the alternative hypothesis two sided and we have demonstrated the test for normal distribution.

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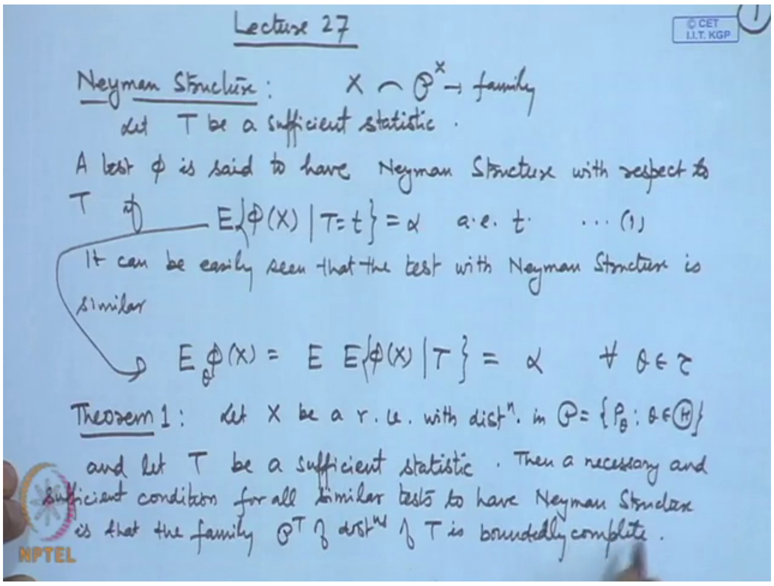

Now, we will further discuss this thing; normally we have seen that when we are discussing exponential families there is a concept of sufficient statistics, then there is concept of (( )) parameter etcetera. We will today show that we can incorporate this concepts to derive the UMP unbiased test, when we are dealing with the multi parameter exponential families. So, in this regard first of all I introduce Neyman structure, so what is the test with Neyman structure? So, as usual we have a random variable or random vector x, which is having a distribution and we say and the family of distributions is x.

Let t be a sufficient statistic here, so we say that a test phi is said to have Neyman structure with respect to t, if the conditional expectation of phi x given t is equal to alpha almost everywhere t, now what does this condition represent? See, if I consider simply expectation of phi x then this is nothing but the power function for theta in the null hypothesis parameter said this denotes the probability of type one error and for theta belonging to the alternative hypothesis said this represents the power of the test.

Now, if we consider expectation of phi x given t; if t is sufficient statistic then what does it mean this term will be independent of theta, now what does it mean that on every value of t is equal to t which we can call orbits of t on every orbit of t this will have power is equal to alpha, so this is a much stronger condition and here we can say of course, you can observe that let me call it to 1, it can be easily seen that the test with Neyman structure is similar, because if I consider another expectation here; expectation of phi x that is equal to expectation of expectation phi x given t, now this is equal to alpha so for all theta belonging to tau that is the boundary of the null and alternative hypothesis said.

So, here what we are saying is that the condition of the power or you can similarity is brought down to the level of the sufficient statistics that is the orbits of the sufficient statistic, so now many times what happens that it is easier to obtain the most powerful test are the UMP test among the test which is having Neyman structure, and then since for a every t it is most powerful, so overall it will be most powerful.

So, here frequently another idea that is used is the completeness idea, the distribution the definition of the completeness and examples of the complete family of distribution and the completed statistics we have discussed earlier in the point estimation, in connection with a derivation of the uniformly most powerful in connection with uniformly minimum variance unbiased estimation etcetera.

So, I will not be repeating those a steps again, I just advise the students to go back to the lectures on a point estimation and again revise the concept of the completeness, here what I will do I will try to incorporate are you can use the concept of completeness in deriving the UMP unbiased test, and in particular the Basu's theorem is also used here; the Basu's theorem is regarding the independence of two is statistics, if one statistic is sufficient and boundedly complete and another statistic is having a distribution free from the parameter then the two statistics are independent.

So, this thing will be there, I am not going to repeat this a steps here the students are advised to refer to my earlier lectures which are related to the completeness, now I will give a result here; let me give some numbering here theorem 1, let x be a random variable are random vector with distribution in p and let t b a sufficient is statistic, then a necessary and sufficient condition for all similar test to have Neyman structure is that the family say p T of distributions of T is boundedly complete, note here that full completeness is not required here bounded completeness is enough here.

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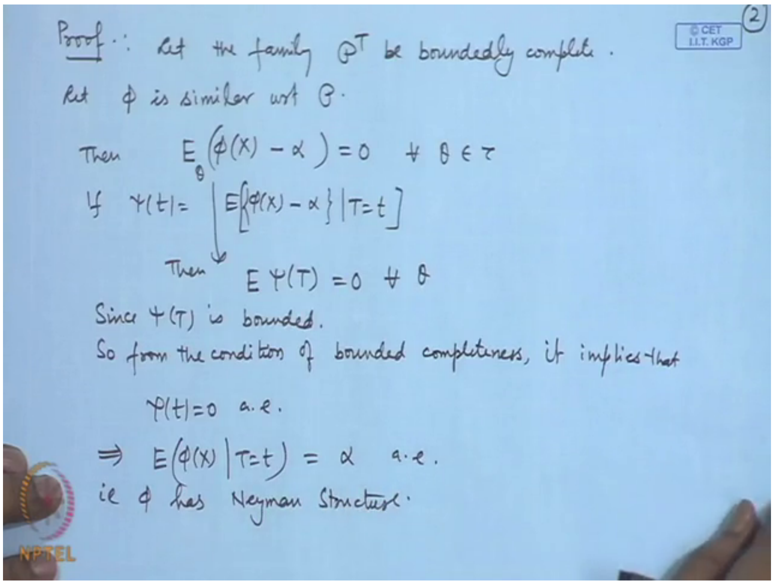

Let me give a proof of this year. So let the family p T be boundedly complete; and let us assume that phi is similar, so if it is similar we are able to write down that expectation of phi x minus alpha is equal to 0 for all theta belonging to… If it is similar, then we can say it is equal to this for every belonging to tau, now if we are having psi T is equal to expectation of phi x minus alpha given T is equal to t let me give this notation here, then what we are saying is this is statement will imply expectation of psi T is equal to 0 for all theta.

Now, this is a test function, so this lies between 0 to 1, because this is simply denoting the probability of rejecting h naught, and alpha is the number between 0 and 1 this is also the probability level we are fixing, so this psi t is bounded.

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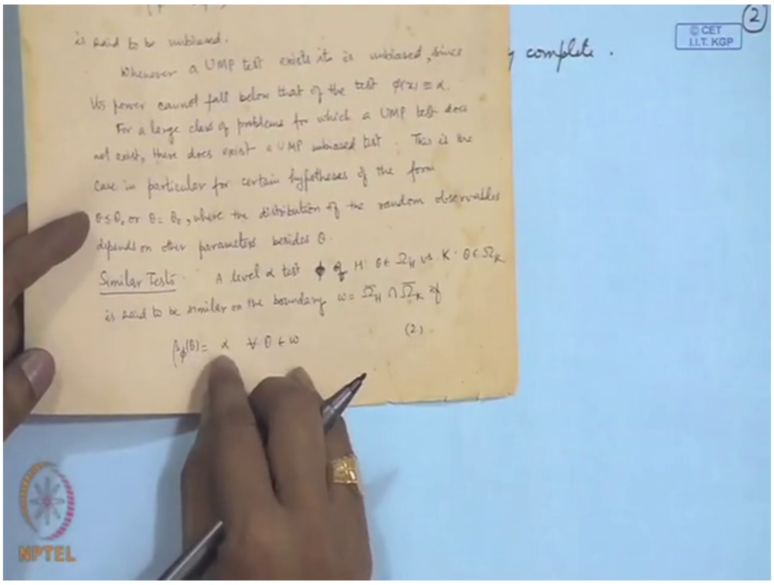

So, from the condition of let me just revise the theorem of similarity here, that we are having x beta phi theta is equal to alpha that was equal to expectation of phi x is equal to alpha, so from the condition of bounded completeness it implies that psi t is equal to 0 almost everywhere, which implies that expectation of phi x given T is equal to t, it is equal to alpha almost everywhere; that means, the test has phi has Neyman structure.

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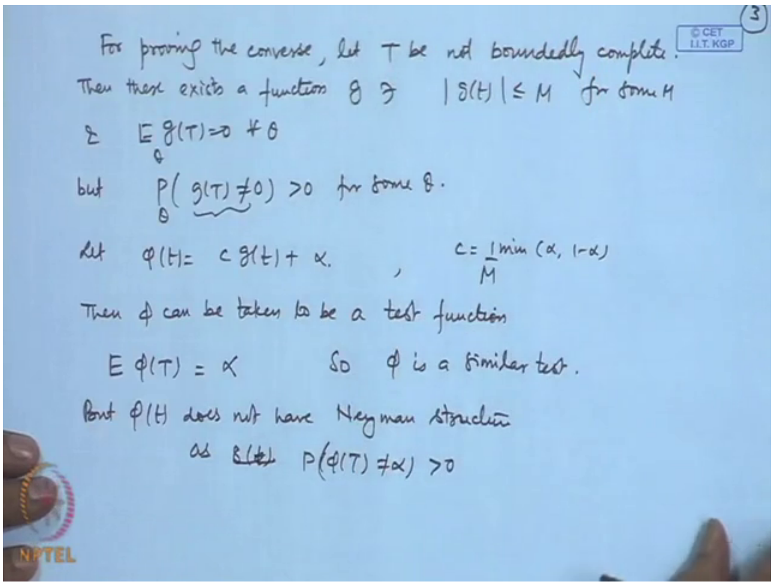

Now, let us look at the converse, for proving the converse let T be not boundedly complete then there exists a function say g such that g is bounded and expectation of g t is equal to 0 for all theta. But probability that g t is naught equal to 0 is possitive for some theta, now let us assume this phi t to be equal to a constant times g t plus some alpha, and here c I am chossing to be minimum of alpha and 1 minus alpha divided by m, then what can be say about phi? c is the minimum of this thing alpha 1 minus alpha by n, and g t is founded by n so this phi t become say test function, then phi t can be taken to be a test function, also what is expectation of phi T? Since expectation of g t is 0 this is simply equal to alpha so phi is a similar test.

But phi t does not have Neyman structure, because I am assuming that g t is not equal to 0 as g t, so that means we are assuming that probability that phi t is not equal to alpha is positive, so phi t does not have Neyman structure if I assume that it is not bounded a complete then phi t does not have Neyman structure, so the converse part is also prove; that means, t should be boundedly complete, now these results are useful for deriving the UMP unbiased test for multi parameter exponential families.

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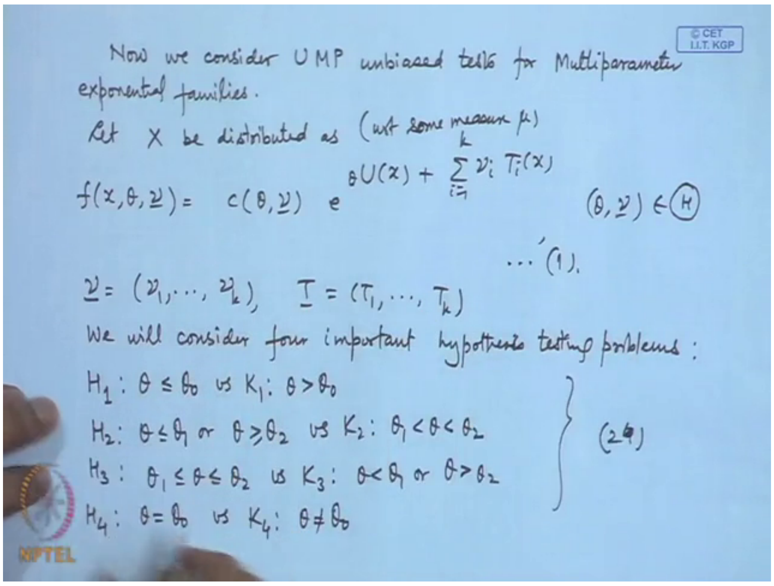

So, let us consider now, now we consider UMP unbiased tests for multiparameter exponential families, so let us consider the multiparameter exponential family as let x be distributed as so we are writing f x theta and some nu that is equal to c theta nu e to the power theta u x plus sigma nu i T i x i is equal to 1 to k, now this is with respect to some measure mu, because we may deal with the discrete or continues are make distribution.

So, this is a general form of the probability density, here you have theta u x plus sigma nu i T I, and this theta nu this belongs to some parameter space say script theta let me call it 1, will use the abbreviated notation nu for nu 1 nu 2 nu k, T for T 1 T 2 T k, and now if we remember yesterday's reference I have introduced four important type of hypothesis, let me repeat them here; we will consider four important hypothesis testing problems.

So, we will follow the notation that I introduce yesterday, H 1: theta less than or equal to theta naught versus k 1: theta greater than theta naught versus k 1 theta greater than theta naught, H 2 theta less than or equal to theta 1 or theta greater than or equal to theta 2 versus k 2: theta 1 less than theta less than theta 2, H 3: theta 1 less than or equal to theta less than or equal to theta 2 versus k 3: theta less than theta 1 or theta greater than theta 2, H 4 theta is equal to theta naught versus k 4: theta is not equal to theta naught.

So, let me give reference two to all this four important types of hypothesis, in the case of one parameter exponential family we have shown that UMP test exist for H 1 and H 2 and UMP unbiased test exist for H 3 and H 4, but know we are dealing with multi parameter exponential family here I am writing theta as one of the parameters, but there are other parameters also like nu 1 nu 2 nu k, these are termed usually as nu sense parameters for example, if I write down the normal distribution with parameters mu and sigma square then in the exponent I will be able to write e to the power minus x square by 2 sigma is square plus theta plus mu by sigma is square x.

So, if I have n observation then it will become sigma x i square and sigma x i there, so I will have two parameters from mu and sigma is square I can write mu by sigma is square and minus 1 by 2 sigma square, so either of them can be considered as theta and other one can be considered as nu, so this is an example of a two parameter exponential family, the one which I have return here this is a k plus 1 parameter exponential family, now we make certain assumption on the parameter space also.

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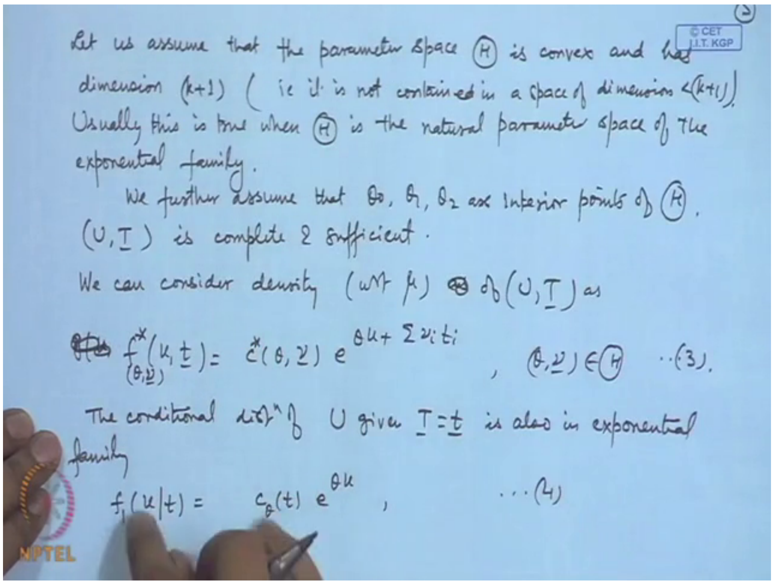

Let us assume that the parameter space theta is convex and has dimension k plus 1, now this assumption is required if you remember the result for the k plus 1 parameter exponential family when we have this type of thing then the parameter is space if it contains k plus one-dimensional rectangle then u T 1 T 2 T k is a complete and sufficient statistics, sufficiency is of course, clear from the factorization theorem, but this will also be complete therefore, this assumption that the dimension of this parameter is space is full that is required; that means, we are not assuming.

So, we are saying that it is not contained in a space of dimension less than k plus 1, usually this is true when theta is the natural parameter is space of the exponential family, we have seen one example where we are dealing with the two normal distributions and the means where same, when the means became same the dimension become 1 less and therefore, the completeness was lost here, then when we are dealing with testing problems we have mentioned certain points like theta naught theta 1 theta 2.

So, we assume that these are in the interior; that means, there are points which are less or more than these, so we further assume that theta naught theta 1 theta 2 are interior points of theta, so u T this is complete and sufficient, so we can restrict attention to density with respect to measure mu as of u T, c theta nu e to the power theta u plus sigma nu i t i.

So, this constant may change here I mean put here c star, earlier I written c in the case of f density, so here I change it 2 c star and of course, the parameters theta and new or occurring here, theta and nu belongs to theta, the conditional distribution of u given T is equal to t is also in exponential family, so I can write the notation here say f 1 u given t that is equal to say c theta t e to the power theta u and some coefficient will come, now if you look at this here T has become fixed here so this is nothing, but a one parameter exponential family, in the one parameter exponential family if I am considering the tests H 1 and H 2 I have UMP test, and for H 3 and H 4 I have UMP unbiased tests.

Let me relate this things here, note here that there will be a little modification in the coefficients here, because the densities with the will be with respect to different measures, here we have started with mu then we are dealing with x then we are dealing with u and t then the measure gets little bit modified. So, I have changed here c star, and when we are considering the conditional distribution of you given t I further modified this coefficient, so the measure will be accordingly whatever variable we are considering here.

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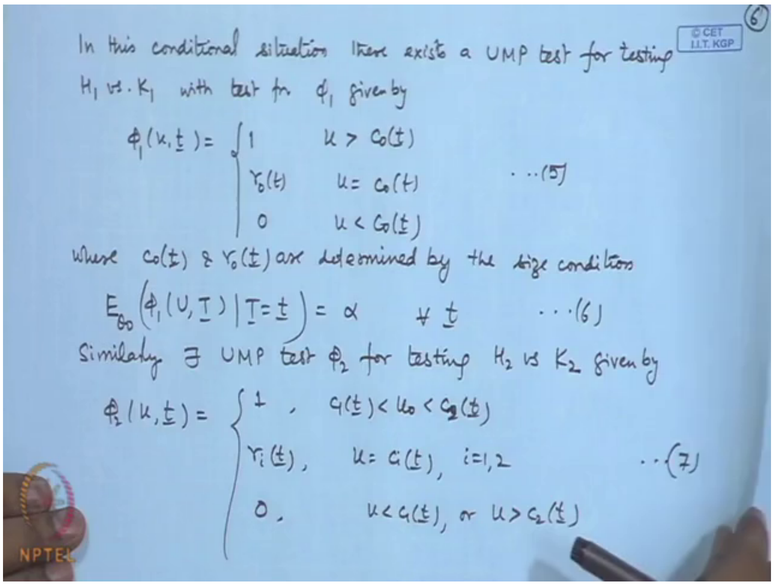

So in this conditional situation there exists a UMP test for testing H 1 versus k 1 with test function phi 1 given by it is 1 when u is greater than some coefficients c naught but this may depend upon t, this is gamma naught t when u is equal to c naught t it is 0 u is less than c naught t, where c naught t and gamma naught t are determined by the condition expectation of phi 1 u T given T is equal to t it is equal to alpha for all.

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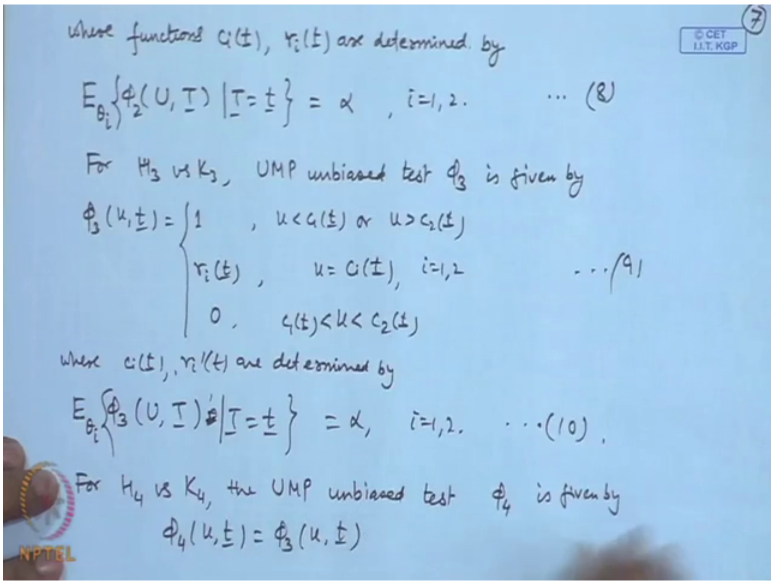

So, here you can see the modification from the original one, in the original we are considering simply one parameter exponential family and therefore, the distribution the test was 1 if u is greater than c naught gamma naught if u is equal to c naught and 0 if u less than c naught, but know there is a dependence on t and this size condition is also conditional now, in a similar way if you are considering H 2 similarly their exists UMP test say phi 2 for testing H 2 versus k 2 given by phi 2 u t is equal to let me describe this thing detail so that it is clear the dependence on t, and once again the constant that is the function c 1 c 2 gamma 1 gamma 2 are determined by c i t and gamma i t are determined by phi of phi 2 of u T given t is equal to t this is equal to alpha for i is equal to 1, 2.

Now, for H 3 problem and H 3 versus k 3 problem and H 4 versus k 4 problem we have seen the one parameter exponential family we had UMP unbiased tests, so if we consider the conditional here conditional distribution that is u given t then for this again we will have the UMP unbiased test that will be the conditional test here. So, for H 3 versus k 3 UMP un biased test phi 3 is given by that is equal to1 for u less than c 1 t are u greater than c 2 t it is equal to gamma i t, if u is equal to c i t for i is equal to 1 2 and it is equal to 0 when you is line between c 1 and c 2, where once again this things are determined by expectation of theta i phi 3 u t is given T is equal to t this is equal to alpha for i is equal to 1 2, for H 4 versus k 4 the UMP unbiased test that will be phi 4, actually phi 4 will be same as phi 3.

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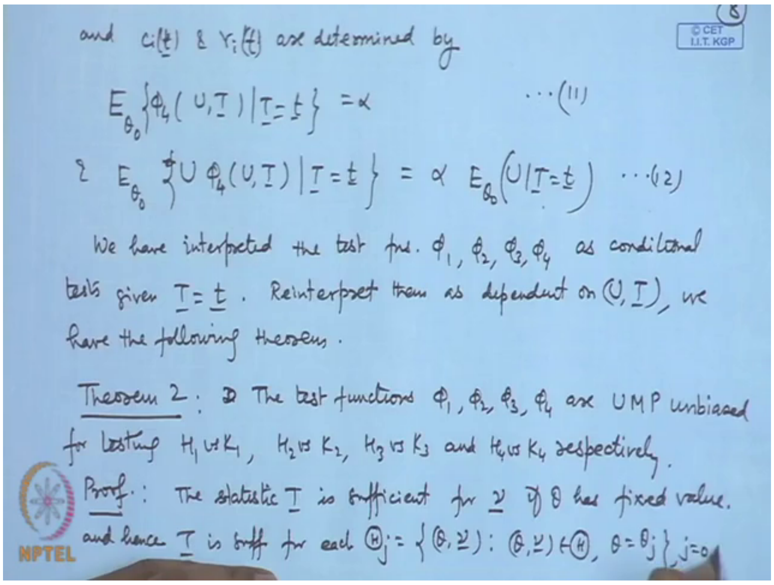

And c i is and gamma i they are determined by expectation of theta naught phi 4 u T given T is equal to t it is equal to alpha. And now you can see here that this size conditions are all conditional, so if I take the expectations I will get the conditions without the conditional here; so what we can say a we have interpreted the test functions phi 1 phi 2 phi 3 phi 4 as conditional tests given T is equal to t.

Now, we reinterpret them as dependent on u T we have the following theorem, so I will call it theorem 2 say this is regarding the UMP tests here, note here one point I have given the test functions phi 1 and phi 2 the conditional tests as u m p, and the phi 3 and phi 4 as UMP unbiased, but when I consider them as unconditional test all of this tests will become UMP unbiased.

So, the statement is in the given theorem here, so the test functions phi 1 phi 2, so phi 1 is defined by these two conditions, phi 2 is defined by these conditions, phi 3 is defined by this conditions etcetera. So, the test functions phi 1 phi 2 phi 3 phi 4 are UMP unbiased for testing H 1 versus k 1, H 2 versus k 2, H 3 versus k 3 and H 4 versus k 4 respectively under the given setup; that means, the joint distribution of the initial random variable was multiparameter exponential family infect it was k plus one-dimensional distribution, and the distribution of u and T the sufficient statistics was also in the exponential family in that case we will have this as UMP unbiased tests, let me scratch a proof of this of course, for detailed proof you may look at the book of Lehmann here.

The statistic t is sufficient for nu if theta has fixed value, so this you can easily see if I am writing down the distribution in this one if I fix theta then this part will become random variable here, it is dependent upon the variable only you will have only e to the power sigma nu i t i x, that will show that t 1t 2 t k is sufficient for the parameters nu 1 nu 2 nu k.

So, and hence we can say that T is sufficient for each this is will call subsets of the parameter is spaces, theta nu where theta nu belongs to script theta and theta has be in fixed as some so this is for j is equal to 0, 1, 2, now this points we have considered because in all this tests we are having the cut of points in the hypothesis as theta naught theta 1 and theta 2. So, at least for those points the sufficiency of T is maintained here.

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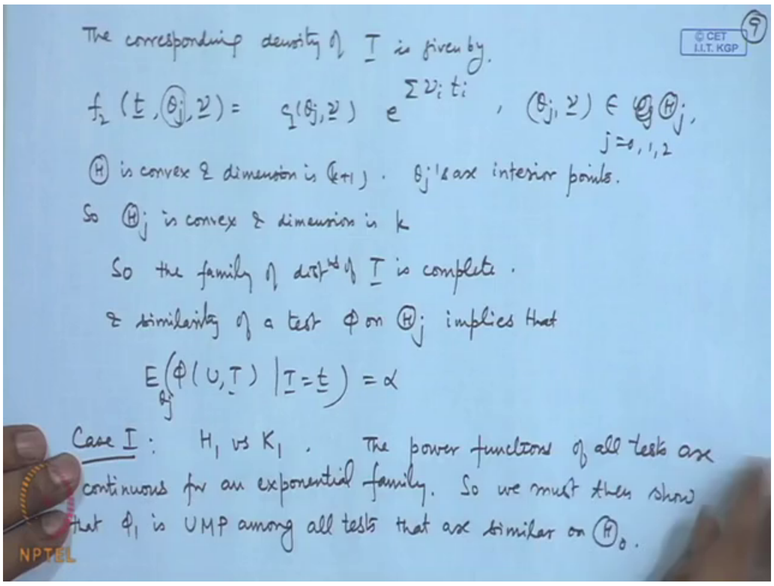

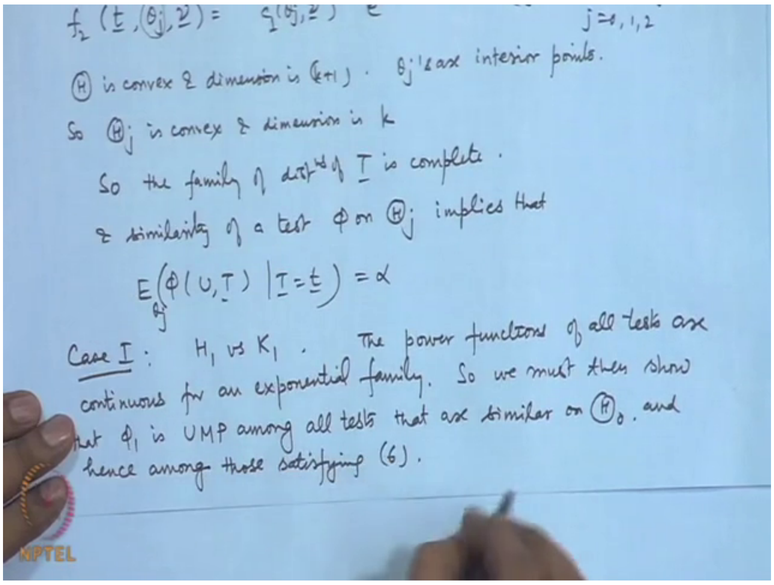

The corresponding density of t is given by… So we can use some notation say f 2 for the distribution of t and of course, theta j will coming it is fixed here, nu that is equal to c theta j nu e to the power sigma nu i t, so this will some coefficient let me put it c 1 here, where theta j nu this belongs to omega sorry theta j for j is equal to 0, 1, 2.

Now, we have assumed theta is convex that is assumed and dimension is k plus 1, and we have assumed that theta j's are interior points so this theta j is convex and dimension is k, so basically what we have done is we have taken one hyper plane there theta is equal to theta j there, so the family of distributions of t so the family of distributions of t is complete, and similarity of a test phi on theta j this will implies that expectation of phi u t given T is equal to t that will be equal to alpha for theta j all right.

So, this is the general description so far, we have derived the conditional tests the UMP test now in the theorem I am cleaning that for the un conditional problem the tests phi 1 phi 2 phi 3 phi 4 are UMP unbiased, so in order to proof this one we will take help of the theorem 1 which I have given today; that means, the test with the Neymon structure and the result which I have given for the similar to test in the previous lecture.

So, we will use both of this results here, now first thing that we notice here is the structure of the multiparameter exponential family here, so for the for the faked value of theta as theta naught theta 1 are theta 2 t is a complete and sufficient is statistic here, so if I have a test function phi to be similar then we should have expectation of phi u T given

T is equal to t is equal to alpha, so now let us consider this phi 1 phi 2 phi 3 and phi 4 separately.

So, let us take case one that is the testing problem H 1 versus k 1. So, another point that yesterday's lemma, which we want to use the power functions of the test functions whatever we are considering must be continues, since we are dealing with the exponential families with the power functions are basically bounded therefore, integral functions and therefore, the expectations of the test functions must be continues.

So, let me give a generally statement, the power functions of all test are continues for an exponential family, so we must then show that phi 1 is UMP among all tests that are similar on theta naught, and hence among those which are satisfying condition 6, the condition 6 let me repeat here this the condition for the Neyman structure this condition here.

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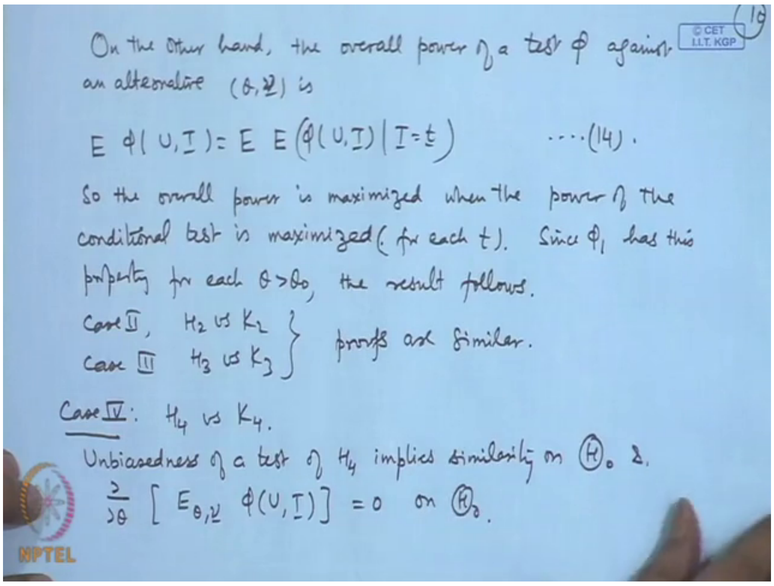

On the other hand, the overall power of a test phi against an alternative theta nu is expectation phi u T that is equal to expectation of phi u T given T is equal to t expectation of this, let me give the number here 13 and this will be here 14. So, the overall power is maximized when the power of the conditional test is maximized for each t, now phi 1 was already having this property since phi 1 has this property for each theta greater than theta naught the result follows, I am not a stating the case 2 and case 3 that is H 2 versus k 2 and H 3 versus k 3 so the proofs are similar. Let me take case 4 that is H 4 versus k 4, here unbiasedness of a test of H 4 implies similarity on theta naught and del by del theta expectation theta nu phi u t that will be equal to 0 on theta naught, now we take this derivative inside the expectations sign.

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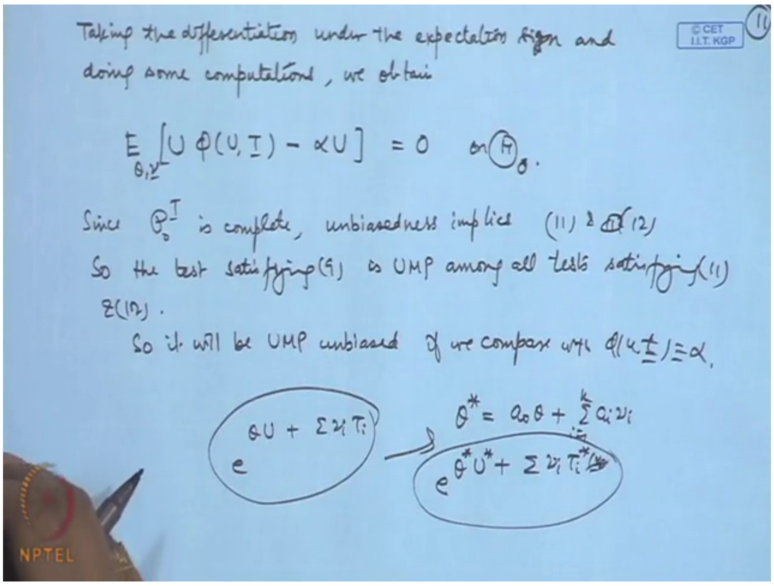

Taking the differentiation under the expectation which will be permissible here because phi is a test functions so it is a bounded between 0 and 1 here. So, and then what we do we carry out little bit of calculation doing some computations we obtain expectation of u phi u T minus alpha u is equal to 0, and now this is u and this is coming because we are considering differentiation you are having the density function e to the power the theta into u, so when u differentiate e to the power theta u with respect to theta u will get e to the power theta u into u and that is why this u has appeared here this is on theta naught, now since the family under theta naught this is complete we already seen this thing unbiasedness implies the conditions 11 and 12, the conditions 11 and 12 which stated for phi 4 so this two conditions will follow because I can write expectation of expectation here.

So, the test satisfying 9 is UMP among all tests satisfying 11 and 12, so it will be UMP unbiased test if we compare with phi u t is equal to alpha, a part which I have to not covered here is the measurable t of this functions, we should actually also show that phi 1 phi 2 phi 3 and phi 4 these are all jointly measurable function we are all functions of u and t.

So, the joint measurable t of this is also required; however, if this proof I am a skipping here and the readers has can actually go through the detailed proof in the book of Lehmann and Romano, we will consider further application of this and then we are writing a distribution in the exponential family so for example, we are considering e to the power theta u plus sigma nu i t i, but one may consider different form of the parameters like we make a consider we parameterization we may consider say for example, theta is star is equal to say a linear combination of theta and nu i's.

So, what we can do? We can do little bit of readjustment of the coefficients the form of the distribution will still remain the same, this will only be exponential family in a slightly different form, we may actually write it as e to the power say theta is star u is star plus sigma nu i t i star.

So, all this things will get little bit modified; however, it remains in the k plus 1 parameter exponential family, what we have demonstrated here that the result for UMP and UMP unbiased tests which were stated for one parameter exponential family can be extended to the case of multiparameter exponential family; that means, we are still testing for one of the parameters the we are having other parameters as the nu since parameters the overall distribution is in the multiparameter exponential family, so there is one exception here what is happening? the UMP test which was there in the one parameter exponential family now it is UMP unbiased.

And the test which are UMP unbiased they also remain UMP unbiased, so in all the conditions we are actually getting UMP unbiased tests, now in particular this helps us to resolve various problems like if you are dealing with the parameters of normal distribution, if we are dealing with the parameters when we are having say for example, if I am considering one poison distribution or two poison distributions, if we are considering binomial distribution two binomial distributions, if we are considering say beta distributions and many of this cases.

So, these are all covered under this that as long as dealing with the distributions are whatever join distribution of the observation is given as low as that is remaining in the multiparameter exponential family it will be following; that means, for testing the problems of the nature H 1 H 2 H 3 and H 4 as I have defined here, for each of this cases we will have UMP unbiased test the form will be given as there. In the next lecture, I will be giving full working out of this tests that is e 1 p unbiased tests for some of these problems that I will be carrying out in the next lecture.