

**Statistical Methods for Scientists and Engineers**  
**Prof. Somesh Kumar**  
**Department of Mathematics**  
**Indian Institute of Technology - Kharagpur**

**Lecture - 14**  
**Parametric Methods - VI**

In the last lecture, I introduced the concept of testing of hypothesis. We saw that Neyman Pearson approach in which they considered the probabilities of type 1 error and type 2 error and based on that the test procedures are devised in which we put a restriction on 1 type of error usually the type 1 error and we call it the size of the test and subject to the tests function satisfying the size of the test condition.

We find out those test which have the maximum power so they are called most powerful test as some solution was proposed for simple versus simple hypothesis cases and later on these procedures were extended to the case of certain type of composite hypothesis and then for certain type of composite hypothesis uniformly most powerful unbiased tests were also devised.

In place of giving the full details of the derivation of the test, I will be basically explaining to you the procedures that the test that have been obtained using this and how to use them.

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Lecture - 14

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Testing for Parameters of Normal Populations

Let us consider  $X_1, \dots, X_n \sim N(\mu, \sigma^2)$

Testing for  $\mu$ .

Case I:  $\sigma^2$  is known.

$H_1: \mu \leq \mu_0$        $\bar{X} \sim N(\mu, \sigma^2/n)$

$H_0: \mu > \mu_0$        $\frac{\sqrt{n}(\bar{X} - \mu)}{\sigma} \sim N(0, 1)$

We consider the test statistic

$$Z = \frac{\sqrt{n}(\bar{X} - \mu_0)}{\sigma}$$

So let us consider testing for the parameters of normal populations. So let us consider  $X_1, X_2, \dots, X_n$  following normal  $\mu$   $\sigma^2$  distribution. We consider testing for  $\mu$ . Now let

us consider say case 1 when sigma square is known. Now we consider various kind of hypothesis. We consider say first problem. I will call the hypothesis testing problem says H1 K1 so H1 mu=say mu is <= mu 0 against say K1 mu>mu 0.

In this particular case, we consider X bar that is following normal mu sigma square/n so we consider root n X bar-mu/sigma that follows normal 0, 1. So we consider the test statistic Z=root n X bar-mu 0/sigma.

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Testing for  $\mu$ .

Case I:  $\sigma^2$  is known.

$H_1: \mu \leq \mu_0$        $\bar{X} \sim N(\mu, \sigma^2/n)$

$K_1: \mu > \mu_0$        $\frac{\sqrt{n}(\bar{X} - \mu)}{\sigma} \sim N(0,1)$

We consider the test statistic

$$Z = \frac{\sqrt{n}(\bar{X} - \mu_0)}{\sigma}$$

$P(Z > z_\alpha) = \alpha$

So if I consider this as z alpha then this probability is alpha, so if we consider probability of Z>z alpha=alpha and here we are considering mu=mu 0.

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The uniformly most power test of size  $\alpha$  for testing  $H_1$  vs  $K_1$  is Reject  $H_1$  when  $Z > z_\alpha$  (Accept  $H_1$  if  $Z \leq z_\alpha$ )

It can be shown that sup  $P(Z > z_\alpha)$  is attained at  $\mu = \mu_0$

In place of  $H_1: \mu \leq \mu_0$ , if we take  $H_1^*: \mu = \mu_0$  vs  $K_1: \mu > \mu_0$ , then also the test procedure will be applicable.

(ii)  $H_2: \mu \geq \mu_0$  vs  $K_2: \mu < \mu_0$

Reject  $H_2$  if  $Z \leq -z_\alpha$  (Accept  $H_2$  otherwise)

equivalently  $H_2^*: \mu = \mu_0$  vs  $K_2: \mu < \mu_0$

UMP test of size  $\alpha$

So the uniformly most powerful test of size  $\alpha$  for testing  $H_1$  against  $K_1$  is a reject  $H_1$  when  $Z > z_\alpha$  where  $Z$  is given by  $\sqrt{n}(\bar{X} - \mu_0)/\sigma$ . Actually, it can be shown that if I consider probability  $Z > z_\alpha$  then the supremum of  $\mu \leq \mu_0$  is attained at  $\mu = \mu_0$ . Therefore, this is the most powerful test of size  $\alpha$ , so of course since this is composite hypothesis situation we will say it is the uniformly most powerful test here.

Now we have the variations, in place of  $H_1$  that is  $\mu \leq \mu_0$  if we take  $H_1^*$  that is say  $\mu = \mu_0$  versus  $K_1$   $\mu > \mu_0$  then also the same test procedure will be applicable. Now the main reason is that actually since here the maximization is occurring that  $\mu = \mu_0$  therefore when the null hypothesis  $\mu = \mu_0$  will be coming here and in this case the maximum is occurring at that point and the power is decided by the alternative.

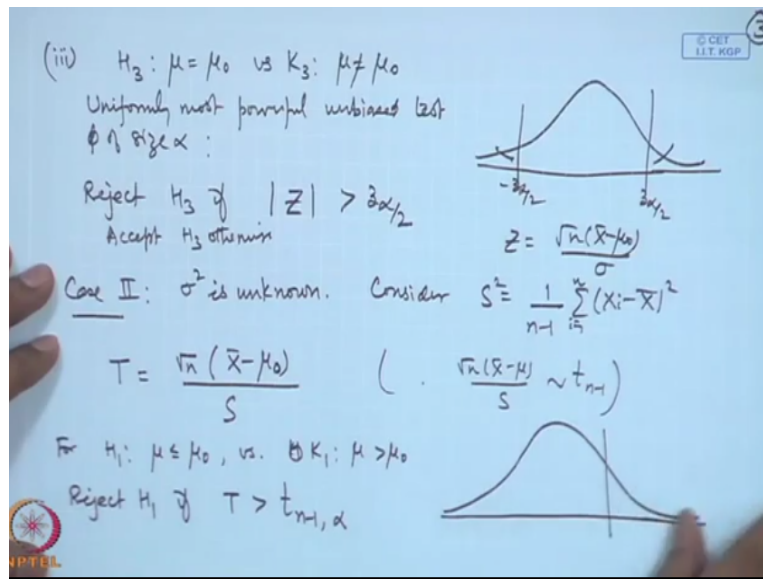
Therefore, the test function will not change and the test procedure will also not change. So you will say accept  $H_1$  if  $Z \leq z_\alpha$ . Here equal to  $z_\alpha$  has no significance because the probability that  $Z = z_\alpha$  will be 0 because  $Z$  is a continuous random variable. Now naturally one may think what happens if we change the null and alternative hypothesis?

For example, here  $\alpha$  is the maximum probability of type 1 error that means we are rejecting and the null hypothesis is true. Now if that is considered to be more serious for the beta, in that case you may like to interchange the hypothesis and we may consider so let me call it say  $H_2$   $\mu \geq \mu_0$  against  $K_2$  say  $\mu < \mu_0$ . I have interchanged the role of null and alternative hypothesis.

But the equality I have included in the null hypothesis. So in this case, we will be considering the rejection on the left side because you will be considering here. So you will consider reject  $H_2$  if  $Z \leq -z_\alpha$  and of course accept  $H_2$  otherwise. See in this case, this hypothesis is also equivalently we may test  $H_2^*$   $\mu = \mu_0$  against  $K_2$   $\mu < \mu_0$ .

Basically, once again if we are considering this one then the probability of say  $Z \leq -z_\alpha$ . When  $\mu = \mu_0$  that will be  $\alpha$  and when we are considering for a general  $\mu$  in this region then the maximum value will be attained when  $\mu = \mu_0$  and therefore the size will be  $\alpha$  so this is the uniformly most powerful test of size  $\alpha$ . Now there may be situations where we may not like to test greater than or less than rather whether a value is equal or not.

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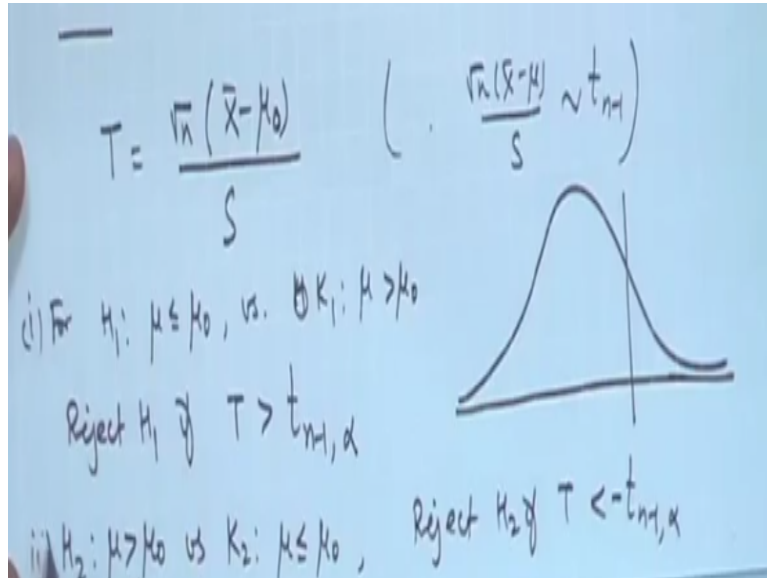
In that case, we formulate the hypothesis testing problem in the following fashion. We consider say  $H_3 \mu = \mu_0$  against  $K_3 \mu \neq \mu_0$ . Now naturally in this case the rejection region will be on both the sides. So we consider say  $z_{\alpha/2}$  and  $-z_{\alpha/2}$ . So you will consider actually this is uniformly most powerful unbiased test of size  $\alpha$ . So that is reject  $H_3$  if modulus of  $Z > z_{\alpha/2}$  where  $Z$  is the same quantity.

That is  $Z = \text{your root } n \bar{X} - \mu_0 / \sigma$ . So you are rejecting in this region and in this region and in the intermediate region you are in favor of the hypothesis accept  $H_3$  otherwise. Now in case  $\sigma^2$  is unknown, then naturally this  $Z$  cannot be used. If you remember the development of the confidence interval, there in place of  $\sigma$  we had used  $S$  there where  $S^2$  was  $1/n-1 \sum (X_i - \bar{X})^2$ .

That is the sample variance. So we consider the situation  $\sigma^2$  is unknown. Then consider  $S^2 = 1/n-1 \sum (X_i - \bar{X})^2$ . So we consider say  $T = \text{root } n \bar{X} - \mu_0 / S$ . So if we consider say  $\text{root } n \bar{X} - \mu_0 / S$  then that follows  $T$  distribution on  $n-1$  degrees of freedom as we have seen in the confidence interval problem. So if we consider this  $\mu$  replaced by  $\mu_0$  then the test statistic will be following a  $T$  distribution.

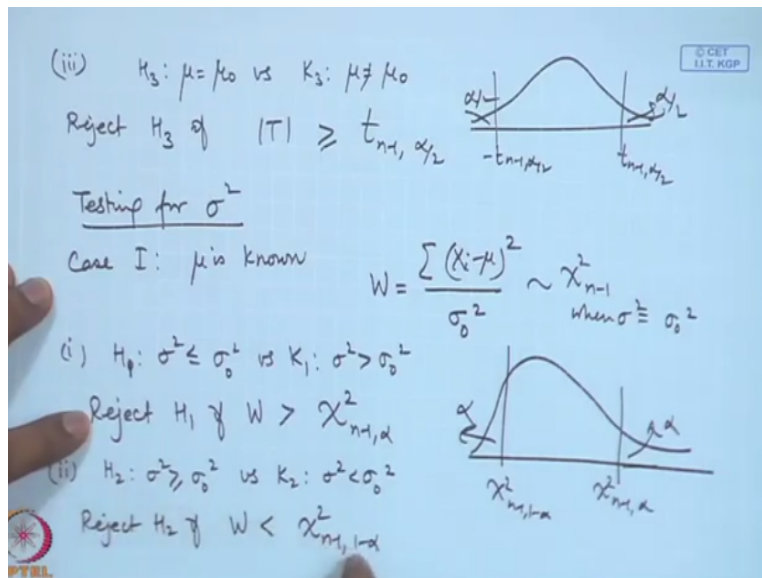
And we can consider the problems so for  $H_1 \mu \leq \mu_0$  against say  $K_1 \mu > \mu_0$  then we will have the test as reject  $H_1$  if  $T > t_{n-1, \alpha}$ .

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If we consider  $\mu > \mu_0$  against  $K_2$   $\mu$  is  $\leq \mu_0$  then the test will be reject  $H_2$  if  $T < -t_{n-1, 1-\alpha}$  that is  $-t_{n-1, \alpha}$  because of the symmetry.

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Then the third situation comes for the 2 sided tests that is for  $\mu = \mu_0$  against  $\mu$  is not equal to  $\mu_0$ . Then we will consider reject  $H_3$  if modulus of  $T$  is  $\geq t_{n-1, \alpha/2}$ . So you will have 2 sided rejection region here. This is  $\alpha/2$  and this is  $\alpha/2$ . So these are the most powerful unbiased tests for the size  $\alpha$  for these problems here. Now one may like to test for the variance also.

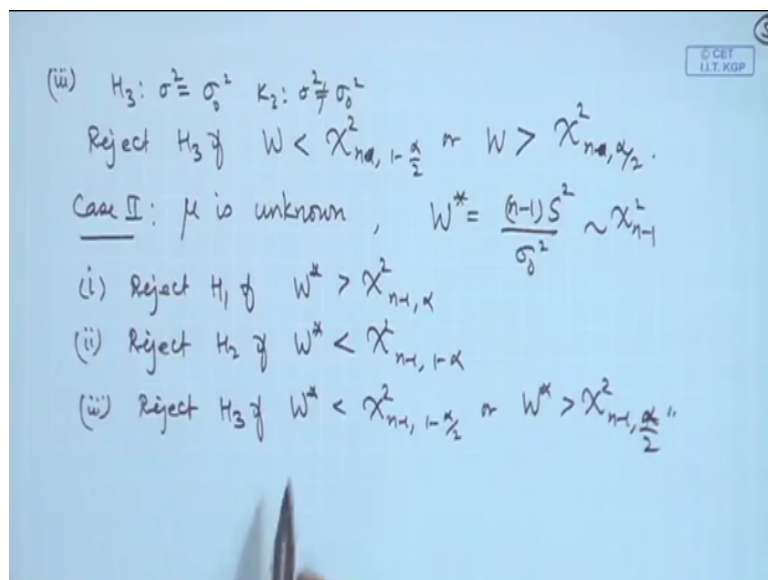
So if we consider the test for the variance, testing for sigma square and again you will have 2 cases, 1 case will be when  $\mu$  is known. If  $\mu$  is known, then we can consider  $\sum (X_i - \mu)^2 / \sigma_0^2$ . So if we consider this as  $W$  then this is following chi square

distribution on  $n-1$  degrees of freedom when  $\sigma^2 = \sigma_0^2$ . So if we consider the hypothesis testing problems based on this.

So for example let us consider say  $\sigma^2 \leq \sigma_0^2$  against say  $\sigma^2 > \sigma_0^2$  then we will consider the rejection region as reject  $H_0$  if  $W > \chi^2_{n-1, \alpha}$  because chi square is  $Q$  distribution and we will have this situation here,  $\chi^2_{n-1, \alpha}$  so this probability is simply  $\alpha$ . As I mentioned earlier, we can also consider  $\sigma^2 = \sigma_0^2$ .

And here  $\sigma^2 > \sigma_0^2$ , still the test function and the test region will be same and we may consider reverse situation  $\sigma^2 \geq \sigma_0^2$  against  $K_2$   $\sigma^2 < \sigma_0^2$  then the test procedure will be reject  $H_0$  if  $W < \chi^2_{n-1, 1-\alpha}$ . This probability is  $\alpha$ , so this is not a symmetric distribution therefore we cannot write minus here.

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And we will have a 2 sided region if we consider  $\sigma^2 = \sigma_0^2$  against  $\sigma^2$  is not equal to  $\sigma_0^2$ . So the test procedure will be reject  $H_0$  if  $W < \chi^2_{n-1, 1-\alpha/2}$  or  $W > \chi^2_{n-1, \alpha/2}$ . So this will be uniformly most powerful test of size  $\alpha$  here in the case 1 and 2 and in the case 3 it will be uniformly most powerful unbiased test of size  $\alpha$  here.

Now in the case when  $\mu$  is unknown then we base our decisions on let us call it  $W^*$  that is  $(n-1) S^2 / \sigma_0^2$  so that follows chi square distribution.

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(iii)  $H_3: \mu = \mu_0$  vs  $K_3: \mu \neq \mu_0$   
 Reject  $H_3$  if  $|T| \geq t_{n-1, \alpha/2}$

Testing for  $\sigma^2$   
 Case I:  $\mu$  is known

$$W = \frac{\sum (X_i - \mu)^2}{\sigma_0^2} \sim \chi^2_{n-1} \text{ when } \sigma^2 = \sigma_0^2$$

(i)  $H_1: \sigma^2 \leq \sigma_0^2$  vs  $K_1: \sigma^2 > \sigma_0^2$   
 Reject  $H_1$  if  $W > \chi^2_{n-1, \alpha}$

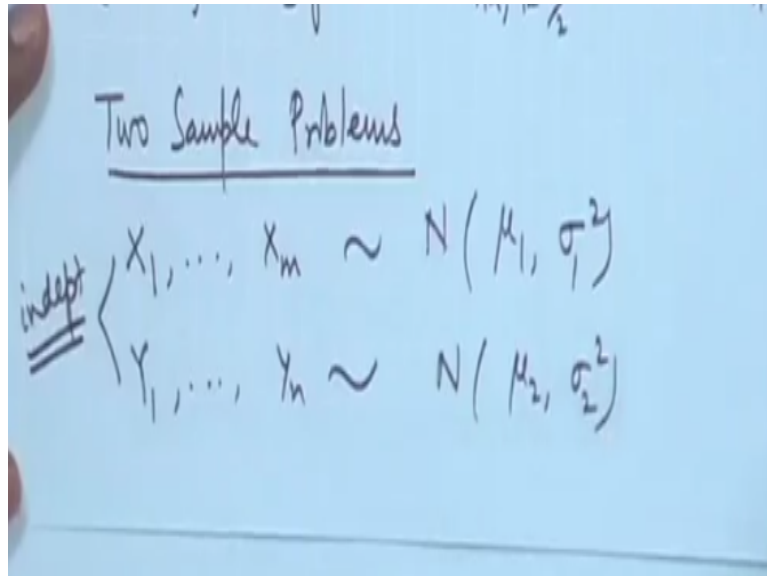
$H_2: \sigma^2 \geq \sigma_0^2$  vs  $K_2: \sigma^2 < \sigma_0^2$   
 Reject  $H_2$  if  $W < \chi^2_{n-1, 1-\alpha}$

Actually, I made a mistake here this should be n here because this is following n, this will be n, this will be n. These are all will be n degrees of freedom. When mu is unknown then you will have n-1 degrees of freedom and then the test procedures will be for first case reject H1, in the second case it will be a reject H2 if  $W_{star} <$  and in the third case reject H3 if  $W_{star} < \chi^2_{n-1, 1-\alpha/2}$  or  $W_{star} > \chi^2_{n-1, \alpha/2}$ .

This is about the testing for the parameters of a 1 normal population. Now this type of methods can be applied actually to other distribution also in which certain nice properties for example if the distributions are in the exponential family, if the distributions are having monotone likelihood ratio even though they may not been in the exponential family, in all those situations this type of testing procedures are applicable.

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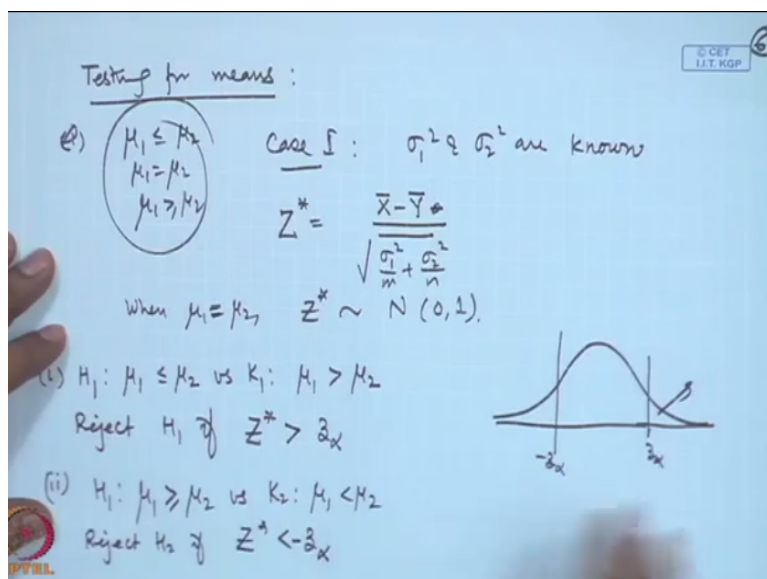




Now I will briefly touch upon the 2 population model for the normal populations. So we consider 2 sample problems like in the case of confidence intervals we have 2 samples available to us, 1 is from say a normal distribution with mean  $\mu_1$  and variance  $\sigma_1^2$  and another independent random sample that is available from normal with mean  $\mu_2$  and variance  $\sigma_2^2$ .

And these 2 samples are taken independently. Now we consider say parameters  $\mu_1, \mu_2, \sigma_1^2, \sigma_2^2$ , this could be our testing problems. Now you can say commonly used problems could be to test whether the mean of the first population is less than the mean of the second population or equal or greater than etc. that means we are interested in the difference here.

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Now naturally this is a problem which can be handled easily using the Neyman Pearson theory. So we consider testing for means. If we consider the testing for means we may consider hypothesis problems of the nature say  $\mu_1 \leq \mu_2$ ,  $\mu_1 = \mu_2$ ,  $\mu_1 > \mu_2$  and so on. These are the types of hypothesis problems that we may have.

So again as before we consider case 1 when  $\sigma_1^2$  and  $\sigma_2^2$  are known. If  $\sigma_1^2$  and  $\sigma_2^2$  are known, then we consider the statistic of the form let me call it  $Z^* = \frac{\bar{X} - \bar{Y}}{\sqrt{\frac{\sigma_1^2}{m} + \frac{\sigma_2^2}{n}}}$ . Now when  $\mu_1 = \mu_2$  then  $Z^*$  follows normal 0, 1. So we utilize this actually. In fact, it can be shown that the maximum of the probability of type 1 error will be achieved when  $\mu_1 = \mu_2$ .

Let us consider various hypothesis testing problems here say  $\mu_1 \leq \mu_2$  against say  $\mu_1 > \mu_2$ . Naturally, if in the alternative case we are saying  $\mu_1 > \mu_2$  that means we will be considering the rejection region on the larger side. So we will consider here that is  $z_\alpha$  so we will consider reject  $H_0$  if  $Z^* > z_\alpha$ . In the second case, here we will be rejecting for the small values of  $Z^*$ .

Now if you consider the small values and then on the left hand side we can consider  $z_{1-\alpha} = -z_\alpha$  so the rejection region will be reject  $H_0$  if  $Z^* < -z_\alpha$ .

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(ii)  $H_3: \mu_1 = \mu_2$  vs  $K_3: \mu_1 \neq \mu_2$ .  
 Reject  $H_3$  if  $|Z^*| > z_{\alpha/2}$

Case II:  $\sigma_1^2 = \sigma_2^2 = \sigma^2$  (unknown)

$$S_1^2 = \frac{1}{m-1} \sum (x_i - \bar{x})^2, \quad S_2^2 = \frac{1}{n-1} \sum (y_j - \bar{y})^2$$

$$S_p^2 = \frac{(m-1)S_1^2 + (n-1)S_2^2}{m+n-2}$$

When  $\mu_1 = \mu_2$ ,  $\sqrt{\frac{mn}{m+n}} \left( \frac{\bar{x} - \bar{y}}{S_p} \right) \sim t_{m+n-2}$

And once again for the 2 sided problem, we may consider this as  $H_0$  here so this will be  $H_0: \mu_1 = \mu_2$  against  $K_3: \mu_1 \neq \mu_2$ . So you will consider reject  $H_0$  if modulus of  $Z^* > z_{\alpha/2}$  that means we will be rejecting on both the sides of the normal curve that

means if the value is in this zone or in this zone that is  $-z_{\alpha/2}$ . Now we can see that the second case when  $\sigma^2 = \sigma_1^2 = \sigma_2^2 = \text{say } \sigma^2$  but this is unknown.

If this is unknown, then we formulate the test statistic. Now let me briefly mention about the large sample cases also. See if we look at the case that I discussed in the beginning here we are considering the approximation by the normal  $0, 1$ . Now suppose the original distribution need not be normal.

But if we are considering the testing for the mean and we have large sample in that case we can consider by applying central limit theorem that this will be approximately normal  $0, 1$ . So the test procedure that I have mentioned here will still be applicable for the large sample cases. However, when  $\sigma^2$  is unknown in that case this procedure will not be applicable.

Similarly, in this problem when I considered comparison of  $\mu_1, \mu_2$  when  $\sigma_1^2$  and  $\sigma_2^2$  are known, in that case even if the original populations need not be normal then by central limit theorem this result will be applicable. However, when  $\sigma_1^2, \sigma_2^2$  are unknown, then this result central limit theorem will not be applicable and we are about to go for the exact procedures.

So let us consider here if we remember our notations that we developed for the confidence intervals that is we considered  $S_1^2 = \frac{1}{m-1} \sum (X_i - \bar{X})^2$  and we considered  $S_2^2 = \frac{1}{n-1} \sum (Y_j - \bar{Y})^2$  and  $S_P^2$  was taken as  $\frac{m-1 S_1^2 + n-1 S_2^2}{m+n-2}$ . Then based on this we had considered that when  $\mu_1 = \mu_2$  then you have  $\frac{\bar{X} - \bar{Y} - \mu_0}{\sqrt{\frac{mn}{m+n} S_P^2}}$ .

This has T distribution on  $m+n-2$  degrees of freedom. Therefore, we can write down the tests for all the 3 situations and let me just repeat it again.

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Case II:  $\sigma_1^2 = \sigma_2^2 = \sigma^2$  (unknown)

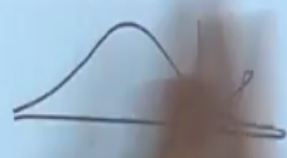
$$S_1^2 = \frac{1}{m-1} \sum (x_i - \bar{x})^2, \quad S_2^2 = \frac{1}{n-1} \sum (y_j - \bar{y})^2$$

$$S_p^2 = \frac{(m-1)S_1^2 + (n-1)S_2^2}{m+n-2}$$

When  $\mu_1 = \mu_2$ ,  $T_1 = \frac{\bar{x} - \bar{y}}{\sqrt{\frac{S_p^2}{m} + \frac{S_p^2}{n}}} \sim t_{m+n-2}$

$H_1$  vs  $K_1$  ( $\mu_1 \leq \mu_2$  vs  $\mu_1 > \mu_2$ )

Reject  $H_1$  if  $T_1 > t_{m+n-2, \alpha}$



We are having the testing problems that is  $H_1$  versus  $K_1$  that is  $\mu_1 \leq \mu_2$  versus  $\mu_1 > \mu_2$ , in this case your rejection region will be on the right hand side that is  $t_{m+n-2}$  alpha. So your region will be reject  $H_1$  so let us call this quantity say  $T_1$  so this is equal to  $T_1 > t_{m+n-2}, \alpha$ .

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Reject  $H_2$  if  $T_1 < -t_{m+n-2, \alpha}$

Reject  $H_3$  if  $|T_1| > t_{m+n-2, \alpha/2}$ .

Case III: When  $\sigma_1^2$  &  $\sigma_2^2$  are completely unknown.

$$T_2 = \frac{\bar{x} - \bar{y}}{\sqrt{\frac{S_1^2}{m} + \frac{S_2^2}{n}}}$$

When  $\mu_1 = \mu_2$ ,  $T_2$  has approximate  $t_p$  dist<sup>n</sup>

where  $p = \frac{(S_1^2/m + S_2^2/n)}{(S_1^2/m + S_2^2/n)}$

$p$  to be integer part of the right hand expression

In the second case, you will be on the left hand side so you will say reject  $H_2$  if  $T_1 < -t_{m+n-2}, \alpha$  and for the 2 sided case, it will be reject  $H_3$  if modulus of  $T_1 > t_{m+n-2}, \alpha/2$ . Now the third case is when sigma 1 square and sigma 2 square are completely unknown. If they are completely unknown in this particular case, we consider say  $T_2 = \frac{\bar{x} - \bar{y}}{\sqrt{S_1^2/m + S_2^2/n}}$ .

When  $\mu_1 = \mu_2$  then  $T_2$  has approximate t distribution on some  $p$  degrees of freedom where  $p$  is given by  $S_1^2/m + S_2^2/n$  whole square  $/ S_1^2$  to the power  $4/m$  square  $\cdot m - 1 + S_2^2$  to the power  $4/n$  square  $\cdot n - 1$ . We usually take  $p$  to be integer part of the right hand expression.

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$$T_2 = \frac{\bar{X} - \bar{Y}}{\sqrt{\frac{S_1^2}{m} + \frac{S_2^2}{n}}}$$
 When  $\mu_1 = \mu_2$ ,  $T_2$  has approximate  $t_p$  dist<sup>n</sup>  
 where  $p = \frac{(S_1^2/m + S_2^2/n)^2}{(S_1^4/m^2 + S_2^4/n^2)}$   
 $p$  to be integer part of the right hand expression  
 The test procedures for  $H_1$  vs  $K_1$ ,  $H_2$  vs  $K_2$  &  $H_3$  vs  $K_3$  can be written based on  $T_2$ .

So the test procedures can be formulated. The test procedures for  $H_1$  versus  $K_1$ ,  $H_2$  versus  $K_2$  and  $H_3$  versus  $K_3$  can be based on  $T_2$ . So I am not describing here for example in the first case it will be rejecting  $H_1$  if  $T_2 > t_p$  alpha. Similarly, in the second case, it will be reject  $H_2$  if  $T_2 < -t_p$  alpha and in the third case it will be reject  $H_3$  if modulus of  $T_2 > t_p$  alpha/2.

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**Case IV: Case of Paired Observations**  
 $X_i \rightarrow$  score on tests of  $n$  students before the coaching  
 $Y_i \rightarrow$  score on tests of  $n$  students after the coaching  
 $(X_i, Y_i) \sim \text{BVN} \left( \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix} \right)$   
 $d_i = X_i - Y_i \sim N(\mu_1 - \mu_2, \sigma_d^2)$   
 $\bar{d} = \frac{1}{n} \sum d_i$ ,  $\sigma_d^2 = \frac{1}{n-1} \sum (d_i - \bar{d})^2$ ,  $\sigma_d^2 = \sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2$   
 $T_3 = \frac{\sqrt{n} \bar{d}}{\sigma_d}$

We had also considered a case of paired observations. In the confidence interval, I had described the situation that is where  $\mu_1$  and  $\mu_2$  are resulting from the same set of individuals or items for example it could be the certain learning procedure and we look at the

ability of the candidates before conducting the learning procedure and after conducting the learning procedure after a certain time.

For example, you could say  $X_i$ s are the scores on tests of  $n$  students okay before the coaching you can say and  $Y_i$ s are the scores on tests of  $n$  students after the coaching. In this case, we can consider say  $X_i, Y_i$  this is following a bivariate normal distribution with some mean say  $\mu_1, \mu_2$  and variances  $\sigma_1^2, \sigma_2^2$  and co-variances  $\rho \sigma_1 \sigma_2$ .

So if we want to compare  $\mu_1$  and  $\mu_2$  we may as well consider say  $d_i = Y_i - X_i$  or  $X_i - Y_i$  say so then this will follow normal  $\mu_1 - \mu_2$  so I call it  $\sigma_D^2$  where  $\sigma_D^2$  is nothing but  $\sigma_1^2 + \sigma_2^2 - 2\rho \sigma_1 \sigma_2$ . Now it is immaterial, we can actually consider our observations to be  $d_i$  and we can calculate  $\bar{d}$  that is  $\frac{1}{n} \sum d_i$ .

And we can consider  $s_d^2$  as  $\frac{1}{n-1} \sum (d_i - \bar{d})^2$  and we can formulate the test statistic let us call it  $T_3 = \sqrt{n} \bar{d} / s_d$ .

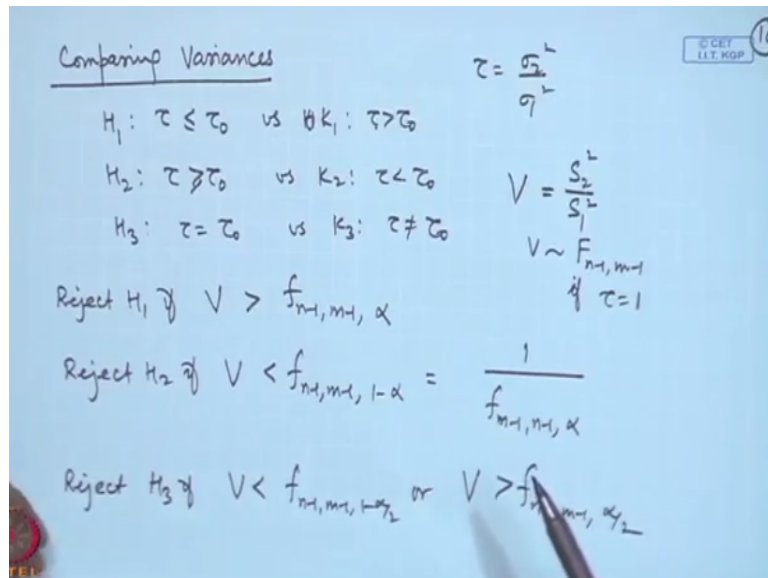
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$(Y_i) \sim N(\mu_2, \sigma_2^2)$   
 $(X_i, Y_i) \sim N(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho \sigma_1 \sigma_2)$   
 $d_i = X_i - Y_i \sim N(\mu_1 - \mu_2, \sigma_D^2)$   
 $\sigma_D^2 = \sigma_1^2 + \sigma_2^2 - 2\rho \sigma_1 \sigma_2$   
 $\bar{d} = \frac{1}{n} \sum d_i$   
 $s_d^2 = \frac{1}{n-1} \sum (d_i - \bar{d})^2$   
 $T_3 = \frac{\sqrt{n} \bar{d}}{s_d}$   
 $H_1: \mu_1 \leq \mu_2$  vs  $K_1: \mu_1 > \mu_2$   
 Rej  $H_1$  if  $T_3 > t_{n-1, \alpha}$   
 $H_2: \mu_1 > \mu_2$  vs  $K_2: \mu_1 \leq \mu_2$   
 Rej  $H_2$  if  $T_3 < -t_{n-1, \alpha}$   
 $H_3: \mu_1 = \mu_2$  vs  $K_3: \mu_1 \neq \mu_2$   
 Rej  $H_3$  if  $|T_3| > t_{n-1, \alpha/2}$

So the test procedure then for  $H_1$  that is  $\mu_1 \leq \mu_2$  versus  $K_1 \mu_1 > \mu_2$ . Once again you can see here it will be reject  $H_1$  if  $T_3 > t_{n-1, \alpha}$ . Similarly, if I consider  $H_2 \mu_1 > \mu_2$  versus  $K_2 \mu_1 \leq \mu_2$  then it will be reject  $H_2$  if  $T_3 < -t_{n-1, \alpha}$ . Similarly, if I consider the 2 sided testing problem,  $\mu_1 = \mu_2$  against  $\mu_1 \neq \mu_2$ , then the test procedure will reject  $H_3$  if modulus of  $T_3 > t_{n-1, \alpha/2}$ .

So we have considered various cases for the comparison of the means of 2 normal populations. Let us also consider a case for comparison of the variances of 2 normal populations.

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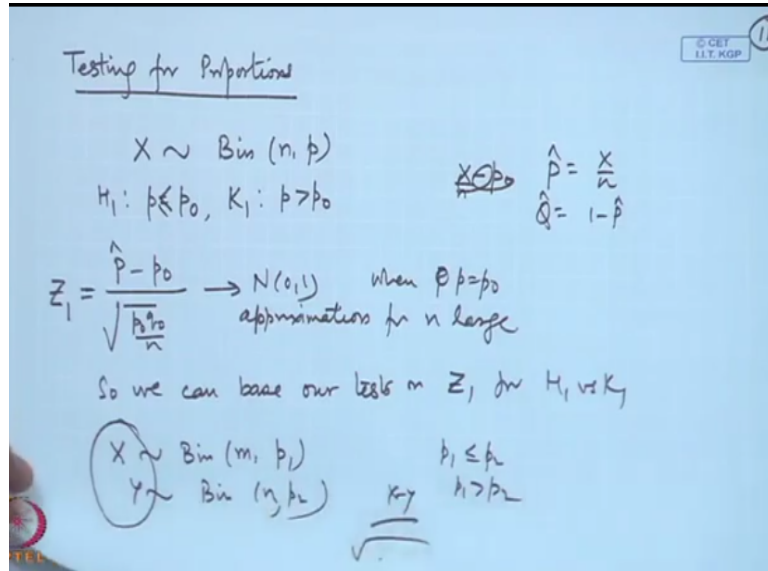
Comparing variances, so that means we may have a testing problem of the nature so let us write say  $\tau = \sigma_2^2 / \sigma_1^2$ . So we may consider say  $\tau$  is  $\leq \tau_0$  against say  $\tau > \tau_0$ ,  $\tau < \tau_0$ ,  $\tau = \tau_0$  against  $\tau$  is not equal to  $\tau_0$ . In all these cases, we may consider say  $S_2^2 / S_1^2$ . Let us call it say  $V$ . Now hence  $\sigma_1^2 = \sigma_2^2 \tau = 1$ .

So then  $V$  will have  $F$  distribution on  $n-1, m-1$  degrees of freedom if  $\tau = 1$ . Therefore, we can use this for the testing here. In the first case it will be reject  $H_1$  if now you can see here you have to reject for the large values of  $\tau$  so large values of  $\tau$  will correspond to the large values of  $V$  so if  $V > f_{n-1, m-1, \alpha}$ . In the second case, reject  $H_2$  if  $V < f_{n-1, m-1, 1-\alpha}$  which is of course equal to  $1 / f_{m-1, n-1, \alpha}$ .

In the third case, it will be 2 sided regions, if  $V < f_{n-1, m-1, 1-\alpha/2}$  or  $V > f_{m-1, n-1, \alpha/2}$ . Of course, we may also consider the case when  $\mu_1$  and  $\mu_2$  are known. In that case, the only thing is that in place of  $S_2^2 / S_1^2$  you can consider  $\sum (Y_j - \mu_2)^2 / \sum (X_i - \mu_1)^2$  and this  $F$  statistic will be replaced by  $f_{nm}$  in rather than  $n-1, m-1$ .

So without spending too much time on that I will just skip that portion, so this is the case for the comparison of the variances. Now equivalently we may have testing problem for the proportions also.

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Testing for proportions, for example if I am considering say  $X$  following binomial  $n, p$  and we may like to test about say  $p = p_0$  or  $p \leq p_0$  as before so we may consider the tests based on  $X - p_0$ , let us write it as  $\hat{p} = X/n$  and  $\hat{q} = 1 - \hat{p}$ . So we may consider basing our tests on this. We can consider  $\hat{p} - p_0 / \sqrt{p_0 q_0 / n}$  and we can consider the normal  $T$  for this thing.

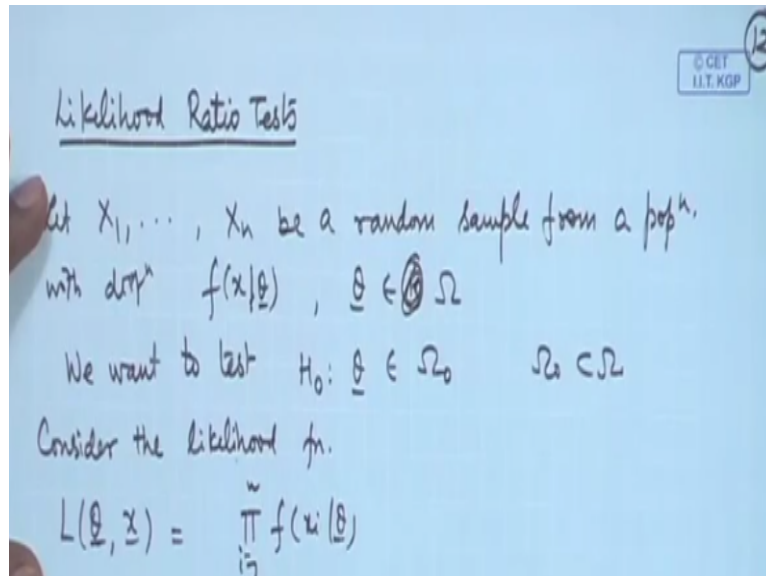
That is when  $p = p_0$  then this is approximately normal  $0, 1$  okay. This is approximation for  $n$  large so let us write it say some  $Z_1$  so we can base our tests on  $Z_1$  for hypothesis  $H_1$  versus  $K_1$  or similarly we can consider  $p > p_0$  against  $p \leq p_0$  etc. all those kinds of cases can be considered. We can also consider this situation  $X$  following say binomial  $m, p_1$  and  $Y$  following binomial  $n, p_2$ .

And we may like to compare  $p_1 \leq p_2, p_1 > p_2$  etc. So we can consider based on the differences  $X - Y$  and then we can consider the  $p_1, q_1$  etc. So all those things can be done. I am not spending too much time on this problem here. Now the test that I have discussed here they are based on the Neyman Pearson theory. However, there was another approach which was considered by R.A. Fisher and others.



That is based on the likelihood ratio. In fact, Neyman Pearson came to the  $f_1/f_0$  form based on the likelihood ratios only; however, the approach in a more general form can be described like this.

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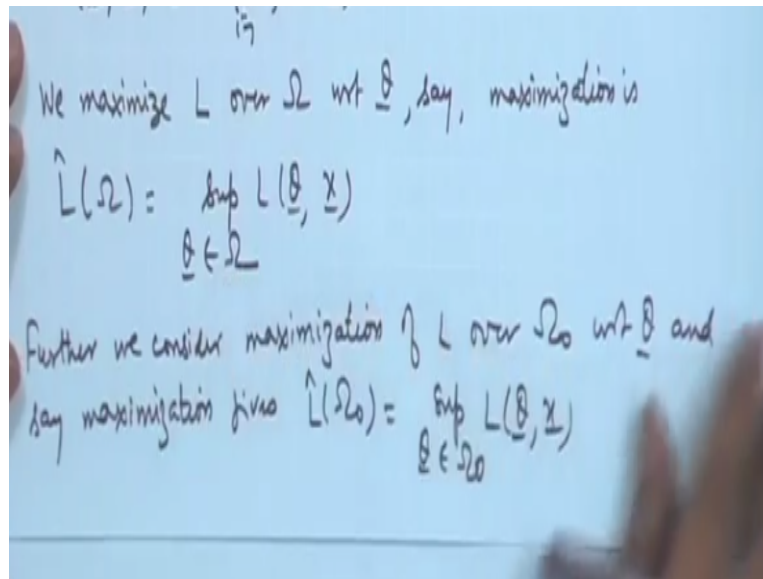
So let me mention this thing, likelihood ratio tests. Let us consider say  $X_1, X_2, X_n$  be a random sample from a population with some distribution. So it could be say  $f_x$  theta we just write in general. Here theta belongs to some parameter space theta. We want to test  $H_0$  theta belongs to say  $\omega_0$ . Let me just change the notation here this omega let me write here. So this  $\omega_0$  is the subset of  $\omega$ .

As you have seen in all these problems like in the binomial problem  $p$  was lying between 0 to 1 so the parameter space was 0 to 1 but in the null hypothesis we are restricting attention to 0 to  $p_0$ . If you consider the previous problems of normal populations etc for example here, you are writing  $\sigma^2 = \sigma_0^2$  but here your  $\mu$  range is from  $-\infty$  to  $\infty$  and  $\sigma^2$  can be  $> 0$ .

So full parameter space is there but in the null hypothesis you are saying  $\mu = \mu_0$  is from  $-\infty$  to  $\infty$  but  $\sigma^2 = \sigma_0^2$ . So you are specifying a region. In the Neyman Pearson theory, it was essential to specify an alternative hypothesis but in the case of likelihood ratio test it is not required. That procedure is based on a simple argument that we consider maximization of the likelihood function under the full region and under the null hypothesis space.

And then we compare them so the logic is as follows. Consider the likelihood function  $L(\theta; x)$  where  $x = \text{product of } f(x_i; \theta)$ .

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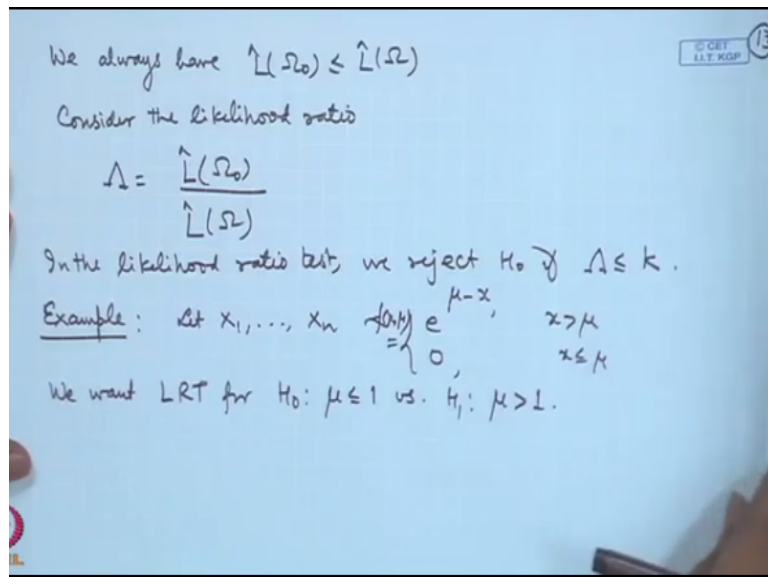
We maximize  $L$  over  $\Omega$  with respect to  $\theta$  say maximization is  $\hat{L}(\Omega) = \sup_{\theta \in \Omega} L(\theta, x)$ . Further we consider maximization of  $L$  over  $\Omega_0$  with respect to  $\theta$  and say maximization is  $\hat{L}(\Omega_0)$ , I call it  $\hat{L}(\Omega_0)$  that is equal to supremum of  $L$  for  $\theta$  belonging to  $\Omega_0$ .

Now you see if the hypothesis  $\Omega_0$  is true that means  $\theta$  belonging to  $\Omega_0$  is true then the maximization of the likelihood function over this will be almost the same as the maximization over the whole space. You can of course notice from a simple mathematical argument that  $\hat{L}(\Omega_0) \leq \hat{L}(\Omega)$  because this is maximization over a subset and this is maximization over the whole space.

So we always have  $\hat{L}(\Omega_0) \leq \hat{L}(\Omega)$ . So naturally if  $\hat{L}(\Omega_0)$  is closer to  $\hat{L}(\Omega)$  that means we have more confidence in the hypothesis  $\Omega_0$  that means the likelihood that  $H_0$  is true is more likely. However, if  $\hat{L}(\Omega_0)$  is much less than  $\hat{L}(\Omega)$  then we have doubts over the correctness or being over  $H_0$  being true.

So therefore if we formulate the ratio  $\hat{L}(\Omega_0)/\hat{L}(\Omega)$  then for the smaller values of that we would tend to believe that  $H_0$  is not true. So this is the basic idea for formulating the likelihood ratio test.

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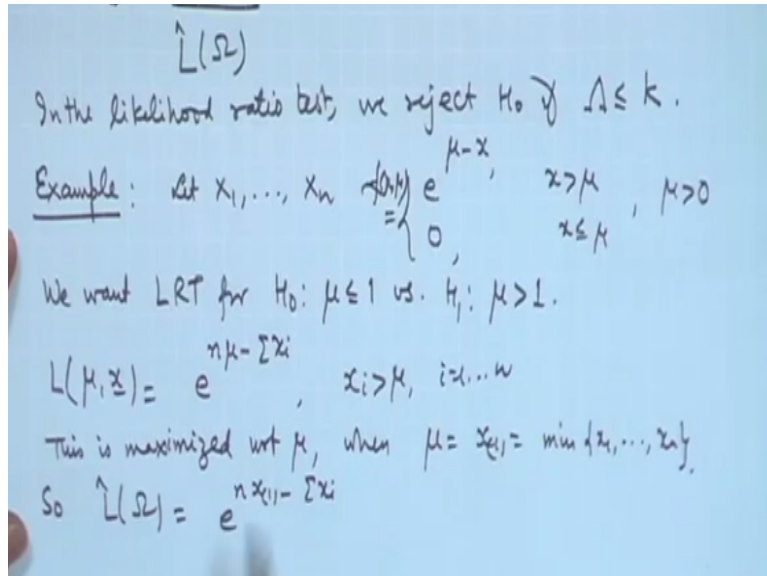


So consider the likelihood ratio so that is let us call it say some lambda that is equal to L hat omega 0. In the likelihood ratio test, we reject H0 if lambda is <= some K. Now once again the question about the choice of K comes and therefore we can choose K to fix the size. We may actually look at what is the probability of rejection? So that is known as the significance testing.

We consider the probability of this and we look at the (()) (44:57) by which we will be actually accepting. For example, if I consider say alpha=0.1 or alpha=0.5 and we look at whether we will be actually rejecting. So the minimum value of 2 which we will be considering that will be called p value of the test. Let us consider an example here. Say X1, X2, Xn follow exponential distribution with parameters say mu.

This is fx mu okay. Now let us consider say we want likelihood ratio test for say mu <= 1 against say H1 mu > 1.

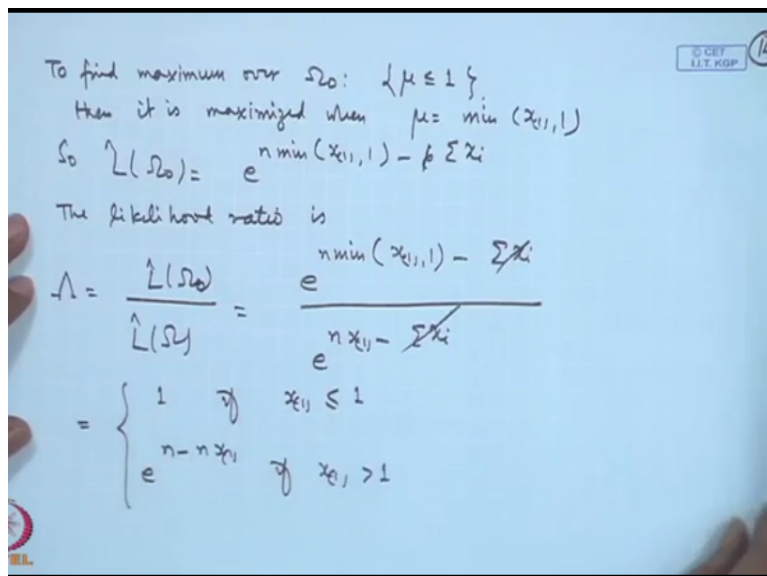
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So we consider here the likelihood function that is equal to  $e$  to the power  $n\mu - \sum x_i$  and here it will be  $x_i > \mu$  for  $i=1$  to  $n$ . So naturally this is maximized, here we can consider say  $\mu > 0$  you may consider this as a typical situation where the life times of components are following exponential distribution with parameter  $\mu$  but here  $\mu$  denotes the minimum guarantee time.

The rate is 1 here so this is maximized with respect to  $\mu$  when  $\mu =$  actually the minimum of  $X_1, X_2, X_n$ . So you get  $\hat{L}(\Omega)$  that is equal to  $e$  to the power  $n X_{(1)} - \sum x_i$  the maximum value of the likelihood function over the parameter space.

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Now let us consider to find maximum over  $\Omega_0$ ,  $\Omega_0$  here is  $\mu \leq 1$ . Then it is maximized when  $\mu =$  minimum of  $x_1$  and 1. Because we are putting 2 restrictions  $\mu \leq$

$x_1$  and  $\mu$  is  $\leq 1$  so the maximum value that  $\mu$  can take is minimum of  $x_1$  and 1. So  $L(\hat{\omega}_0)$  that will become  $e$  to the power  $n$  minimum of  $x_1$  and  $1 - \sigma x_1$ .

So now the likelihood ratio is say  $\lambda = L(\hat{\omega}_0) / L(\hat{\omega}_1)$  so that is equal to  $e$  to the power  $n$  minimum of  $x_1, 1 - \sigma x_1$  /  $e$  to the power  $n x_1 - \sigma x_1$ . So this term naturally cancels out. Now this is equal to 1 if  $x_1$  is  $\leq 1$  and it is equal to  $e$  to the power  $n - n x_1$  if  $x_1 > 1$ . So you can easily see that when the likelihood ratio is 1, you always cannot reject  $H_0$  because this is the best that can happen.

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$$\lambda = \frac{L(\Omega_0)}{L(\Omega_1)} = \frac{e^{n x_1 - \sum x_i}}{e^{n - n x_1}}$$

$$= \begin{cases} 1 & \text{if } x_1 \leq 1 \\ e^{n - n x_1} & \text{if } x_1 > 1 \end{cases}$$

LRT will always accept  $H_0$  if  $x_1 \leq 1$ .

So we can say that LRT will always accept  $H_0$  if  $x_1$  is  $\leq 1$ .

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When  $x_1 > 1$ , we consider the rejection region to be

$$e^{n - n x_1} < k \Rightarrow x_1 > c$$

where  $c$  is to be chosen suitably.

Suppose we want:

$$P(x_1 > c) = \alpha$$

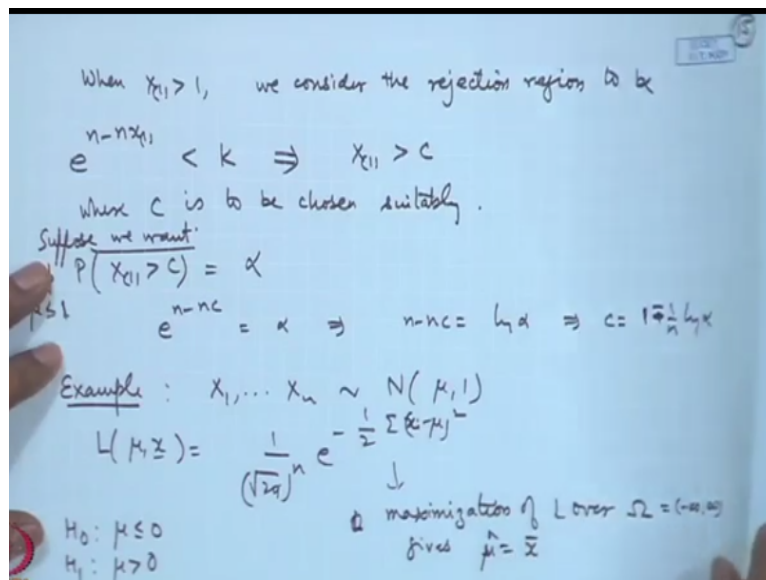
As

$$e^{n - n c} = \alpha \Rightarrow n - n c = \ln \alpha \Rightarrow c = \frac{1}{n} \ln \frac{1}{\alpha}$$

Now let us look at the other region. So when  $x_1 > 1$  we consider the rejection region to be the power  $n - nx_1 < K$ . So if I take log etc then adjust the terms then it is equivalent to something like saying  $X_1 > \text{some } c$  where  $c$  is to be chosen suitably. As an example we may consider say probability of  $X_1 > c = \text{say } \alpha$ . Suppose we want this for supremum  $\mu \leq 1$  suppose we consider this situation.

If we consider this situation, then this is equivalent to the power  $n - nc = \alpha$  that means  $n - nc = \log$  of  $\alpha$  or we can say  $c = 1 - 1/n \log \alpha$ . So you are actually rejecting for a value slightly higher than 1 okay. So this is a typical application of a likelihood ratio test and also you can see I can show you through an example for the normal distribution that how does it compare with the standard test that we obtain using Neyman Pearson theory.

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Let us consider another example say I consider  $X_1, X_2, X_n$  following normal  $\mu, 1$  situation and we consider the likelihood function  $= 1/\sqrt{2\pi}$  to the power  $n$   $e^{-1/2 \sum (x_i - \mu)^2}$ . Now I consider the hypothesis testing problem say  $\mu \leq 0$  against say  $\mu > 0$ . Now if I consider the maximization of  $L$  over  $\Omega$ , here  $\Omega$  is actually  $-\infty$  to  $\infty$  gives  $\hat{\mu} = \bar{x}$ .

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So  $\hat{L}(\Omega) = \frac{1}{(\sqrt{2\pi})^n} e^{-\frac{1}{2} \sum (x_i - \bar{x})^2}$

maximization of  $L$  over  $\Omega_0 = (-\infty, 0)$ , given  $\hat{\mu} = \min(\bar{x}, 0)$

$$\hat{L}(\Omega_0) = \begin{cases} \frac{1}{(\sqrt{2\pi})^n} e^{-\frac{1}{2} \sum (x_i - \bar{x})^2} & \text{if } \bar{x} \leq 0 \\ \frac{1}{(\sqrt{2\pi})^n} e^{-\frac{1}{2} \sum x_i^2} & \text{if } \bar{x} > 0 \end{cases}$$

So  $\lambda = \frac{\hat{L}(\Omega_0)}{\hat{L}(\Omega)} = \begin{cases} 1 & \text{if } \bar{x} \leq 0 \\ e^{-\frac{1}{2} [\sum (x_i - \bar{x})^2 - \sum x_i^2]} & \text{if } \bar{x} > 0 \end{cases}$

Always accept  $H_0$  if  $\bar{x} \leq 0$

And therefore you will get  $L(\hat{\Omega}) = \frac{1}{\sqrt{2\pi}} \pi$  to the power  $n$   $e$  to the power  $-1/2 \sum (x_i - \bar{x})^2$ , but if we consider maximization over  $\Omega_0$  where  $\Omega_0$  is actually  $-\infty$  to  $0$  then we will get  $\hat{\mu} = \bar{x}$  if  $\bar{x} < 0$  then it will be  $\bar{x}$ , but it will be  $0$  if  $\bar{x} > 0$  so that will give us minimum of  $\bar{x}$  and  $0$ . If that is happening then  $L(\hat{\Omega}_0)$  that will become equal to  $\frac{1}{\sqrt{2\pi}} \pi$  to the power  $n$   $e$  to the power  $-1/2 \sum (x_i - \bar{x})^2$  if  $\bar{x} < 0$ .

And it is equal to  $\frac{1}{\sqrt{2\pi}} \pi$  to the power  $n$   $e$  to the power  $-1/2 \sum x_i^2$  if  $\bar{x} > 0$ . so we can put  $\leq 0$  here it does not matter. Now the thing is so the ratio that is  $L(\hat{\Omega}_0)/L(\hat{\Omega})$  if you see that is equal to  $1$  if  $\bar{x} \leq 0$  and it is equal to this ratio  $e^{-\frac{1}{2} [\sum (x_i - \bar{x})^2 - \sum x_i^2]}$  if  $\bar{x} > 0$ . That means always accept  $H_0$  if  $\bar{x} \leq 0$ .

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When  $\bar{x} > 0$ , we reject  $H_0$  when

$$e^{\frac{1}{2}(\sum(x_i - \bar{x})^2 - 2x\bar{x})} < k$$

or  $\sum(x_i - \bar{x})^2 - 2x\bar{x} < c$

$$\sum x_i^2 - n\bar{x}^2 - 2x\bar{x} < c$$

or  $\bar{x}^2 > c_1$

$(Q\bar{x}P > c_2) \checkmark$

Since  $\bar{x} > 0$ , this reduces to  $\bar{x} > c_3$ .

Notice the difference from NP test for  $\bar{x} < 0$  case.

Now in the other case you will formulate the region here when  $\bar{x} > 0$  we reject  $H_0$  when  $e^{\frac{1}{2} \sum (x_i - \bar{x})^2 - 2x\bar{x}} < K$ . So if I take log here and adjust this  $1/2$  here, it is becoming  $\sum x_i^2 - n\bar{x}^2 - 2x\bar{x} < \text{some } c$ . Now this can be further simplified here. We can consider this as  $\sum x_i^2 - n\bar{x}^2 - 2x\bar{x} < c$ .

So this cancels out so we get actually  $\bar{x}^2 > \text{some } c$ . So rejection region is turning out to be 2 sided something like modulus  $\bar{x} > \text{some } c_1$ . Let us call it  $c_1$  this as  $c_2$  here so actually we can again see here, here I am considering  $\bar{x} > 0$  this is equivalent to  $\bar{x} > c_2$  okay.

Since  $\bar{x}$  is positive this reduces to  $\bar{x} > \text{some } c_3$  kind of thing. Now if you compare it with the Neyman Pearson test, there it would have been  $\sqrt{n} \bar{x} > z_{\alpha}$ . Now here it is like this only in this particular portion, but when  $\bar{x} < 0$  we are always accepting  $H_0$  so that is the difference from the Neyman Pearson test. So notice the difference from NP test for  $\bar{x} < 0$  case.

But  $\bar{x} >$  then it is but for all practical purposes you can see because  $\alpha$  will be sufficiently small, therefore  $z_{\alpha}$  value will be very close to high and therefore the 2 tests will be practically the same. In the parametric methods, I have concentrated mostly on the point estimation, confidence interval and testing of hypothesis problems. So there are other cases also when we do not have the parameter specified.

That means the distribution is not specified and we consider distribution free methods; however, that will be slated for a different zone. Now we will be moving over to another topic in this statistical methods so that I will be starting from the next lecture.