Modeling Transport Phenomena of Microparticles
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Lecture – 13
Viscous flow past a spherical drop

Hello! So in the previous lectures we discussed about applications of arbitrary solutions of stokes flows and then some applications like the mechanics of swimming microorganisms. So now justifying the title of this course that is 'Modeling transport phenomena of microparticles', we are going to discuss another application that is a 'Modeling transport of viscous drops' okay.

So typically when you hear about the word viscous drops, you can as a layman you can imagine several applications. In real life we come across viscous drops in variety of situations right, like micro to macro level. So drops and bubbles these are sometimes, time and often they are used one for the other and vice versa but there is some basic differences which one can formally define okay.

So when it comes to application like some gas bubbles in some soda okay, so that is like some gas is trapped inside a volume okay and external there is another fluid right. So basically these two are immiscible. So in such case that is called a bubble, so whereas if suppose there is a liquid trapped in a volume so then it is a it is a drop so that is a basic difference. So you can imagine drops mean you have a lot of examples in real life.

So today we are going to discuss about transport mechanism of such viscous drop. Of course to start with we have a spherical scenario okay and then how the corresponding transport can be modelled. So in order to model typically we talk about spherical drop, let us say in another fluid. So then since these two fluids are immiscible so we have to know what are the corresponding interface mechanism right.

So we spend some time on discussing the corresponding interface mechanism okay.

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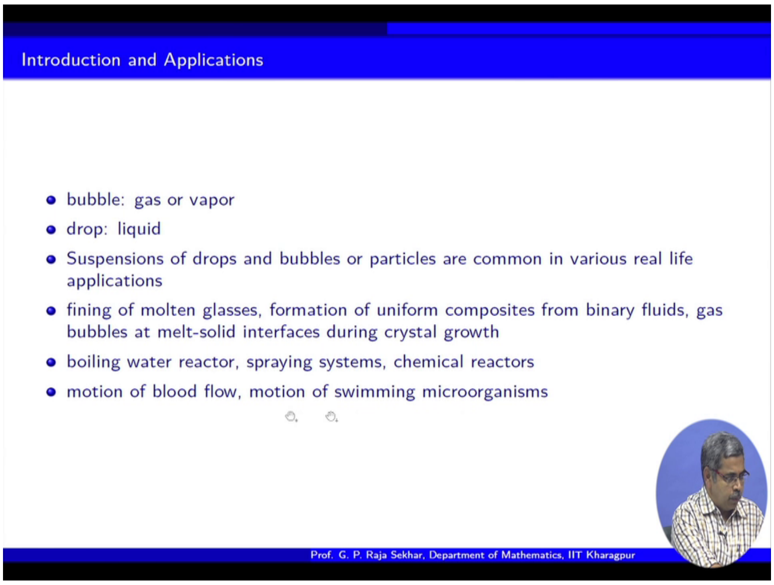

So as I indicated bubble means gas or vapour trapped in a volume inside another immiscible fluid then drop so a liquid which is trapped in another fluid okay. So there are several examples. Then you see several applications where suspensions of drops or bubbles okay and even in our most of the propelling systems rockets etcetera so where liquid fuel is used.

So you come across the impact of drops and suspension of drop very much and these are also used in various mixing technologies okay. So these are some other additional applications. For example formation of uniform composites, that is where I said the mixing technology. So to have a uniform compositor one has to have a fine mixing technology. So and there should not be any deposit so they use some bubbles etc. and then finally when they burst so you get a complete uniform texture okay.

So these are some spraying systems, chemical reactors. So you can plenty one can give and also these one can see some drop droplet phenomena when you have a motion of swimming microorganisms or blood flow okay.

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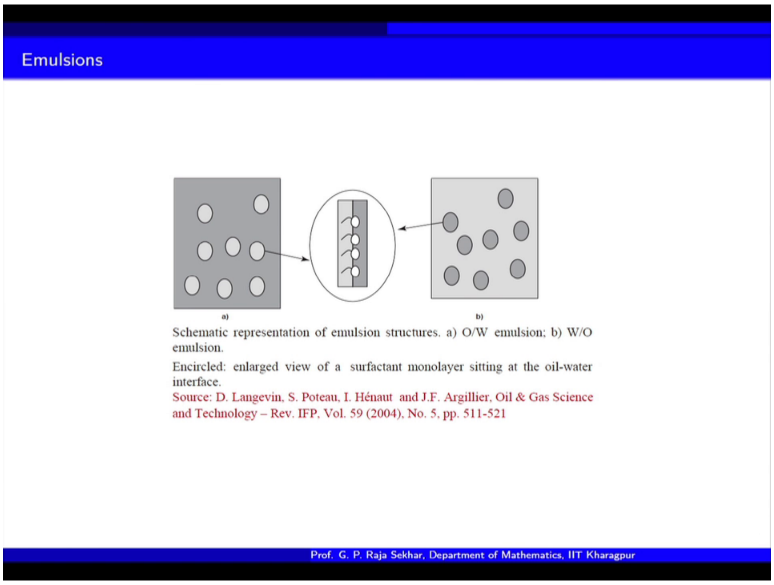

So most interesting where lot of applications are oil industry. So typically you can see oil in water or water in oil okay. So this is the other scenario and sometimes these droplets have see this is water okay oil in water. So this oil so you can see so there will be some deposits. So these are the surfactant layers okay. So that means when you have a droplet you can just visualise.

If you have a drop in an immiscible flow so then if you put some like contaminant concentration something, immediately we will see some particles will go and accumulate on the surface of that okay. So they form a surfactant mono layers right. So definitely they will impact the migration of the drop okay. So that is the applications.

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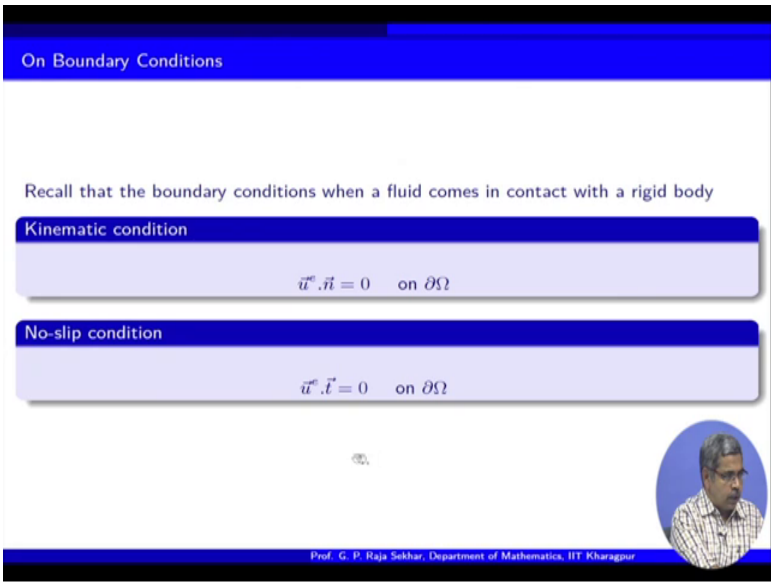

Now let us have a quick recall of the interface conditions in case of a fluid coming in contact with the rigid body. So we discussed this is a kinematic condition where normal velocity is 0 if the assuming that the body is at rest okay. Similarly, this is the no slip condition so the tangential velocity is a 0. Now we need to discuss similar conditions when you have a fluid comes in contact with another immiscible fluid okay.

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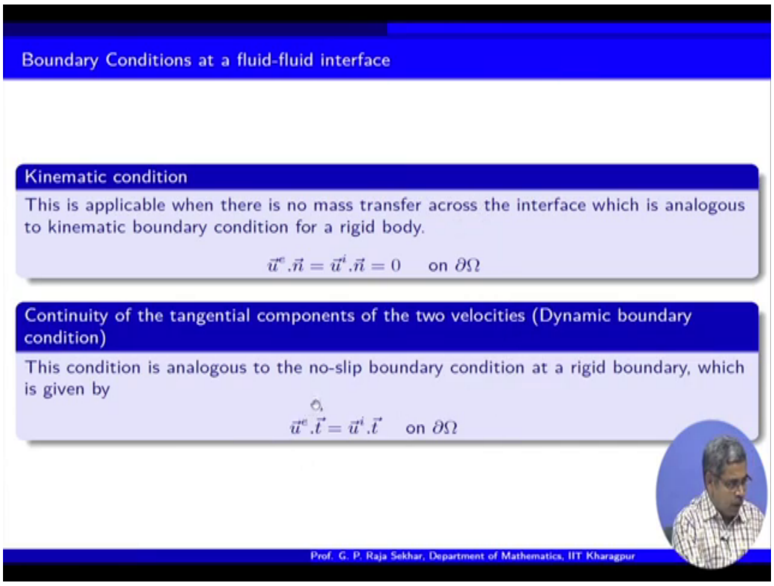

So in this scenario since there is no mass transfer happening across because both are immiscible, so there is no mass transfer across the interface. Therefore, the corresponding normal velocity exterior is 0 and interior is 0 okay. So this is not a good notation. We should have written two of them independently.

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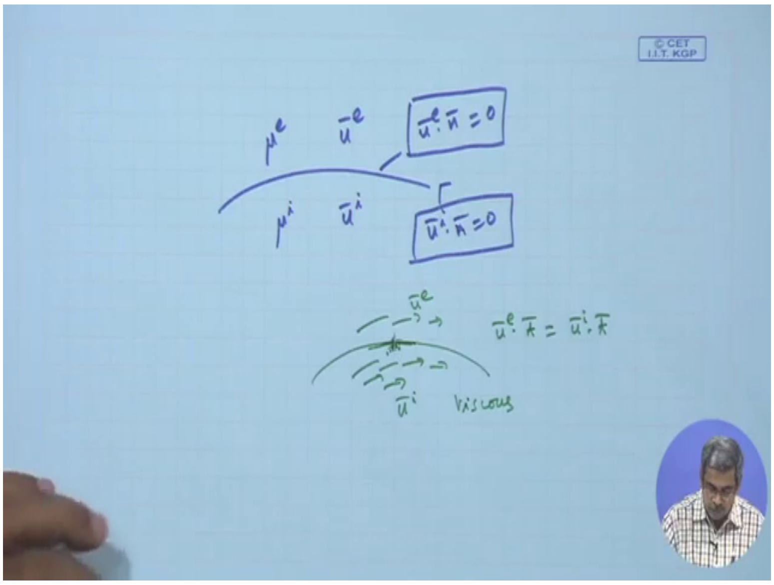

So suppose you have an interface where you have some exterior fluid okay with velocity ue, interior let us say this is the velocity. And if this is the boundary, so then since there is no mass transfer because this is of viscosity Mu i this is of viscosity Mu e. So then what we are assuming is here ue dot n okay. So of course the corresponding directions of the n matters but since we have homogeneous boundary condition so that is immaterial okay. So this is a kinematic boundary condition for each of the fluid velocities okay.

So now similar to the dynamic boundary condition what would be the condition here assuming the interface is a stationary okay. So dynamic boundary condition is intact. So since you have a fluid motion taking inside so it is expected that the corresponding tangential velocities are continuous okay. So this analogous to the no slip condition at a rigid boundary. So you may be questioning this a bit initially without accepting as it is.

So how to justify in some sense. Suppose so you have fluid inside is also moving because it has an interior velocity and then this fluid is also moving with an exterior velocity. Since the flow is viscous this is very important for us, since the flow is viscous, so the fluid layer which is close to the boundary. So we will have the velocity of this boundary okay and the fluid layer which is adjacent so they will have the velocity of this.

And then these two have to be continuous because we assume that the interface is very thin and there is no mass accumulation at this stage. So hence we get t equal, so this is the corresponding interface condition. Now since this is the viscous environment so are these the boundary conditions? So no naturally we have to discuss about the corresponding stress okay.
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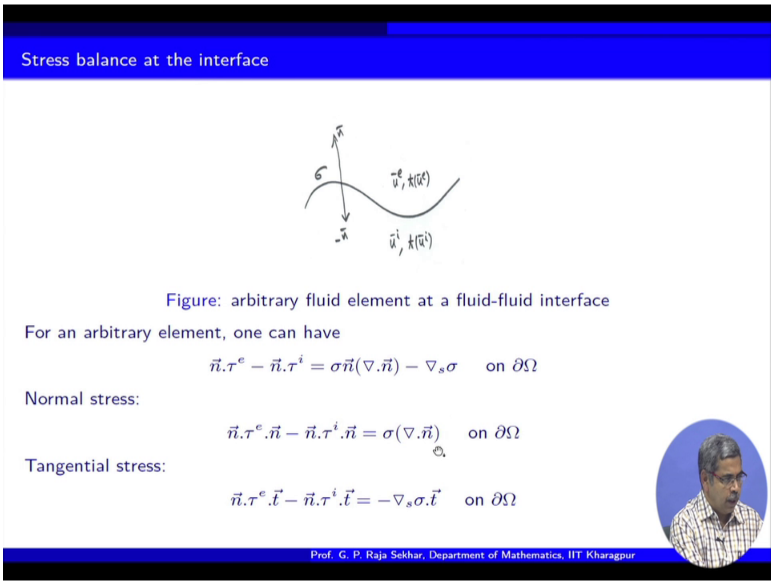

So let us consider such an interface with normal and the corresponding interior we have given negative. So let us say the velocity is ui and the corresponding stress is indicate like this okay. And the corresponding stress is this okay. So when you have such interface what happens to the surface mechanism? So if you ask the question it is natural to think even without much of physics we say okay.

Surface tension plays a vital role because the surface tension plays a role when you have interface not within the domain right. So naturally, the stresses have to be balanced while considering the surface tension effects okay. So that is the stress balance essentially tells. So if some balances the stresses, so what you will see is the balance is something like this tentatively.

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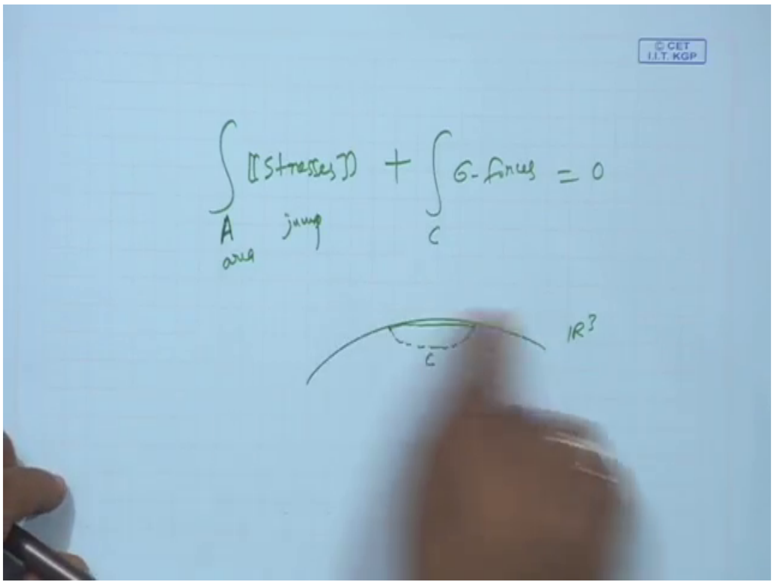

So you have an area so where you are balancing the stresses okay. That means jump exterior to interiors. This is a jump you can say exterior to interior plus also you have to balance the surface tension forces okay, and this total will be 0. So this is the fluid stresses due to the velocity and pressure whereas this is the corresponding forces due to the surface tension. So if this is on an area this will be on a curve bonding the area.

So it is essentially something like this. If you have and let us say this is in 3-D okay, so and if you take a small surface element okay then there will be a bounding curve C okay. So on this you get the corresponding surface tension forces. So if one formally balances this, we get the corresponding stress balance. So I am not giving you the formal proof because it involves a lot of calculations okay.

But one can refer any standard book on viscous drops so some of them will be including the references. So one can get this derivation. So what it tells is for an arbitrary element one can have the external stress minus the internal stress is balanced by essentially surface tension force is a surface activity. Hence this gradient is having a special understanding. So this is the surface gradient okay.

So if it is a 3-D surface let us say and if you are on y equal to constant, this gradient is with respect to xy. If you are on x equal to constant this gradient is with respect to yz, etc., okay. And the curvature also affects okay. Curvature will also influence therefore the surface tension and the corresponding interaction of the curvature so that is this okay. So if you decompose this normal stress is balanced by the surface tension times the curvature.

And then tangentially is balanced by the surface force which is a strictly surface gradient of the surface tension because it is a scalar okay. So if one assumes that there is no like surface tension is a constant, it is not changing, then what you would have this will be 0. So one would expect the continuity of the tangential stresses and in case of normal stress you get a constant multiplied by the curvature which depends on the specific boundary at hand okay.

So this is additional tool that one has to consider when we have a fluid-fluid interface.

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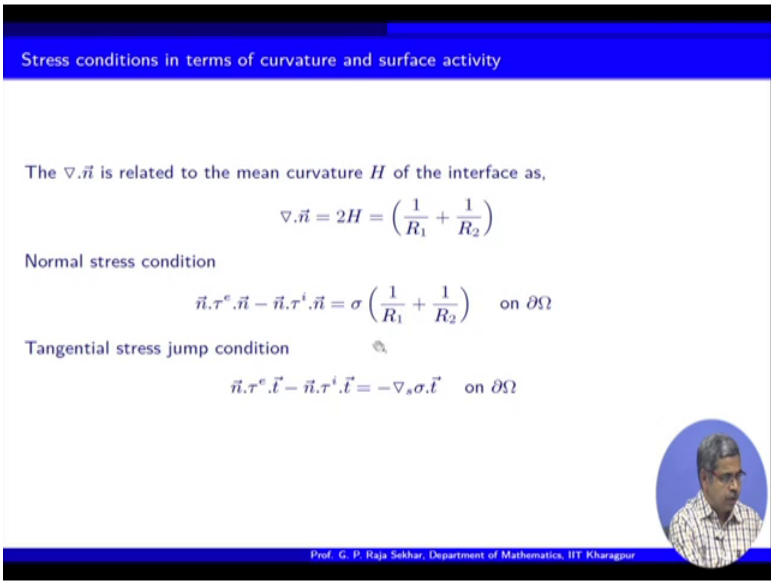

So now to proceed further this is related to the mean curvature H of the interface by this okay. So this is very standard notation. Then normal stress can be expressed in this form and the tangential stress can be expressed in this form okay. So now let us say it is a spherical so then correspondingly this will be simplified okay to 2 by a okay because both the r1 and r2 coincide. So you get the corresponding okay.

So therefore, this is in summary we have kinematic condition, normal exterior is 0, normal interior velocity is 0, tangential velocity is continues, then we have normal stress balance and we have tangential jump condition we call it because there is it is not continuous there is a jump. Of course here also normal stress there is a jump which depends on the curvature. So these are the boundary conditions which are used.

We will see some scenarios where the surface tension is also varying okay. But in this lecture we assume that the surface tension is a constant okay. So now let us consider one problem which is just an extension of for Stokes flow past a rigid sphere that we extend it to Stokes flow past a drop okay, spherical drop.

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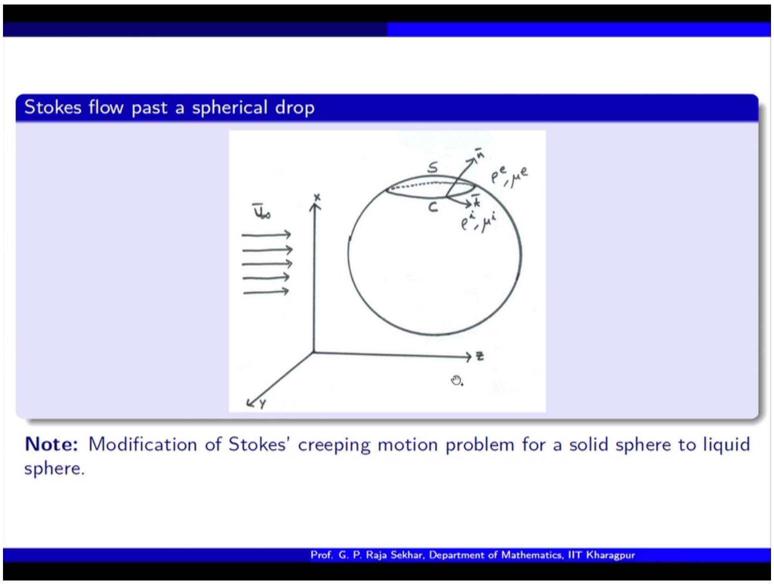

So we are considering Stokes flow past a spherical drop. So you have a drop where you have fluid exterior with the viscosity Mue and density Rho e and interior fluid you have this custom Mu i and the density Rho i. And we have a far-field ambient velocity. So once you have a far-field ambient velocity in presence of the drop it gets disturbed. This flow gets disturbed. So our aim is to compute the flow disturbance okay. So in this approach there are two cases.

One is you have an ambient flow then flow past a drop. On the other hand you can consider you attach a moving frame to the drop, so then drop is moving with a velocity where you have ambient zero velocity okay. So both are equivalent acceptor there is a shifting of the frame takes place okay. So in any case so both of the solutions are equivalent okay. So as I remarked, this is nothing but modification of the rigid sphere problem okay.

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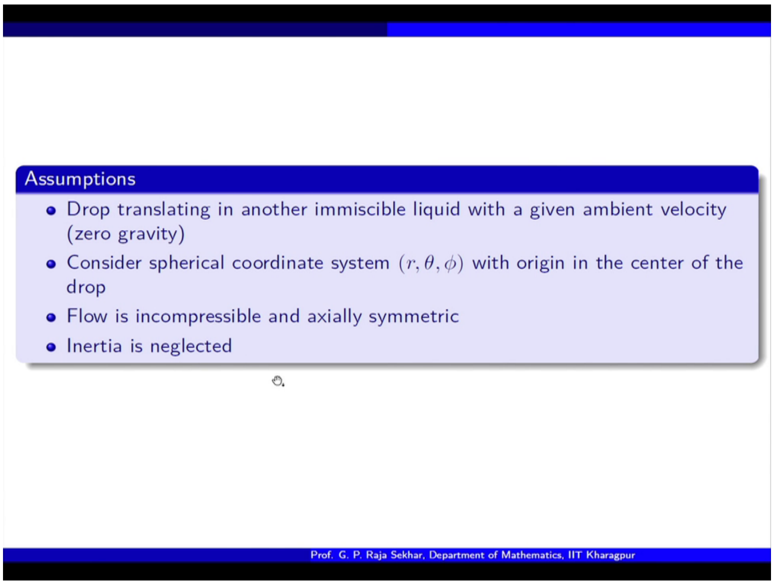

So let us say go for the assumptions drop is translating in another admissible fluid with a given ambient velocity. There is no gravity and we are considering r, Theta, Phi, with origin as the centre of the drop. Even though I have shown the frame like this we are assuming it is at the centre of the order. Flow is incompressible and axisymmetric and inertia is neglected. So essentially we are boiling down to stokes equations okay.

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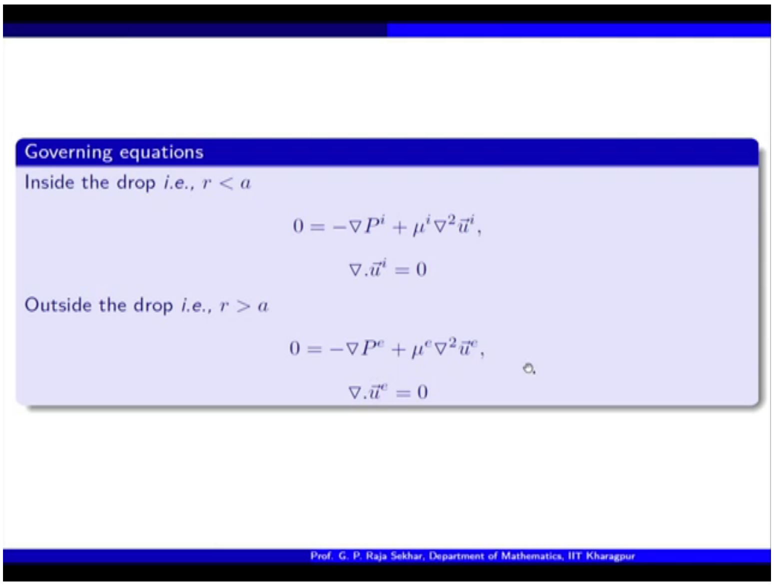

So we assume for r < a inside the drop we have stokes equations. R>a we have stock secure equations and the superscripts denote flow exterior to the drop and interior to the drop. Now since we have assumed axisymmetric, we introduce corresponding stream function and we reduce the corresponding scalar equivalent equation outside-inside.

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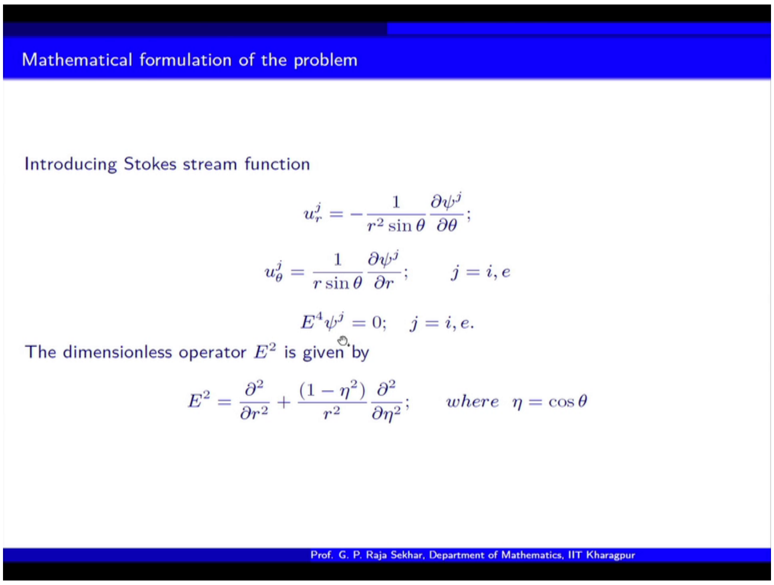

So since we have already discussed a stokes flow past a rigid sphere, so this is exactly following the same methodology. One can get this. So basically we assume once we have a eliminated pressure we get the stream function, what is the stream function equation. So we adopt that okay and the corresponding separable solution in spherical polar coordinates is also very much available.

So for example we must refer the book by Happle and Burner okay which is listed in the references okay.

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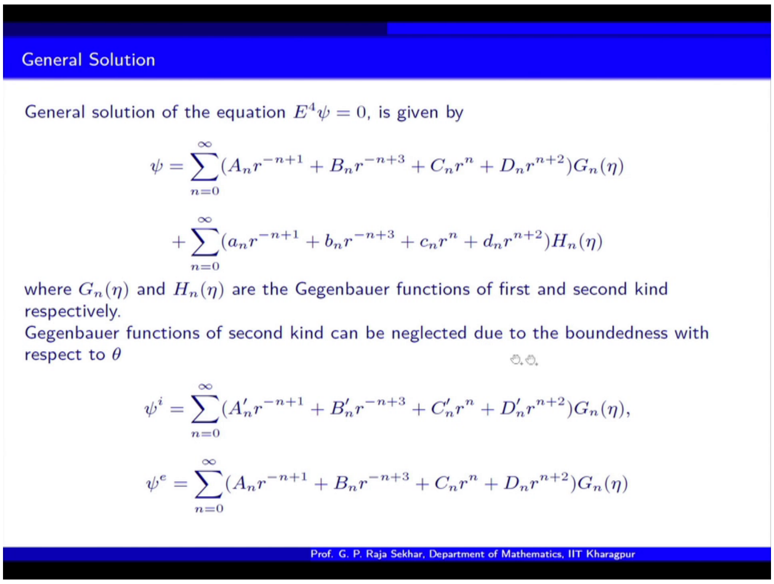

So we move on to the general solution to start with. So you can see the general solution is expressed in terms of something called Gegenbauer functions. So these are essentially these are linear combinations of Legendre polynomials okay. So for separable solution. And the corresponding methodology one should refer some books on PDE or Happle and Burner. So here is Gn + Hn are Gegenbauer functions of first and second kind.

But one may observe that Gegenbauer functions of second kind they are bounded okay with respect to Theta. So due to the boundedness with respect to theta we neglect okay. We need boundedness with respect to theta and these are to be neglected. So the left over will be bounded solution okay. So correspondingly for interior we adopt some coefficients and exterior we adopt some other coefficients.

Because this is the general structure so let us call it for interior with primes and for exterior no primes. So these are the general solution at this stage okay.

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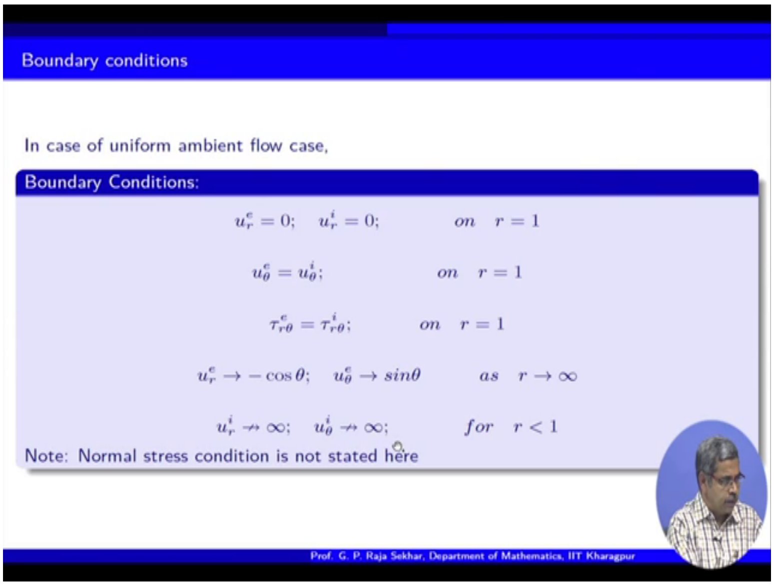

Now what are the boundary conditions since we are a spherical case, u dot n is nothing but the radial component. So radial component of the exterior flow is zero, radial component of the interior flow is zero, then tangential component is continuous. Which means here only the theta component because due to axisymmetry there is no Phi component. Then the corresponding shear stress; that is tangential stress is continuous okay.

If you recall we had jump and the jump is nothing but the surface gradient of the surface tension. So now, here when we are writing this we are assuming that surface tension is constant. It is not depending on any quantity. For example, surface tension can vary with temperature okay or surface tension can vary with concentration or a combination of both. So in which case the surface gradient has to be there.

But for the present problem we are considering surface tension is a constant, therefore, the corresponding substrate gradient is 0 therefore, the corresponding tangential stress is continuous okay. And we have far field conditions. So this is uniform flow along z direction. So corresponding radial component and corresponding tangential component. You might be wondering we did not use normal stress so far.

So we did not use any of them we are simply stating so that is a remark. Normal stress condition is not stated here so we will pay some attention to this why we have not stated here okay. And this is for interior flow of the boundedness condition. We are not allowing any similarities okay.

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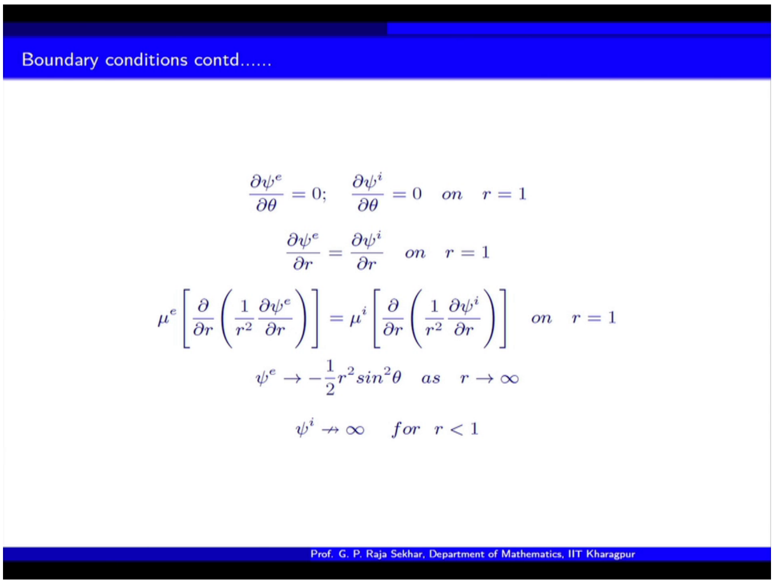

Now we have introduced stream function therefore it makes sense to get the corresponding boundary conditions in terms of stream function. So that this is the normal velocity, tangential velocity continuity and then tangential stress continuity on the boundary okay. Far field is given by this corresponding far field condition. So corresponding to uniform flow this will be the equivalent stream function okay.

So that is as I indicated we are not commenting at this stage the normal stress balance under the boundedness okay. So we go for the corresponding general solution, then use these boundary conditions okay. If you recall for Stokes flow what we have done? Once we have a far-field behaviour, then all the functional dependency of theta is controlled by the far field behaviour.

So correspondingly no point in considering the other functional forms okay. So that means you have a complete basis but we know that for solution only particular solution is contributing. So the remaining we are throwing because even if you consider and do the algebra you get a homogeneous system which gives you a trivial solution okay. So hence it makes sense to first impose the far field condition and then obtain restricted solution okay.

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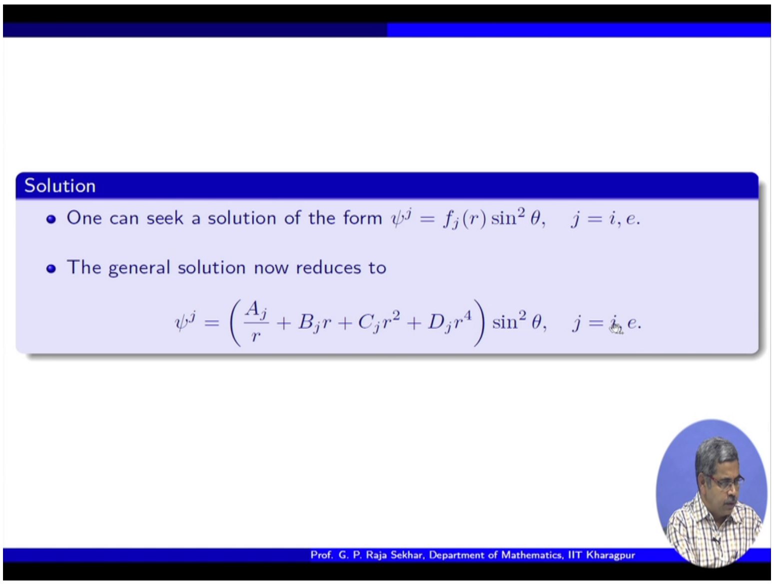

So that is what we do. Since far field is behaving like Sin power 2 Theta, we assume each interior and exterior is of this form. This we have done very much for the rigid sphere case. Therefore, we have such structure okay for the interior and exterior. So from prime we have to be consistent j i means interior e means exterior. Now we use the far field condition then we get De is 0 and Ce is -1/2. So how did we do this?

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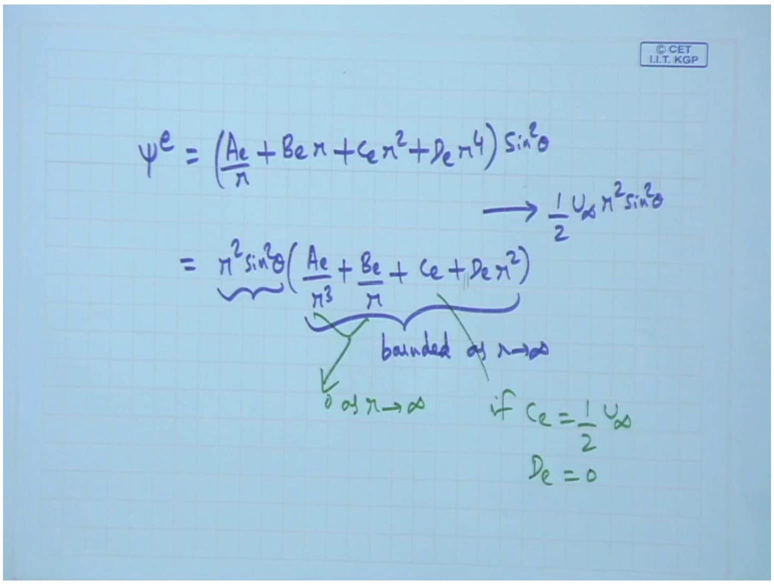

So this we have shown earlier so we can discuss. So exterior Psi is given by Ae/r Ber Cer power 2 so Der power 4 Sin power 2 Theta. But we have a far-field behaviour so this must go to 1/2 U infinity r power 2 Sin power 2 Theta okay. So what we are doing? So if we take r power 2 Sin power 2 Theta common, so this will be cube okay. Now far field has a functional dependency like this so far field should behave like this that means we want this to be bounded okay.

So in order as r goes to bounded as r goes to infinity. So r goes to infinity so these two go to zero. They go to 0 as r goes to infinity okay. So then Ce if Ce is a 1/2U if Ce is 1/2U infinity, then the flow goes to exactly what we are looking for. But this is the one which is making unbounded. So therefore, this implies De is zero okay. So this gives you De is 0 and Ce is -1/2 for U can be 1 here if your non dimensionalize.

So simply we get C is half okay. So then interior solution we need a regularity okay.

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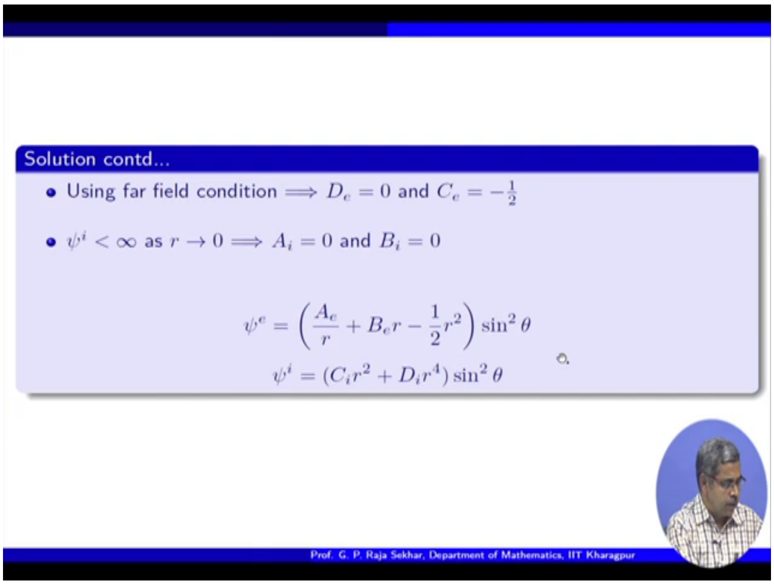

So correspondingly for interior solution we consider Ai is 0, Bi is 0. Therefore, the reduced stream functions exterior and interior are this. So we do not have U infinity because we are working in normalized the situation non dimensionalized okay. Now rest of the boundary condition what is the boundary condition?

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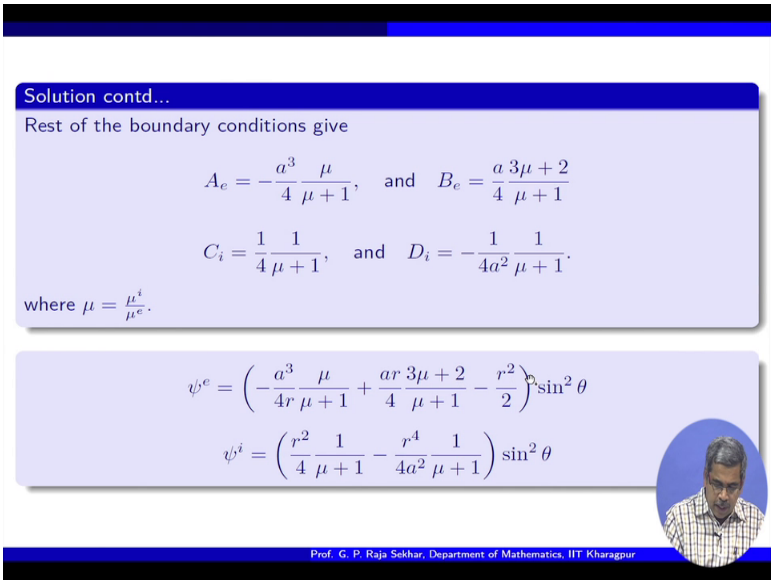

Tangential stress balance so we have once we have this we use the tangential stress balance okay. So we have to compute Dow Psi Dow r, then 1 over r power 2, then Dow by Dowr and mind you so there is a coefficient viscosity Mue and this is Mu i. So therefore, the terms not immediately cancel because you have they would not come up as a common because there is a ratio of Mue and Mui involved.

So this algebra can be done on this because you have only one two three four coefficients. That is very straightforward okay. So we can determine these four coefficients. You can see as I indicated Mu is involved in this coefficients which is the ratio of the viscosity. So that is interior to the exterior. So again I am mentioning this algebra is very straightforward. Computing the tangential stress balance is very straightforward and nothing much.

It just two partial derivatives on Psi okay. So you can do it very easily to realize that we get this coefficient. So once we have this coefficients we have the corresponding stream function exterior and interior.

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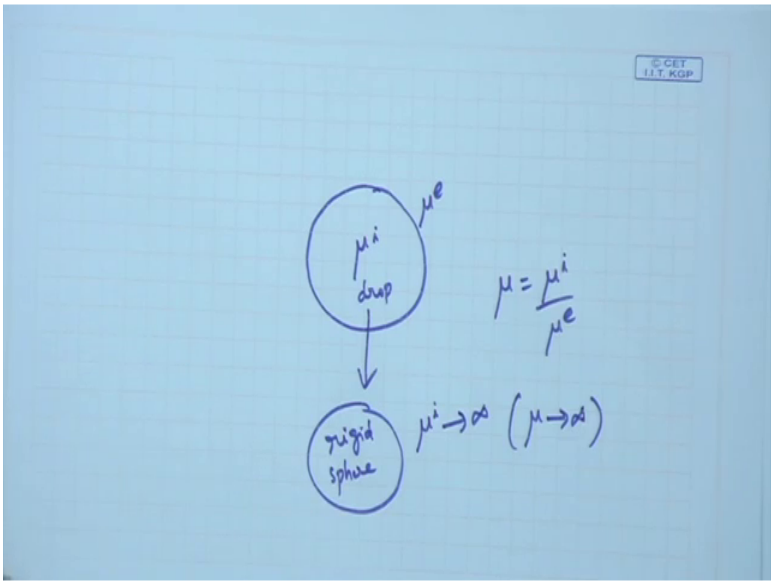

Now this is the case of drop having viscosity Mu i and Mu e okay and we have Mu which is ratio of Mu okay Mu e. So if you want to get the limiting case of this is a drop. If we want a limiting case of rigid sphere then we go for interior should be highly viscous. That means Mu i goes to infinity. In other words Mu goes to infinity. If you take this limit we can retrieve the corresponding solution for the stokes flow past a rigid impermeable sphere okay.

So now you should ask the question still we did not use the normal stress balance okay and we managed the solution right. So what is the scenario here? Our assumption is who is keeping the drop spherical right? Surface tension is keeping the drop spherical right. So under this assumption the normal stress balance is automatically satisfied okay. So if you deform the surface so then the normal stress balance becomes an additional condition and using that one can determine the corresponding deformation okay.

So this involves some algebra but I will just try to explain a bit so before that you can see this is for a particular viscosity ratio streamlines are given.

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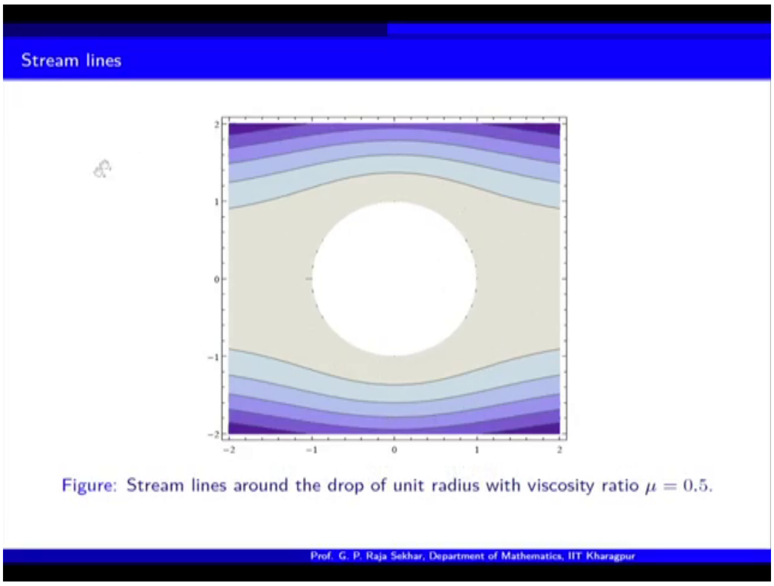

So you can draw for various viscosity ratios and see how the streamlines behave to get some physical insights okay.

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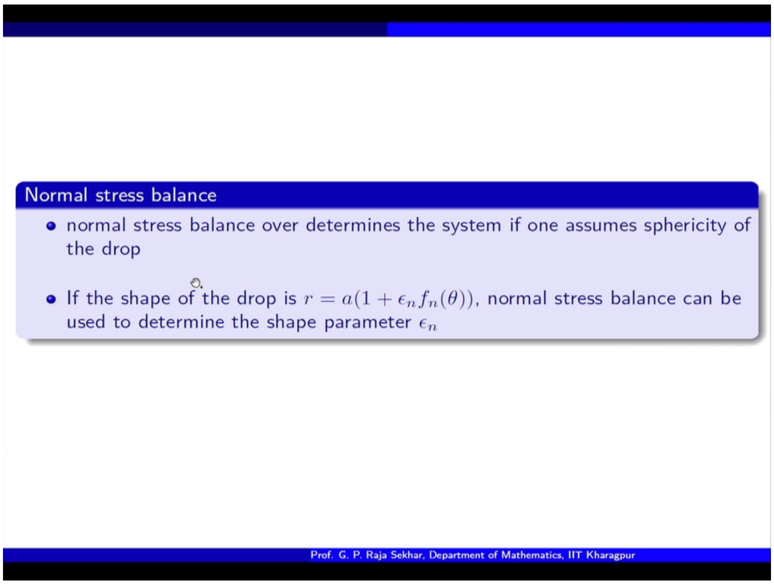

So as indicator normal stress balance over determines the system if one assumes the sphericity of the drop. Whereas, if we assume that the drop is deforming then normal stress balance can be used okay. So when I say that so this is spherical.

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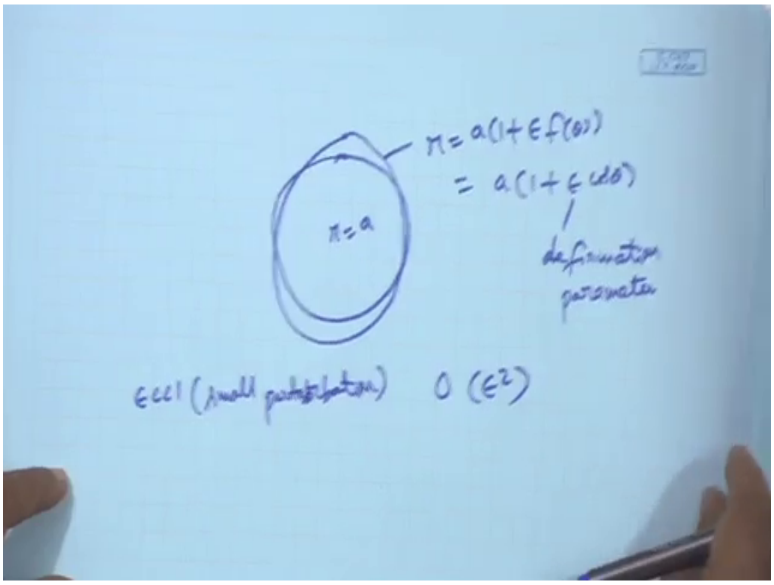

Now suppose something like that so this is some function of theta. So example says then we have normal velocity condition, tangential velocity condition and tangential stress condition. So on the spherical part, we have got the solution. We have determined the coefficients but suppose if you assume the drop is deformed like this, where epsilon is the corresponding deformation parametric okay. Then one can use normal stress balance and expand these okay.

How do we expand in terms of this r? We assume that epsilon is that is small perturbation okay. Small perturbation it is slightly deformed so which means we consider terms up to order epsilon square okay. So then how do we expand various polynomial solutions?

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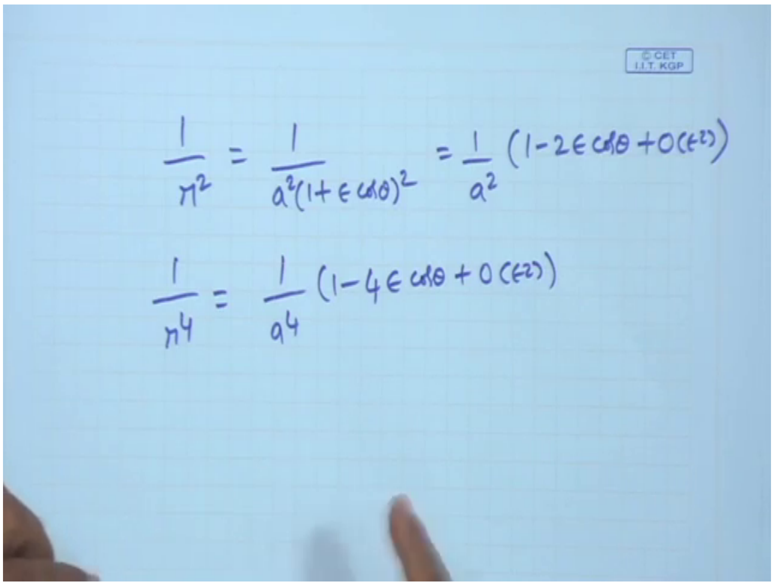

Suppose you have one hour r square, this is nothing but 1/a power 2(1+Episilon Cos Theta) power 2. So this is nothing but 1/a power 2 (1-2Episilon Cos Theta + O(Epsilon power 2)) with the assumption that Epsilon is much smaller. Similarly suppose we have such this will be 1 / a4 (1 - 4Episilon Cos Theta). So this is the case of a small deformation. So to balance the normal stress we need the pressure.

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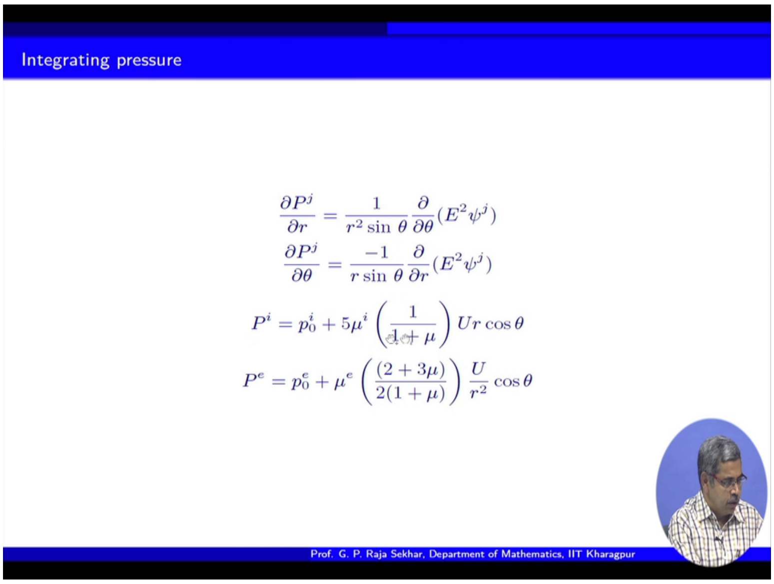

So in terms of stream function from the momentum equations we can get this compact form okay. So you can see this is the r momentum and this is the Theta momentum and we have any way we have the corresponding stream function. So from that one can integrate to get the corresponding pressures okay. So this one has to do the corresponding calculation. Now once you integrate the pressure, pressure is a unique up to a constant.

So these are the corresponding constants okay. And this is viscosity ratio and this is expressed in a dimensional form that is why you are seeing U okay. Now once we have we compute the normal stress exterior and interior. So what is a normal stress expression? Ur okay. So we have integrated the pressure and then we have velocity so one can compute the corresponding normal stress, then balance the normal stress.

You can see why we are getting a 2 Sigma by r because we are talking about the sphere. So therefore, the normal stress balance reduces to 2 Sigma by r and where r equals to A we have this okay.

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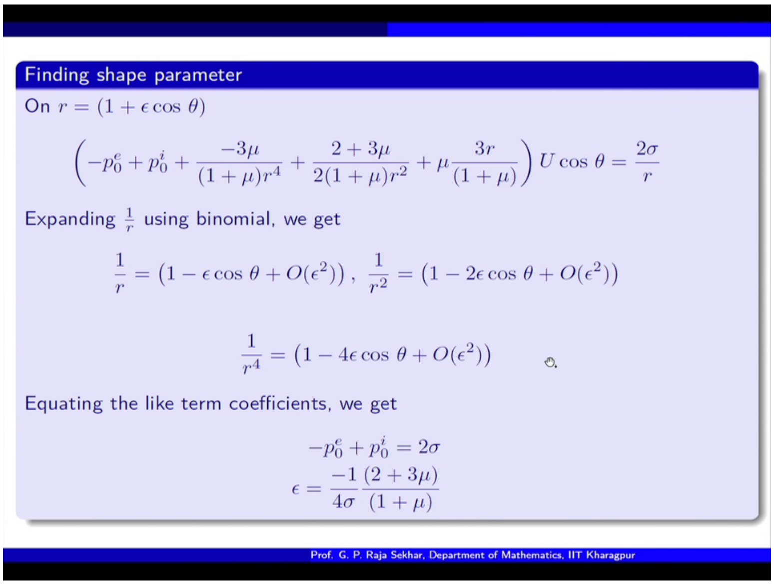

So as I indicated this we are expanding on this deformed sphere okay, deformed surface. We have taken A = 1 essentially this minus this Mu times this equals to sigma by r where r is this. So now as I explained, we go for such a small deformation then correspondingly equate terms having similar functional dependency. So the constant term is equals to 2 sigma while sigma is constant here.

Then the corresponding coefficient of Cos Theta will lead to this relation. Which means we have determined the corresponding shape parameter okay. So what we intend to do here is for a small deformation like this one can determine the shape by balancing the normal stress okay. So this is not very straightforward. I am sure you may not appreciate so much because it is a lot of algebra involved so that would take some time.

But it is doable it is very routine calculation. So we will try to give some relations in the appendix so that you are comfortable in the following and then you can continue this algebra okay. So if you do it then you will be ready to handle some research problems, so that is a basic intention that I did not give you the routine calculations here.

If you do it and then arrive at these expressions you will appreciate and you will be ready to solve some research problems okay. So this is the case of a spherical drop migrating. So when we said here migrating, we can compute the corresponding migration velocity. So how do we compute so we have to compute the drag forces acting on the drop the process is similar to what we have done for our Stokes flow past a rigid sphere.

So once we compute the drag force we balance the net forces to be 0. Then we get the corresponding migration velocity. So I hope you get some idea about migration of viscous drop. Of course spherical in an ambient flow. And in the next lecture we try to consider the case where you have surface tension depending on some activity like temperature or concentration. Until then, thank you!