Modeling Transport Phenomena of Microparticles
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Lecture – 15
Singularities of Stokes Flows

Hello! You are going to learn slightly according to me a slightly difficult topic which is called singularity solutions of stokes flows okay. So we have considered stokes flow past a sphere and then stokes flow past a cylinder by using the stream function and we have gone for separation of variable solution and then we could solve the stokes flow past a sphere problem and then get the corresponding force acting on the sphere okay.

So in order to discuss little more advanced problems in particular flow past microparticles, when I say microparticles these are like bacteria or motility of sperm etc. So these are having very irregular aspect ratios. So as a result solving using usual numerical methods etc., analytical anyway ruled out, but even solving using regular numerical methods will be very challenging. So in order to handle such situations something called singularity solutions of Stokes flows.

So that is very much handy okay but I am giving a caution. So this is slightly difficult topic so please pay attention and since the scope is limited so we will try to cover some relevant topics okay. So what do you mean by singularity? So like some something blows up at a particular location we say that is singularity okay. So to understand a singular solution of stokes flow so let us review just quickly Laplace equation and the corresponding singularities because it is relevant so that once we understand that we can go for Stokes flow okay.

So the first function which comes to our mind is a step function that is the Heaviside function okay.

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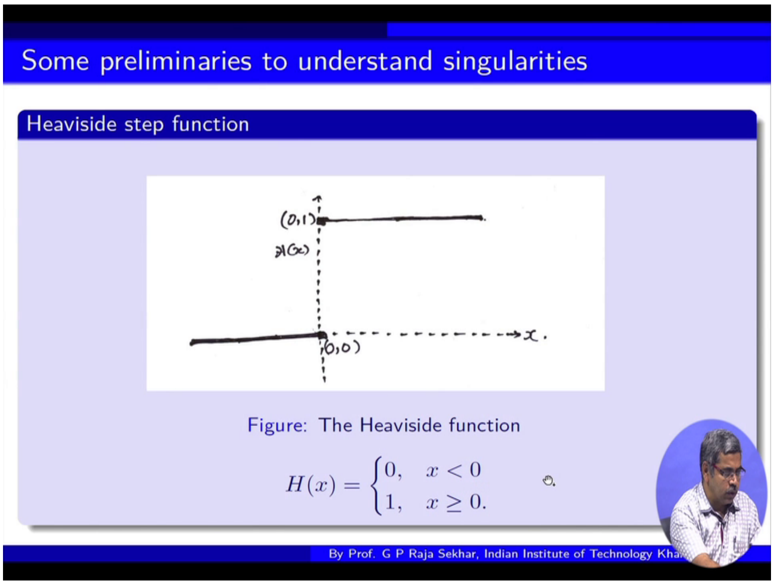

So the definition is as follows. So this is a Heaviside step function. Since we are referring origin so this is the H of x only. So the definition is H of x is zero when x is less than 0 and 1 x is greater than 0 okay. So that is the definition okay greater than or equal to 0. So it is a step function. So this is more relevant.

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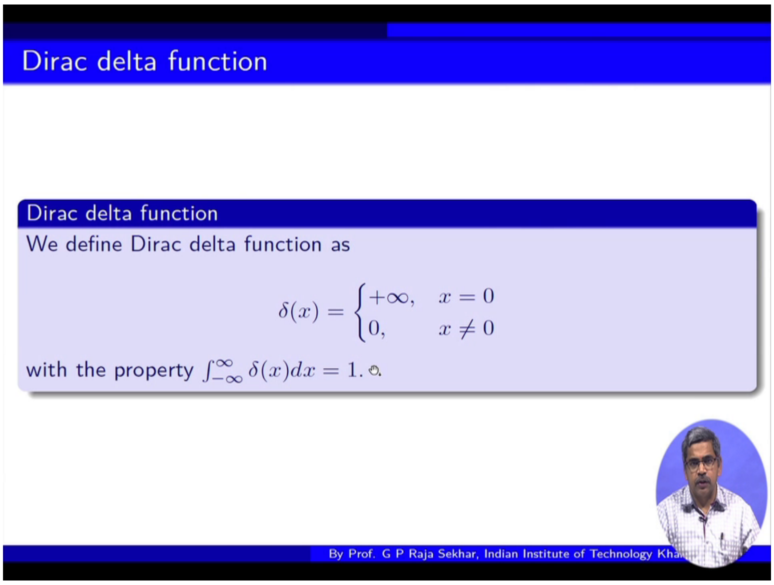

And then the next function that comes to our mind is a Dirac delta function. So it is defined as follows. It is 0 at x not equal to 0 and it is unbounded at x = 0 okay. And with the corresponding property that the total integral is 1. So now you might be wondering what is this function this is delta and Dirac delta function? So it is a point source so it is like we apply an infinite stress okay force at a particular point so the stress generated at a point source okay.

So that will when the point is shrinking 20 whatever the stress generated that is supposed to be infinite. So that is a Dirac delta function. So this can be defined in various ways using various limiting functions okay. So it is hard to realize how this integral is becoming one etc. So let us have a quick look at one of such definitions. But there are many like that.

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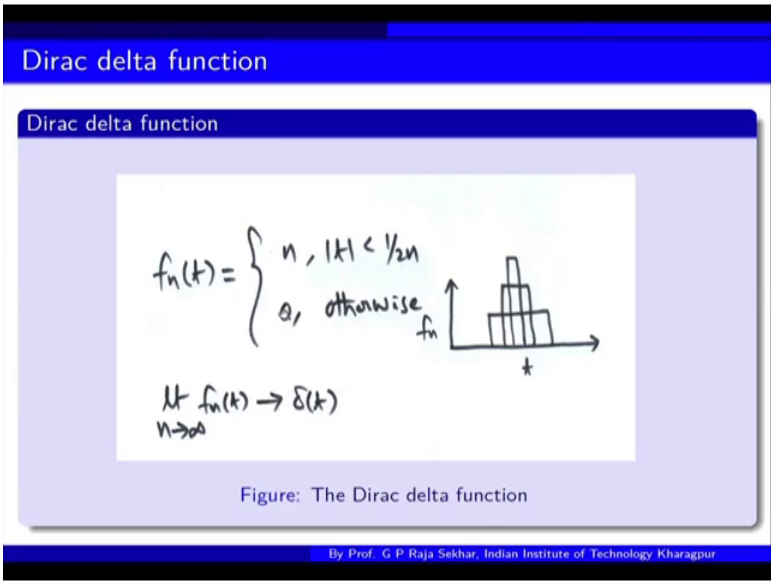

So if you consider a function fn(t), which is defined as n when t is less than 1/2n and 0 otherwise. So here if you see if you consider f1, so let us say these points are minus half and half so then f1 is defined as 1 when Marty's less than half 0 otherwise. So that is this box okay. So then f2 is 2 when t is less than 1 by 4. So it is okay so this box.

So like this if you keep on increasing n the total area remain the same but the strip becomes thinner and thinner and as n goes to infinity fn(t) approaches the Delta function okay. You can see a thin line shoots up to infinity okay. So this is a standard way of defining Dirac Delta function. But there are several such examples for defining Dirac Delta function. One can consider several functions which will give in the limiting case Dirac delta function okay.

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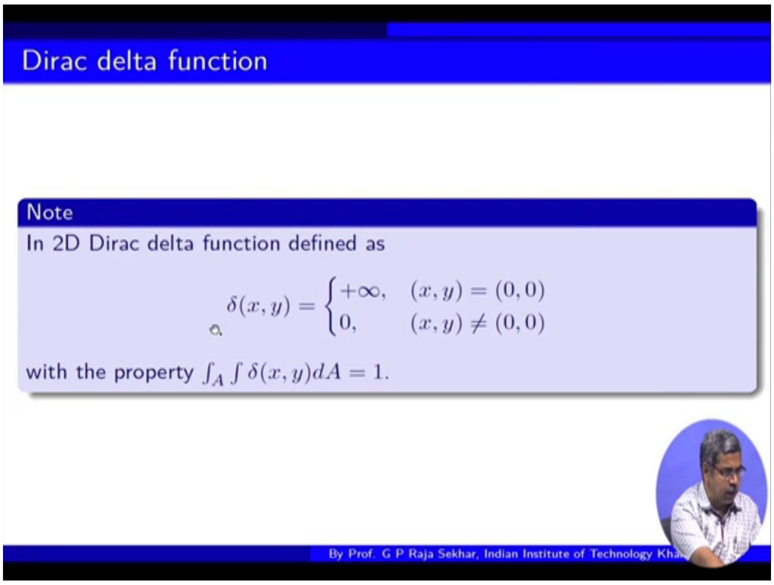

Okay so it carries several properties in 2-D that can be extended the definition of 1-D that we have given. So this is at a point not 2, 0 at this if it is non zero and then if it is zero then it goes up. And again the total area reminds unit okay.

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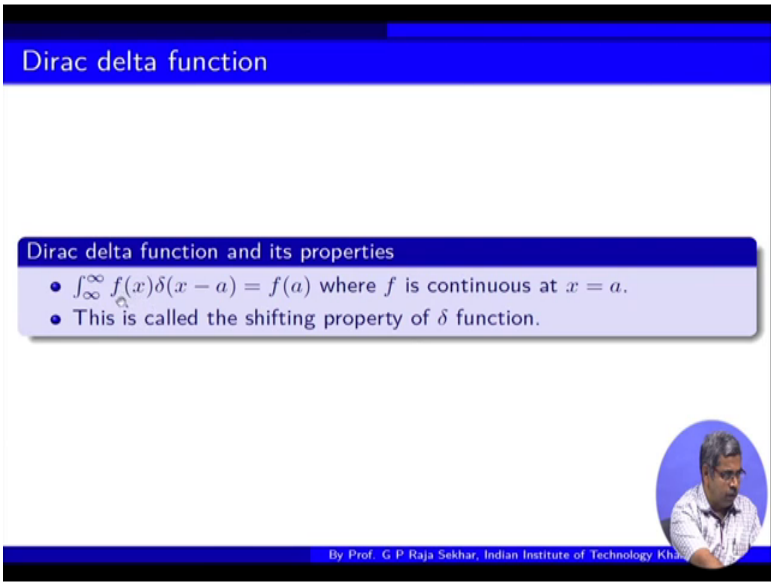

Now it shares an interesting property that is called shift property, f(x) is taken convolution with a delta function then the total value is the function taken at a okay. So you can see the Dirac Delta function definition can be extended to any x0 okay.

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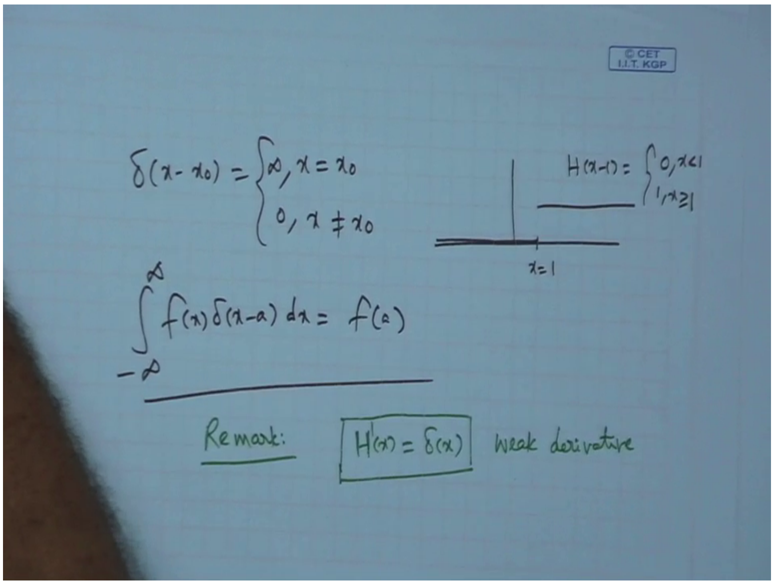

So this is unbounded x equals to x0, 0, so this can be need not be origin so we have shifted. So now correspondingly the definition satisfied is f(x) Delta (x-a) this is the shift property okay. So we have a Heaviside function so that also can be extended. Suppose this is x = 1 so then one can define so this is 0 and then 1. So this is the H(x-1). So this will be 0 when x is less than 1 and 1 x greater than equal to 1 okay.

So that is one can define like this okay. So now these two shares a relation, so I just give a remark the proof will be maybe we supply as an appendix. So they share interesting relation H dash of x is Delta of x. That means the derivative of the Heaviside function is Dirac Delta function. So this requires some concept so this is called weak derivative. So my apologies to use this term but you understand as a notation the derivative of Heaviside function is Dirac Delta function.

Why I said because this is as you see it is a step function so there is a discontinuity right, so now when you talk about the derivative so it is not in usual sense. That is not in classical sense it is a weak sense. So what is the weak derivative etc.? So the concept is involved so for the time being maybe a simplified version we will put it in the notes so that you can understand little bit more okay.

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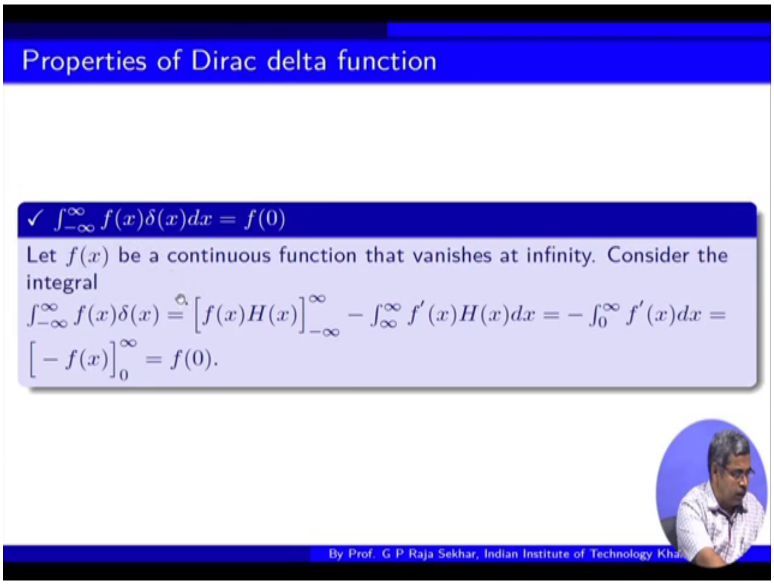

So apart from this there are several properties. This can be done at zero okay we can show it the same shift if it is Delta function this is obviously f of 0. So this can be done very quickly so you can see consider this. Convolution then if you integrate by parts this is f of x and H of x at these limits than this. So this is going to 0 okay because of the definition of the Heaviside function.

Then you are left with this we apply definition of a Heaviside function. Then you get this. So the proof for this is exactly similar line. Only thing it is a shifted by a okay.

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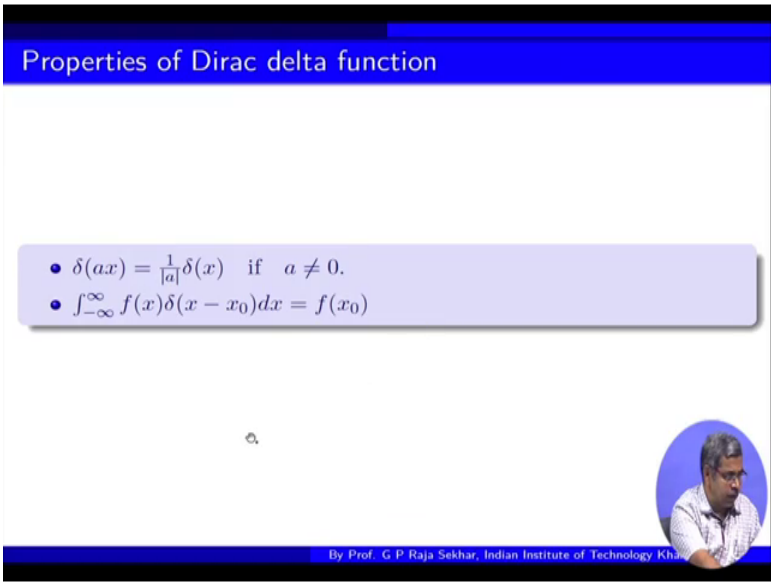

So this is one property similarly there is a scaling property that Dirac Delta function satisfies. ax is 1 over mod a delta x if a is non-zero okay. So these are standard definitions. This is the shift that we have already indicated okay. So now once we discussed Dirac Delta function so we are going to define the singular solutions okay of Laplacian. So before we do that we discussed something called fundamental solution. So what is the fundamental solution?

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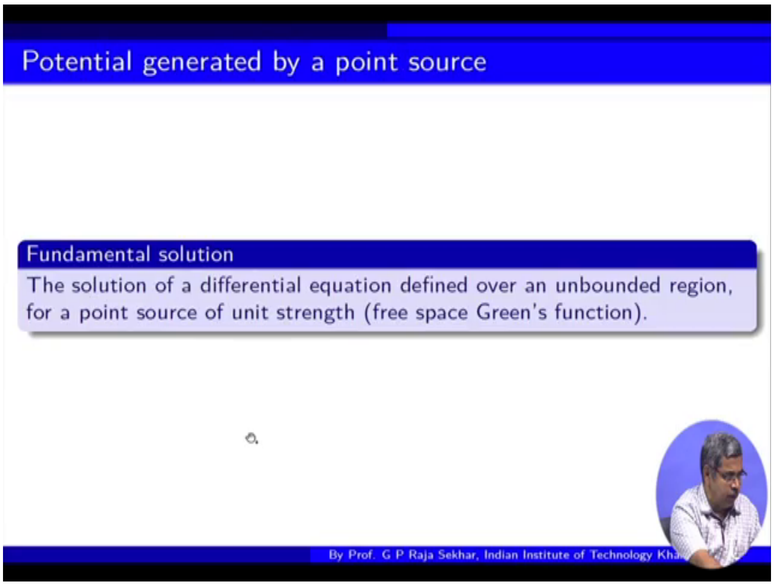

Solution of a differential equation defined over an unbounded region for a point source of unit strength. That is also called the free space greens function. So what do you mean by this?

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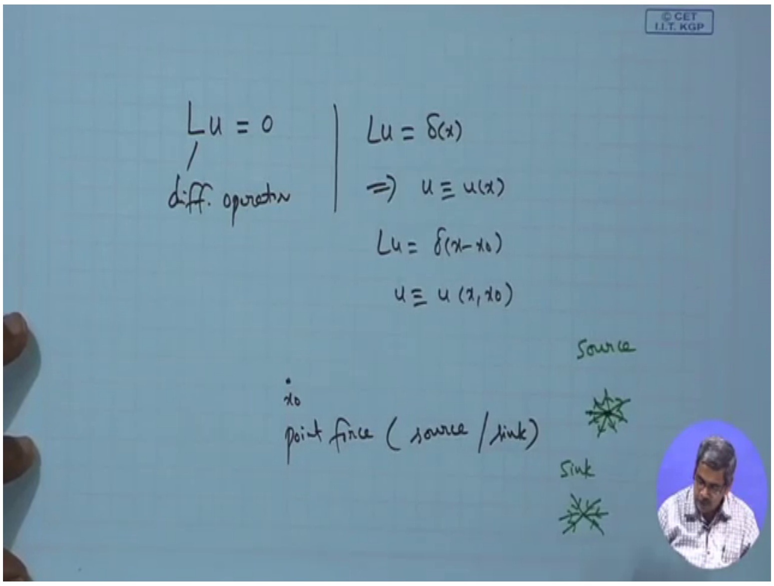

If you take some differential equation, some operator L u = 0 so where L is a differential operator, then the fundamental solution we are saying you consider Lu = Delta of x. If we solve the corresponding equation whatever we get. Suppose if we take Lu is point source then whatever the solution you get that depends on these two.

And if you do not use any boundaries then whatever the solution of this particular equation so that is called the fundamental solution or free space greens function okay.

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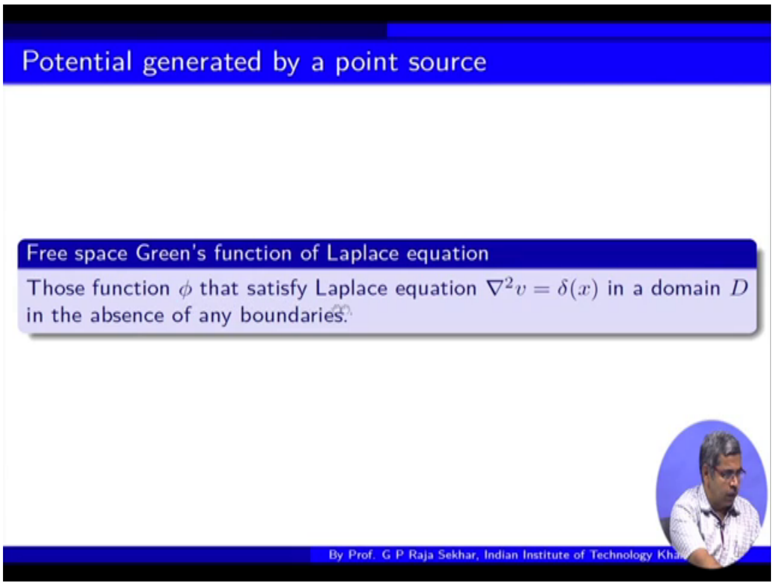

So that is what I indicated, for Laplacian if you consider Laplacian, if you take del square v equal to delta of x, in a domain D in the absence of any boundaries whatever solution we obtain that is called greens function; free space greens functions okay. So now the question is how do we get this okay? So when we say a point force at a particular point x0, so singularity so we expect it is a source or a sink.

So what do you mean by this? If it is a source we expect flow is radially symmetric and something is coming out of a dot and if it is a sink then we expect something is going in okay. And of course one can define if you if source of large intensity the more flow comes out. If it is a sink more flow from the environment will be sucked inside. So this is a general sense okay. This is the point source point sink.

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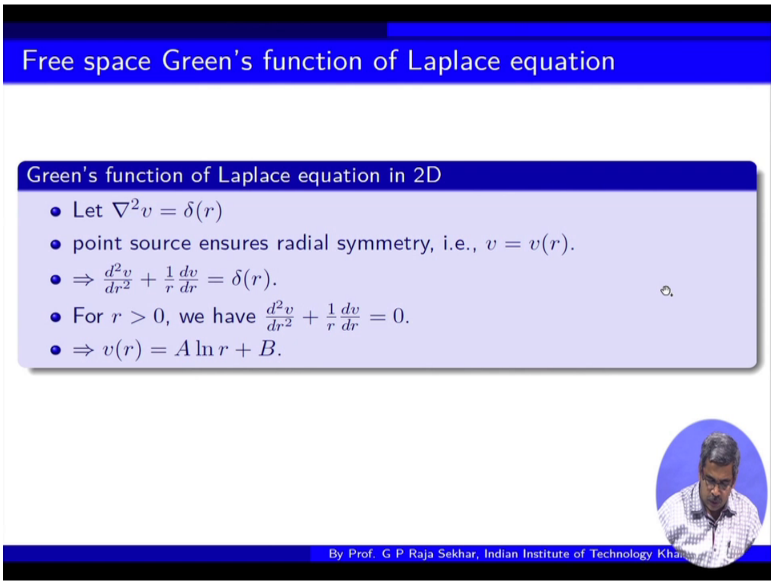

So we have to consider the corresponding solution. So we consider this and as I indicated point source has a radial symmetry. That means only the impact is along the radial direction. So therefore, it ensures that v is functional v of r okay. Then we expand the Laplacian considering only radial symmetry so that there is no data dependency and we are in 2-D. So the Laplacian reduces to this and the right hand side is delta function.

Which means this satisfies homogeneous equation for r greater than 0. That means you if you puncture if you remove r equal to 0 it satisfies homogeneous equation right and one can write quickly the solution. So what we are considering if you have a point source.

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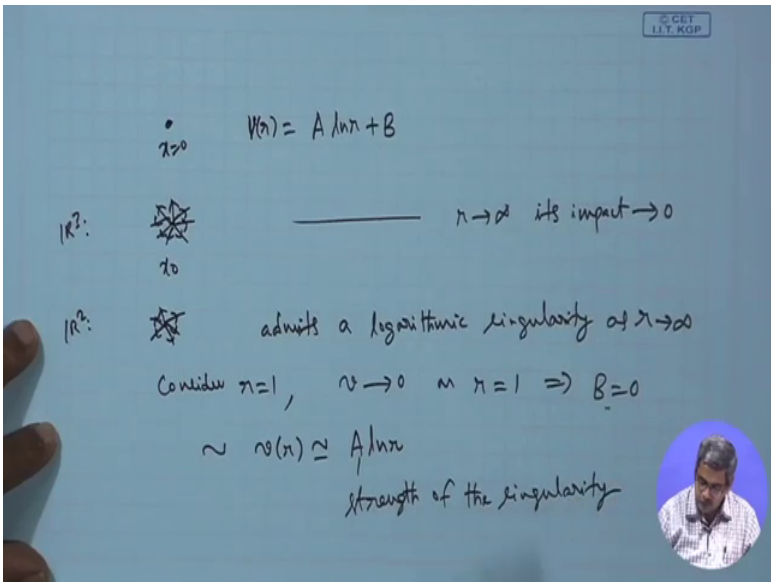

So this is x = 0 and if you consider the solution V of r it behaves like okay. So typically in r3 if you consider some flow at a particular point as we go r goes to infinity, naturally its impact goes to 0 okay. That is typically in 3-dimensions. But in r2 if you consider similar condition, so this is the source we are talking about okay. In r2 if you consider similar condition so it admits a logarithmic singularity as r goes to infinity.

So therefore, how one determines B is we normalize the intensity of this on a unit circle so simply we consider r = 1 and then we say that v goes to 0 on r = 1. So this will give B = 0 okay. So that will indicate that VR is behaving like lnr and what is this A? A is called strength of the singularity. So A is called strength of the singularity okay.

So how one can determine A so we can have a so this I have already indicated we go for r = 1 and then now we normalize. So how to determine the strength?

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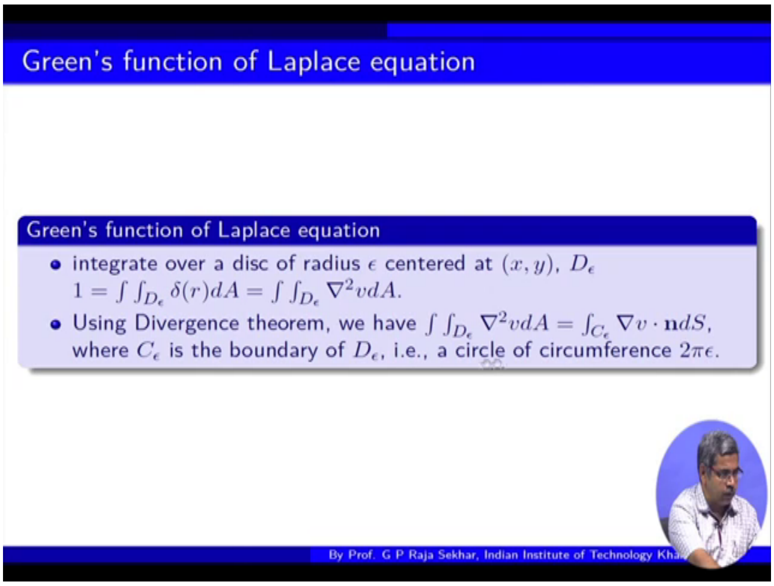

So you consider a disk of radius Epsilon cantered at this point okay, then if you consider this by virtue of the Delta property that is 1 and by virtue of the corresponding fundamental solution that is the Laplacian okay. Now we use Gauss divergence theorem. So if C Epsilon denotes the boundary of the disk D Epsilon, then by Gauss divergence theorem we get this and since it is a circle of circumference 2 Pi Epsilon.

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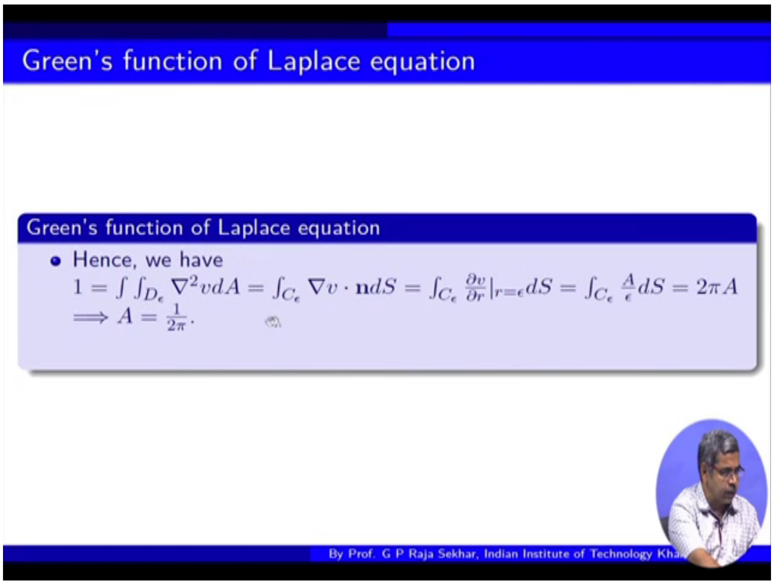

We get simply this is the flux dv dr and the corresponding element is a corresponding line element r D Theta that is considered. So that will be Epsilon D Theta and the dv dr is A by Epsilon. So what we get is 2PiA. So this implies A is 2Pi. So that means we have determined the strength of free space greens function in 2-dimension okay.

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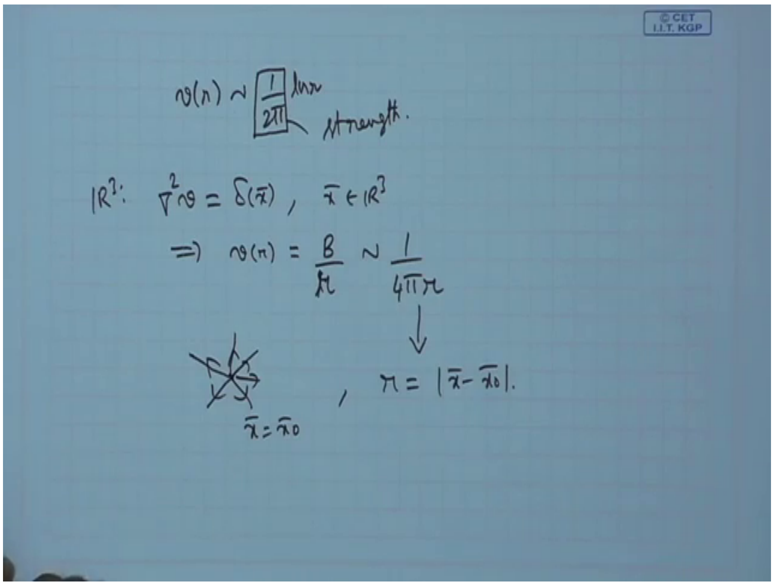

So now without mentioning the proof we also indicate that if we take any point x bar where x bar is in r3, then we get the corresponding solution in 3-dim behaves like B by r and the strength can be calculated using similar arguments okay. So this is the corresponding greens free space greens function in r 3 okay. So these are little since this course requires some knowledge about PDE so I assume once these explanations you will be able to get the corresponding derivations okay.

So this is the 2-dimensional free space greens function and 3-D as I indicated so this is the corresponding free space greens function okay. So now we got the corresponding singular solutions of Laplace equation okay. So we can use these singularities to represent a particular solution about stokes equation okay. So before we do this let us consider point source.

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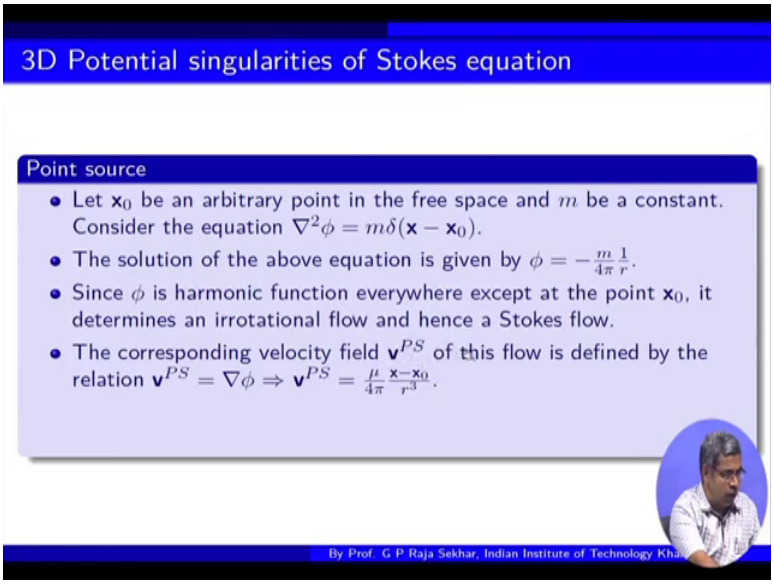

So that is nothing but now instead of considering delta function we are considering of some strength m okay. So then the solution of this will be given by so this plus or minus so it depends sometimes we take plus for so sometimes minus for source. So that is not an issue we can absorb in this and call it to some m prime okay. It indicates only the direction. Source or sink indicates the direction.

So in 3-D this is a point source that is nothing but the velocity potential corresponding to a point source at x = x0. Which means we are talking about x = x0 so in that sense r will be okay. So when we say this r is this okay. So that is the point source, now one can represent phi is harmonic because it is a velocity potential everywhere except to the point x0 because it is a singularity. So it determines an irrotational flow, hence a stokes flow.

So let me explain this statement.

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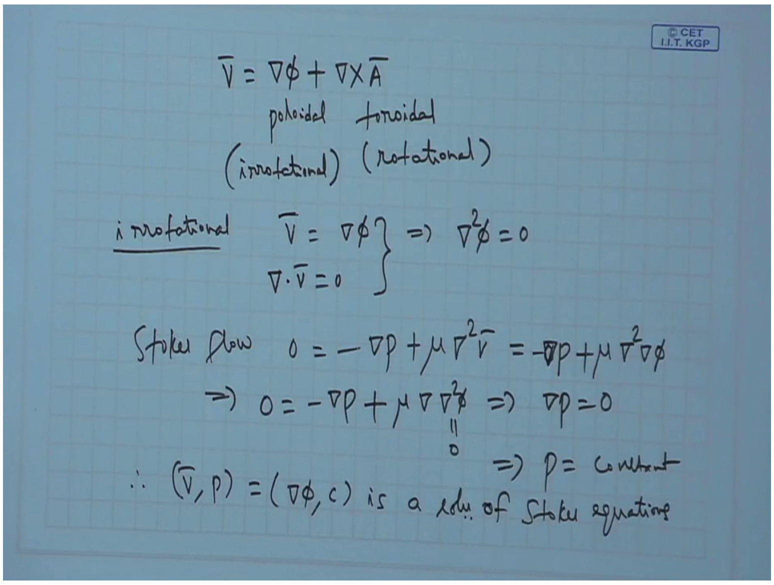

So any vector field can be decomposed poloidal, toroidal. So that means this is the rotational and this is irrotational okay. Now if you ignore rotational and consider only irrotational we have V is Grad phi. So this if we go for usual conservation of mass that is divergent free combining these two we get del square phi is 0 okay. So which means we have phi is harmonica okay.

That means when we consider stokes flow and restrict to rotational flow what we get is. So they commute and this is 0 this implies we get 0 this implies P is constant. So therefore the conclusion is V P given by constant is a solution of stokes equations. And what kind of solution? This is an irrotational solution okay. So this is a particular.

So therefore, what we now think is you have potential flow and you have a singularity corresponding to the potential. Now just now we have shown that any potential also solution of stokes equation with constant pressure. Therefore, once you have a singularity corresponding to potential flow, maybe we can get the corresponding singularities of stokes flow as well right. So this is what we are anticipating.

So let us see how the corresponding singularity solution we can get from Stokes flow okay. So we have the potential source we are calling. So that is due to the Laplacian which is given by gradient of scalar. Now if you have potential we know what is Phi in 3-dimensions. Therefore, the velocity is given by the gradient of Phi, which is this so then we can get the corresponding velocity potential.

So I am calling the M here is Mu okay. So that maybe I should have used a unified notation okay. So how we are getting?

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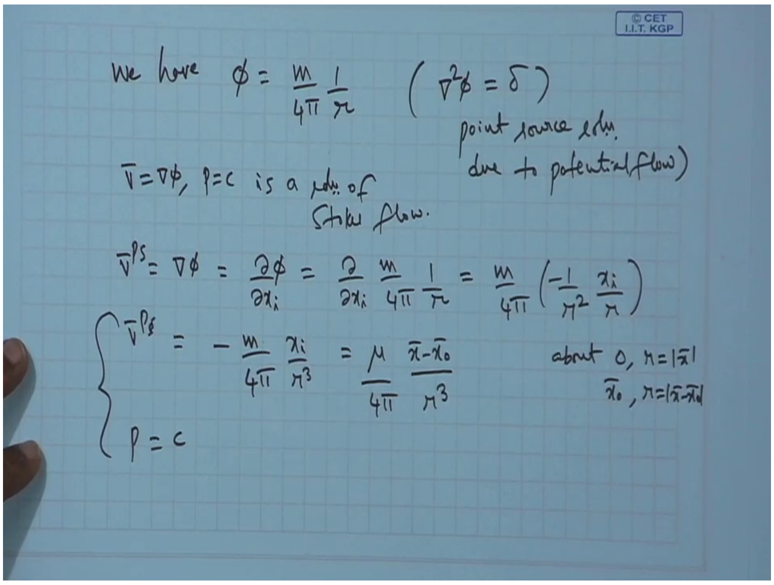

We have Phi = say m by 4 pi and one over r. So this is by virtue of this is nothing but points source solution due to potential flow okay. Now we have shown that V bar equal to Grad Phi, P equal to constant is a solution of stokes flow. Therefore, what will be singular solution of stokes flow. So that is nothing but so this is nothing but in index notation. So this is nothing but okay, so this is we get and derivative of r with respect to xi that will be okay.

So the corresponding velocity due to the potential is given by r cube. So this week I have written it as opening the completing the summation. xi means if it is a about if the singularity is about origin then r is nothing but if it is about then r is nothing but okay. So therefore, this is a nothing but generalizing okay. So this is a corresponding singular solution okay. So once we have the singular solution p is constant. So these two together will be solution of stocks flow okay.

Then the question comes can we compute the corresponding stresses? Yes, we can compute the corresponding stresses okay.

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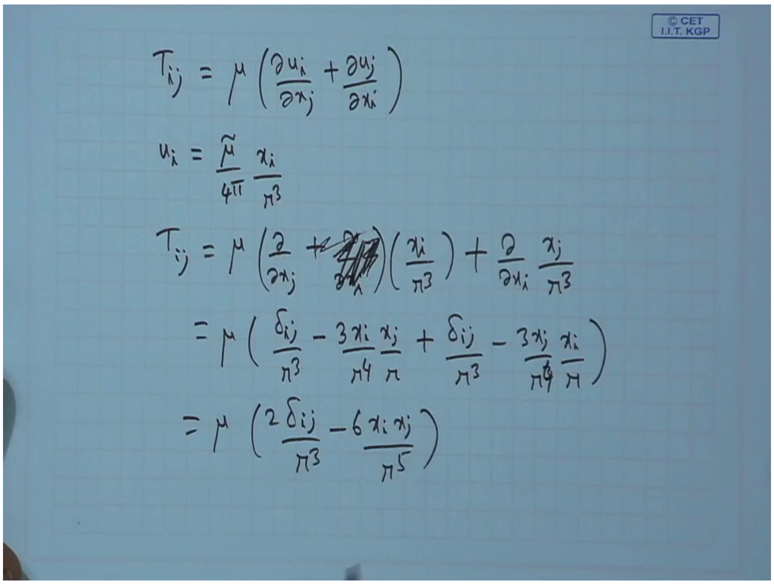

So consider the expression for the stresses Tij is Mu okay. So now we have ui is Mu by 4 Pi xi by r cube. So we can compute the corresponding stresses Tij this Mu is viscosity so let me call this is some strength. Otherwise if it will be misleading some Mu Tilde. So Mu then this is common okay. So I am leaving this part okay.

So we can normalize. So now Dou ui by Dou xj right. So that will be ui is so we have to go Dou by Dou xj, xi operating on xi by r cube. So this is Mu first one if you operate. So this we have to use uv formula right. So if we do this on xi what we get? When xi equal to xj we get 1. So therefore, Delta ij by r cube then on this if we do -3xi by r4, then xj by r okay. So the first one we have to operate on this plus Dou by Dou xi, on uj. So that will be xj by r cube okay.

So this if we do again the same way we get again. So this time r4 here, so what we get the corresponding stress is 2 delta ij by r cube -6 okay. So it is very easy to calculate the corresponding stress okay. So what we have done just now is we have considered the potential singularity okay. So then using the potential singularity we have computed the corresponding singular solution of a stokes flow okay.

So now once you have a singularity if you differentiate the singular solution what happens? That is the next question. So if you differentiate we get the higher order singular solutions. So let us see that.

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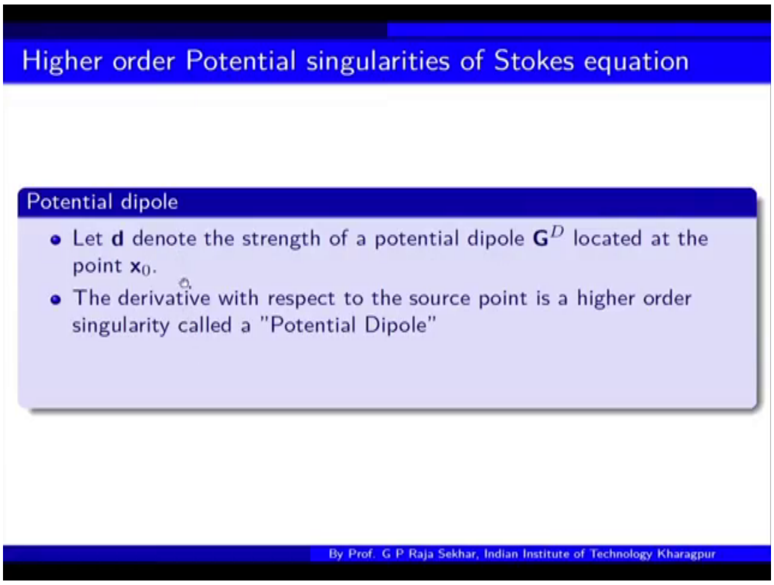

Suppose we consider a potential derivative with respect to source point is a higher order singularity that is called a potential dipole okay. So if D denote the strength of a potential dipole GD located at a point x0. So then the derivative with respect to point source is a higher order singularity. It is called a potential dipole okay. So which means we have the corresponding PS so this if you generalize what is our free potential source?

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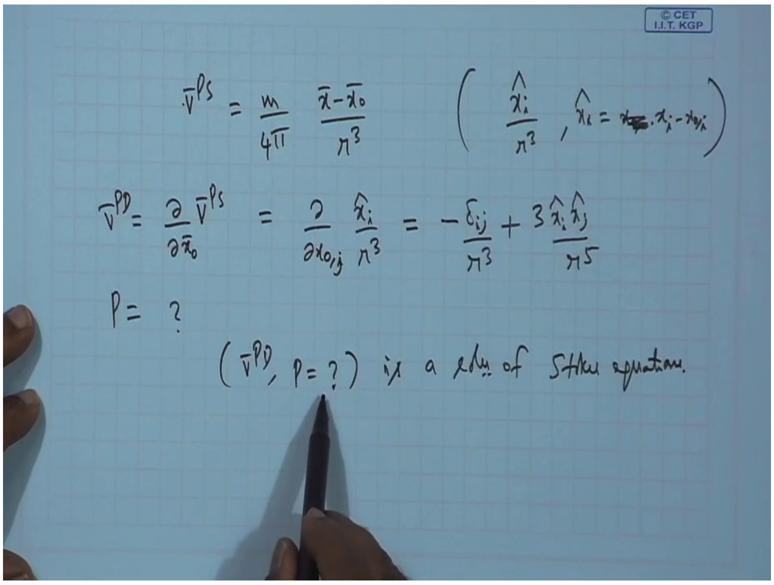

So this is some strength so that we can ignore. Then we have x bar minus this is the general sense in summation notation we are writing it as xi by r cube. So here xi strictly speaking where xi is xi - x0i. That is these are the coordinates with respect to the singularity okay. So now we are talking about derivative of this with respect to the coordinates, so then this is called potential dipole. This is called potential dipole.

So you can see the singularities of order r cube here so if you differentiate it will be much more singular okay. So this can be obtained very much okay. So remember we are differentiating with respect to the coordinate okay. So in summation notation okay so j let us say then V in summation notation. So this if you execute here x0 is sitting in this right so there will be a negative sign when you differentiate this with respect to x0j.

So that here too. So therefore, by virtue of this will produce okay then minus will give first minus then next time when we get in another minus. So therefore, here we get this is the corresponding solution for the derivative singularity. Which means this also represents a potential solution but also represents solution of the stokes flow and the corresponding pressure.

Please think it is very easy that means we are asking potential dipole P given by what is a solution of stokes equation okay. So I have shown you for the point singularities so now for potential dipole you can very easily get what is P okay. So these are very elementary singular solutions of stokes equations. But one can derive an arbitrary solution of stokes equation so that is definitely much more involved.

Once we get so when I say potential that is only we are considering irrotational but if you consider arbitrary flow and then get the corresponding singular solutions that will represent the complete flow structure and that can be used for solving any arbitrary flow. That is much more involved. But with this I am sure you get some idea about singular solutions of stokes equations. Thank you!