Modeling Transport Phenomena of Microparticles

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Lecture-03
Reduced Forms of Navier - Stokes Equation and Boundary Conditions

Okay so in the previous class we discussed about a stress tensor and then we have derived conservation of mass and the linear momentum balance and then from general incompressible Navier-Stokes equation. We discussed some limiting cases before we attempt any problem as you can see the equations are highly non-linear, so we would not be able to solve immediately, even numerically it will pose a lot of challenges for that matter, the existence and uniqueness of these Navier stokes equations mathematically is still an open problem okay.

So in order to solve any problem we have to consider geometry right, once you consider a geometry what is the immediate question which comes to our mind is on the boundaries of those geometries, what is the corresponding physical phenomena that one has to consider? So that is the biggest question to be asked okay. So today we are going to discuss in brief the reduced Navier Stokes equations for a simple configuration and then for those configurations how the corresponding boundary phenomena will be implemented okay.

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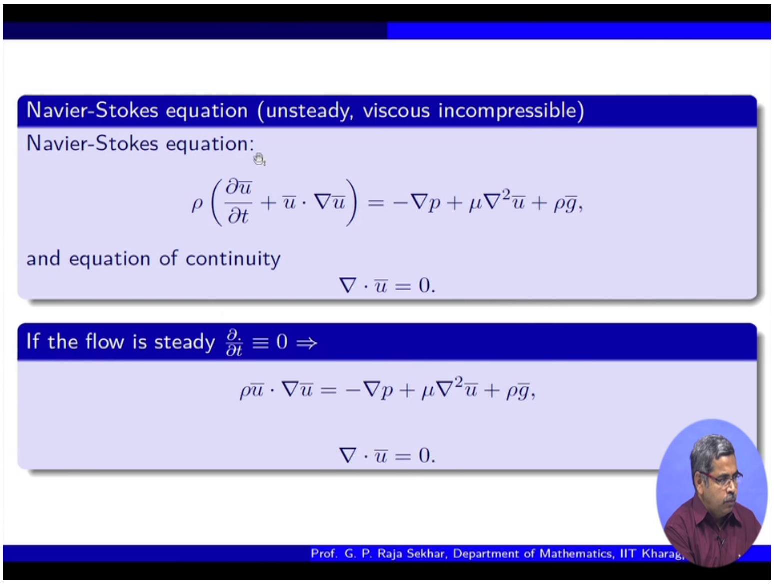

So let us recall quickly the governing equations, so this is an unsteady viscous incompressible so most of the times we start with the incompressible okay. So this is a local rate, this is a convective rate as we have discussed and this is the pressure viscous forces and the body forces and this is the corresponding equation continuity. If the flow is steady that means there are no variations with respect to time we neglect this so we get the corresponding simplified equation okay.

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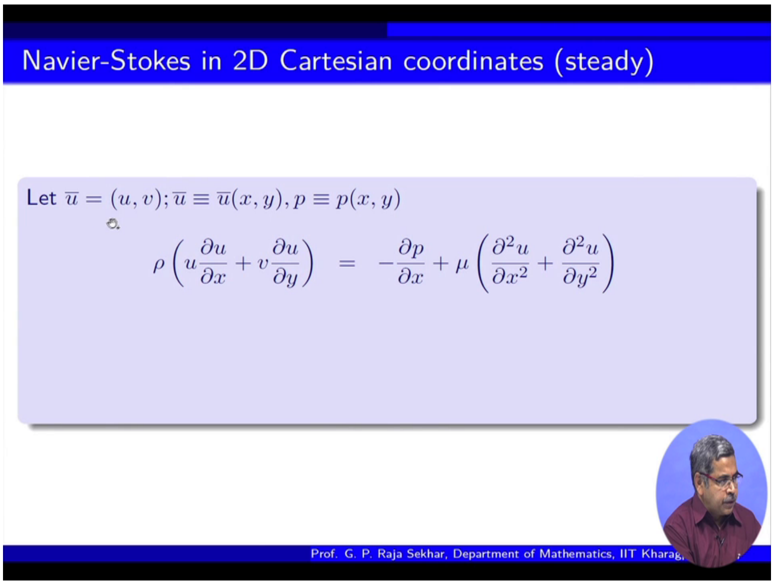

So now let us restrict the flow to two dimensions, that means the velocity vector is u, v okay. so where it is a function of x and y in two dimensions and also the pressure is function of x and y in two dimensions then what we are trying to do is, we are trying to write down the narrow stokes equation in component form.

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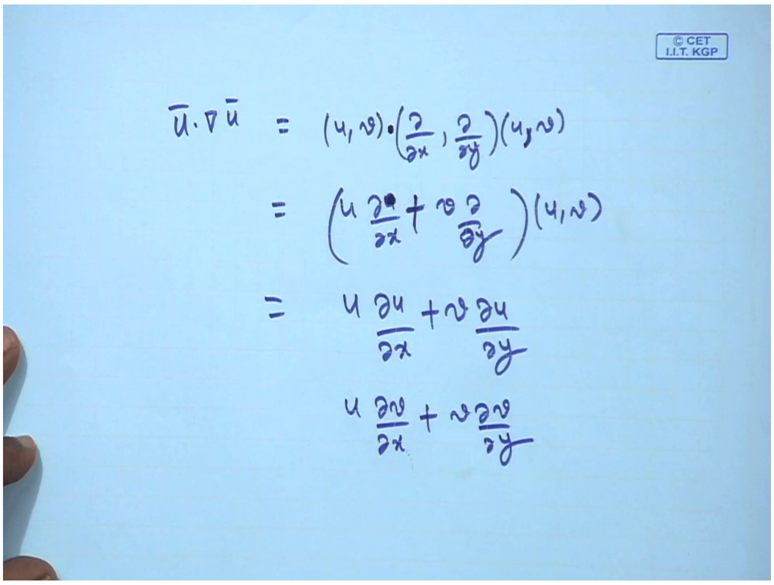

So this steady case we are writing, so what we have to we have to expand this so u bar dot Grad u we have to expand, so this is nothing but on again u, v okay, so this will be okay. One component will be getting from u, so the first component, I am writing okay. The first component will be so this is a dot product, so take u first so this is gradient operating.

So the best way is without u if you do it, you can get it without any mistake. So first you take the dot product of these two, so that will be this operating on u, v. So this will give you u Dow u by Dow x, v Dow u by Dow y, so that is the first component and second component okay.

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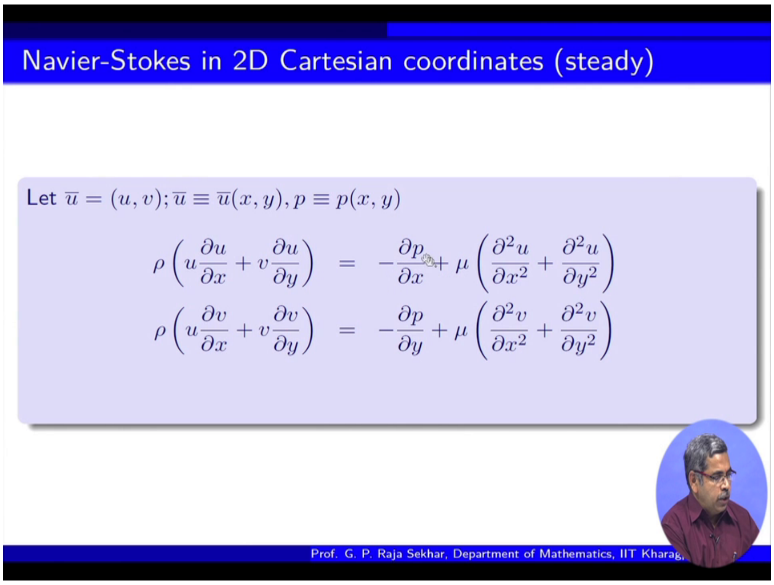

So once we do this that is what we have written first component and corresponding to the pressure gradient you get this and this is simple Laplacian, so very straightforward okay, and this is the corresponding equation of continuity so this is the simplest. So if you want to decompose in cylindrical or 2D plane polar coordinates or spherical so you have to know the corresponding gradient and Laplacian in terms of those coordinate systems and not only that.

So here in Cartesian when we have taken u dot Grad u and also Laplacian straightforward we have pulled out the corresponding components, because the corresponding unit vectors are constants so when you operate on Laplacian. So there are 0 they are treated as constants okay. But when you use polar coordinates or cylindrical spherical, one has to be very careful when you are operating Laplacian.

Because the corresponding unit vectors which are say in cylindrical ER, E Theta, EZ, so they are functions of the corresponding archit fice. So therefore when you operate so you have to consider the corresponding variations okay, so this is very important.

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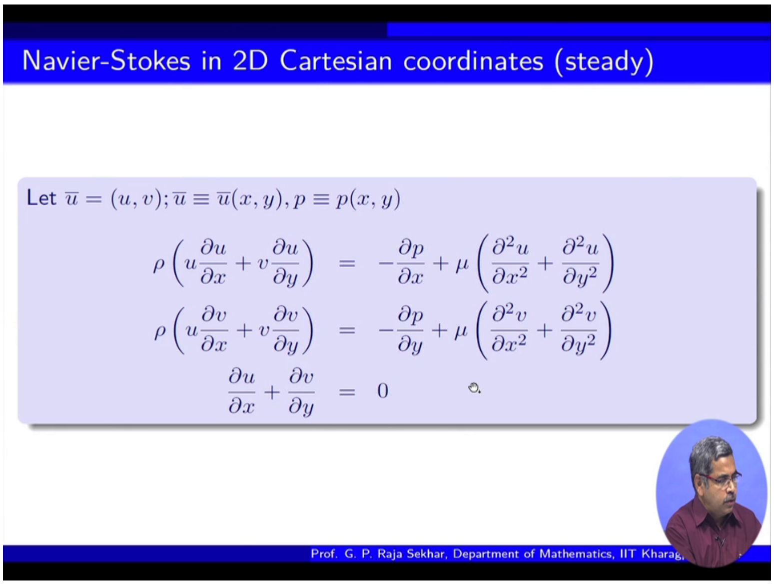

So to start with we have considered only Cartesian 2D so in component form we have written, now let us see further simplification okay.

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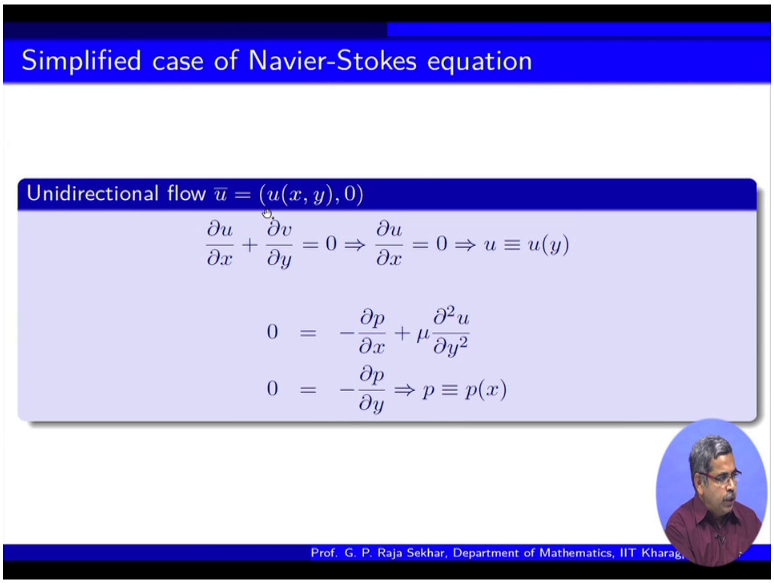

Suppose the flow is unidirectional that means at this stage when we say unidirectional, we are not assuming that u is function of x alone or y alone etcetera, etcetera. When we say unidirectional component along only one direction, it could be 0, B, no problem okay. So here unidirectional means along x direction we have considered that is why it is u, 0, then consider your equation of continuity. So then this indicates that since v is 0, Dow ux is 0 there for u is function of y. So that is the immediate information we are getting okay.

Then write down the corresponding momentum equations.

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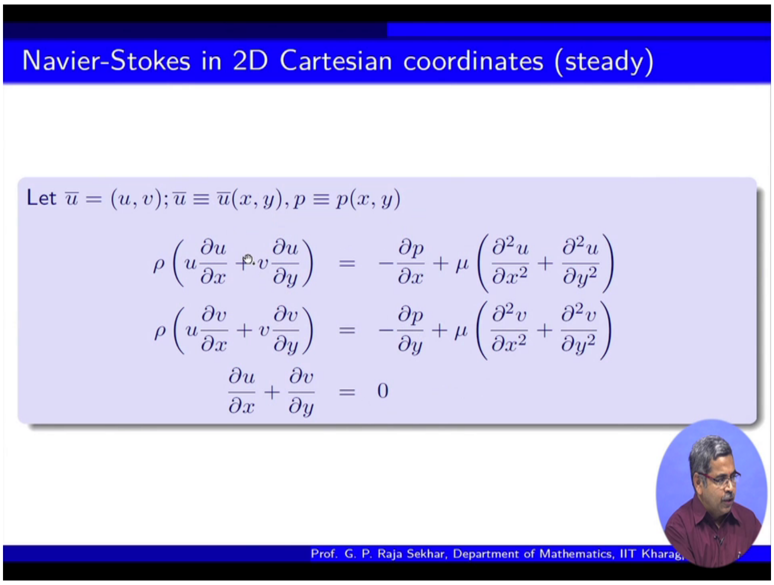

Dow u, Dow x is 0, therefore this is gone, v is 0 therefore this is gone; similarly here since v is 0 this is gone okay. And Dow u, Dow x is therefore this is 0, because Dow u Dow x is throughout, throughout the domain therefore the higher order derivative is also 0 and v is 0. If you assume unidirectional is the corresponding information u is function of y that is also used, then the reduced momentum equations so this one can very easily reduce it, then another additional information that you are getting from the y momentum is p is function of x alone okay.

So these two can be combined to infer an additional information, what is that?

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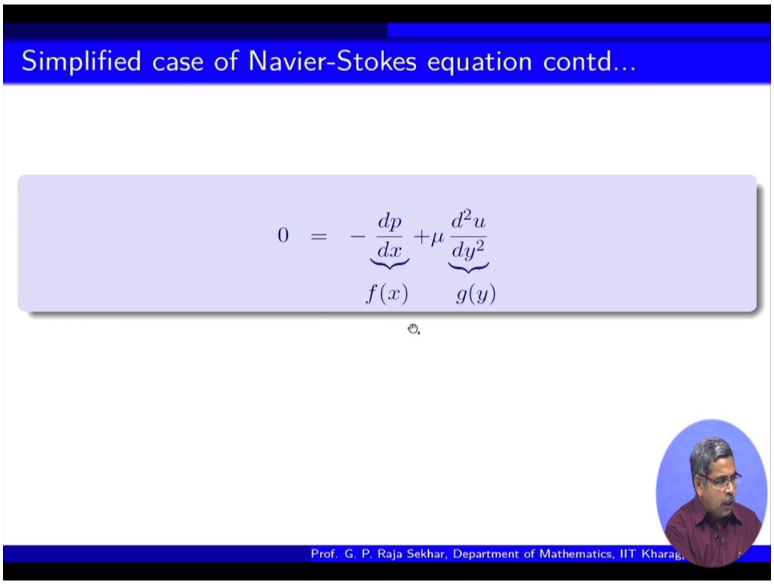

If you see I have changed from partial to total derivatives the reason is straightforward, u is function of y alone therefore, the partial derivatives we have switched over to total derivative, p is function of x alone the same is the case so therefore this becomes a total derivative and since it is this is also I have written like this and if you pay attention this is function of x this is a function of y and you just equate them okay.

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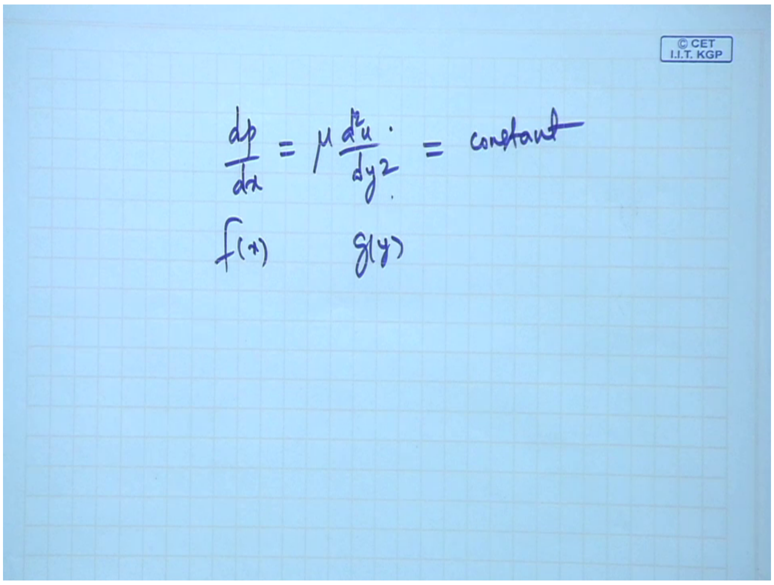

So if you take it the other sides, dp dx equals to Mu and this is function of x, this is function of y both are equal that means each of them has to be constant. Why? Whether this is not on any boundary? This is a invariably throughout the domain okay. So that is the first simplification we get it, so in some sense the pressure gradient is constant, if you take it off this one otherwise we are corresponding second order derivatives of velocity constant okay fine so this is one immediate simplification right.

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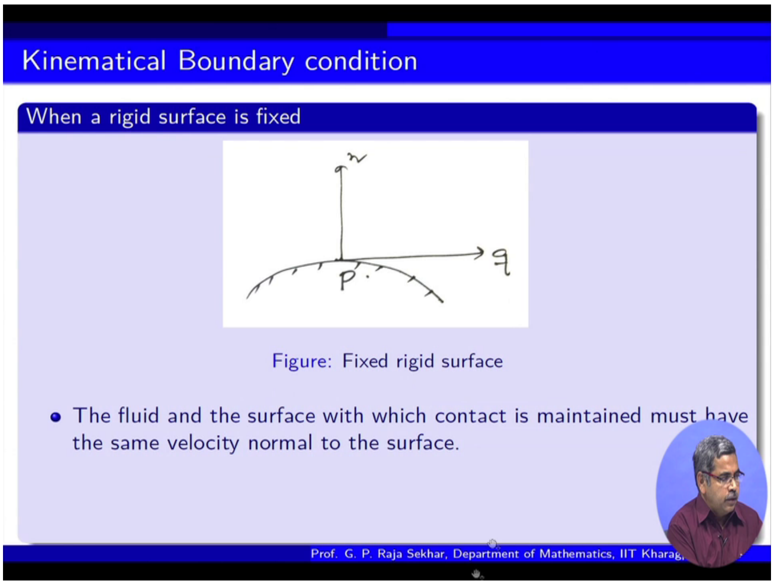

Now we move on to boundary conditions, so this is very important suppose even in a simplest case you want to solve the problem. So for example you have second-order velocity equals to constant, so it is an ODE, you can get the corresponding solution, which involves a two arbitrary constants. Now how do you determine these constants?

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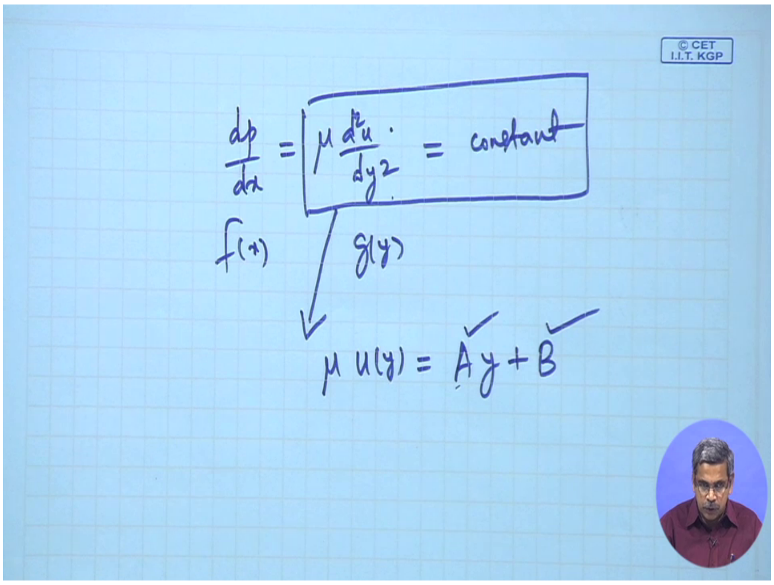

What I am trying to say is, since we have this admits a solution you can say Mu times u of y will be some Ay okay Plus B, du y is constant, so that is what you get right, so we need two arbitrary constants okay. So how do we eliminate? So that is the question so in order to eliminate this constant what should we have to consider geometry, so since we are in Cartesian unidirectional perfect, but you need some geometry. You have to introduce geometry so let us say you are introducing a flow over a plane okay.

So then the question comes what are the corresponding boundary conditions on this plane? So that we use those and eliminate these arbitrary constant, so that is one okay, concern. So now we introduced in a generic sense what are the boundary conditions. So to start with let us say you have a boundary which is a stationary then when you have a fluid molecules other into the surface, you can have two thoughts.

What are the two thoughts one is the flow is very viscous and other thought is the flow is inviscid okay. So correspondingly do you expect some change or you think the phenomena remains the same okay? So that is the question so maybe some people are based on the intuition. They would postulate or no, no or compared to viscous an inviscid, there should be a change.

Somebody may say immediately we cannot see any change we expect, if the surface is impermeable all that we can expect is nothing penetrates, so therefore we expect the velocity is 0 that is the immediate based on general physics one can conclude okay. So let us have a look at it okay, what are these corresponding boundary conditions.

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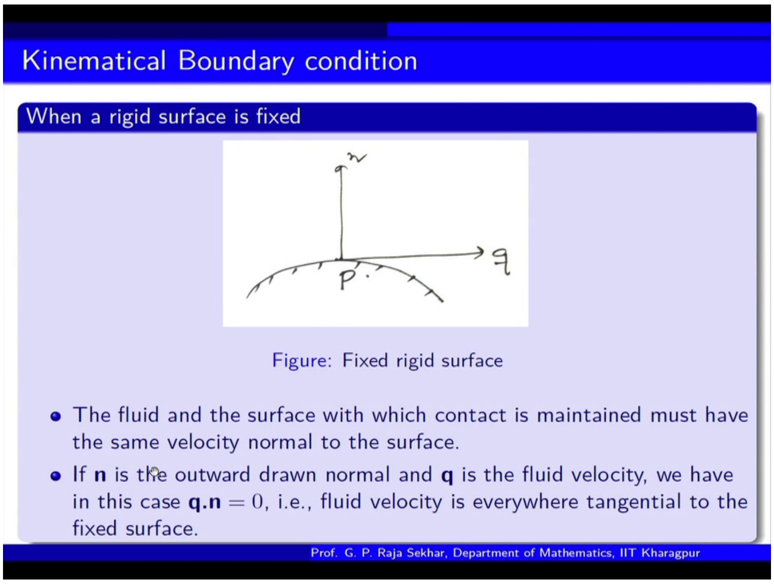

So you consider an interface okay, it is fixed that means it is not moving and this is a normal and this is a corresponding reaction, so the fluid and the surface with which contact is maintained must have the same velocity normal to the surface okay. So what does it mean fluid and the surface with which contact is maintained must have the same velocity normal to the surface, so what is the assumption here as they already indicated the assumption here is, it is impermeable nothing can pass through okay.

So therefore normal there is no flux, so therefore normal velocity is 0 so if n bar is outward normal so then q dot n is 0. In other, other sense fluid velocities everywhere tangential to the fixed surface that means the velocity is only tangential to the surface normal, there is nothing penetrating but on the surface it is moving okay. So that is the immediate this is typically known as kinematical boundary condition okay.

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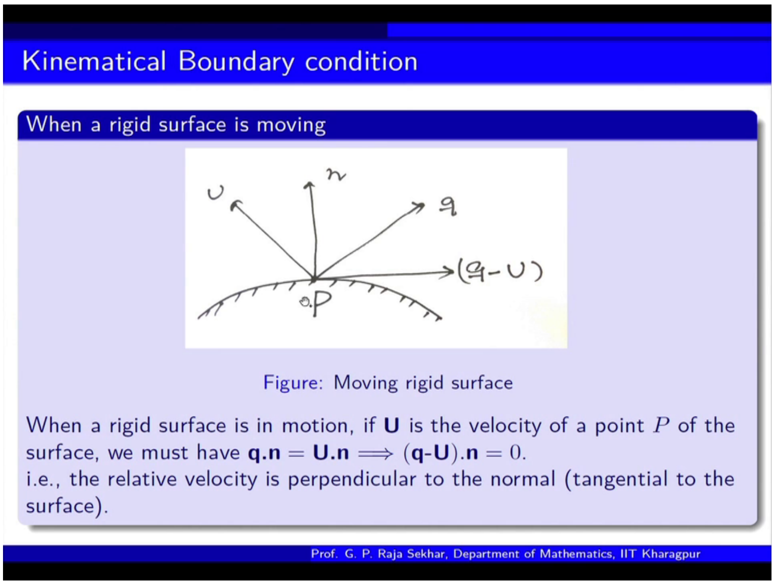

Now let us consider the case, where the surface is moving with the velocity U surface is moving with velocity u and fluid velocity is q okay. So then again we expect since nothing is penetrating the fluid adhering to the surface should move along with the velocity of the surface so correspondingly, if you consider the relative velocity the normal component must be 0.

So that is what we are doing okay, so this is the relative velocity, this is the velocity of the surface and this velocity of the fluid, so the relative velocities normal component is 0 because nothing is penetrating, so previous cases when the surface is fixed and this case is when the surface is moving and this is the corresponding kinematic a boundary condition okay.

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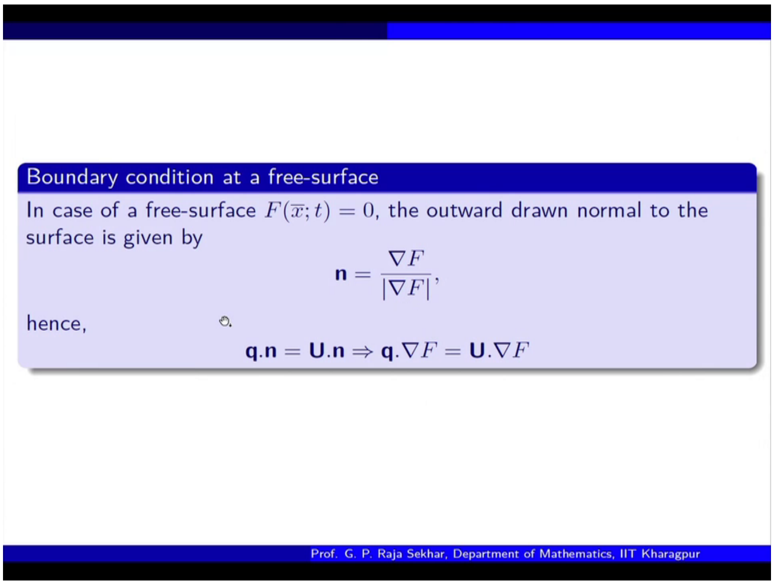

Now something called free surface what do you mean by free surface? So some surface is moving with space and time, it is not like a flat or it is very regular so it is moving, so it is depending on both the position and time so that is called a free surface okay, like a wave so on, sea top etc. So in free surface can be represented by a functional relation of position n time and that is equal, example you can take y equals to some function of x, t, in two dimensions.

Now the outward Dow normal is this very straightforward, hence q dot n is U dot n that implies because we assume the corresponding the magnitude of the gradient is non 0. So this is very straightforward calculation okay.

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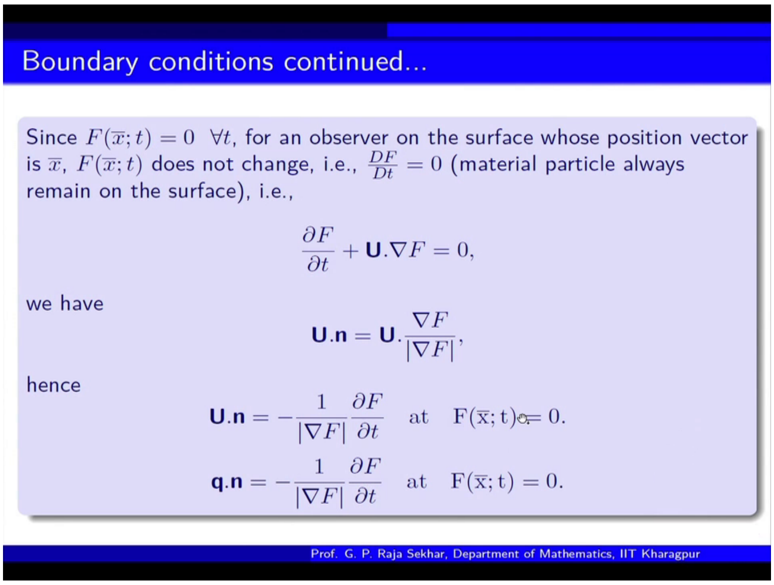

Now one assumption what we use is if you have a free surface whatever particles sitting on that, they will not leave okay, this is the kinematic assumption, what do you mean by that you can see if you drop some particles, so they will be oscillating like that okay. So they are not escaping unless you give some larger external forces are right, so with that assumption the material derivative vanishes okay.

So that means whatever the materials on the free surface so they are intact they are not leaving the surface, so under that so the material particle always remain on the surface therefore the material derivative is 0. This brings in some relation this is we have expanded this is the velocity with which the surface is convecting okay, and the hence we can, this we had already because n bar is nothing but this but from here we are getting.

So we can now we want this normalized by magnitude of the Grad F, therefore with simple algebra we can get okay, at a free surface U dot n is this okay hence q dot n is this we have used this relation in this nothing, nothing great about this okay. So that means if you have a free surface then you compute the normal velocity than if you balance the corresponding kinematic condition what we get is the fluid velocity normal component q dot n must be related to the convection.

How with the corresponding free surface getting converted so that is related in this fashion okay. So you can reduce it if F is stationary then this is 0 therefore, you get the usual kinematic condition for stationary. So this is the first thing.

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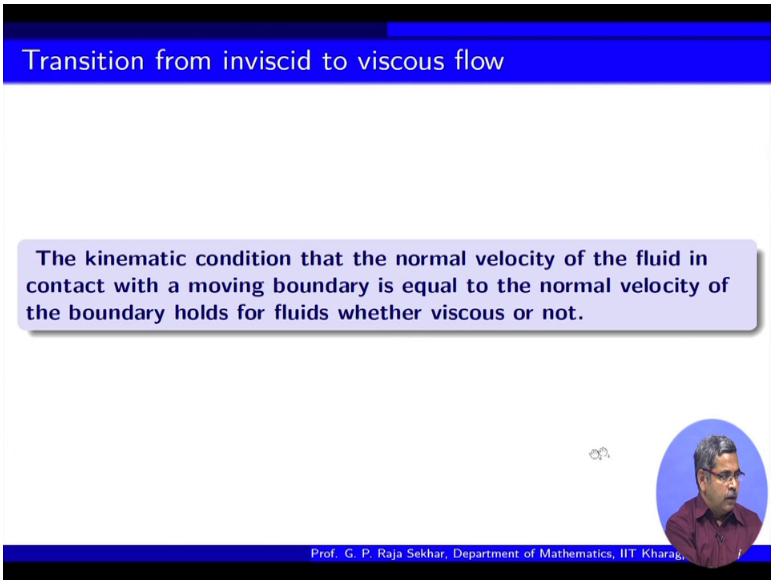

So quick inference the first question you recall what was the first question that we have asked? Whether there will be a difference when you have a viscous and in inviscid? Okay that is the first question so far what we discussed is we are normal is 0, because there is nothing is penetrating assuming it is an impermeable surface okay suppose you have some permeability of the surface, then what you expect is whatever normal flux entering the same amount will come through okay.

As long as there is no surface reactions happening on this nobody is sitting on surface who is eating or producing more, so this is typical assumption all that we get is the normal velocity is continuous, if it is a there is a porous okay. But since we are assuming it is not permeable so we got a q dot n is 0, right now if you consider viscous case do we have this and that is all or anything else that is the question okay.

So at this stage what we are making a remark the kinematic condition that the normal velocity of the fluid in contact with a moving boundary is equal to the normal velocity of the boundary this holds for a fluid whether viscous or not then what is so special about viscous fluid that is the question, right. You imagine you have a plane then you have a viscous flow then what we expect due to the viscous effects, the layers which are close to the plane so they are experiencing more roughness more friction due to the surface roughness.

Then the layers far away from the surface you expect at least based on intuition the impact of the boundary should not be seen far away, right so that means there is something more happening near in case of viscous flow okay. So what is that war so that is what we are moving? When a viscous flow is in contact with the solid the tangential velocity of the fluid and the surface agrees, this is ignorance condition or no slip condition okay.

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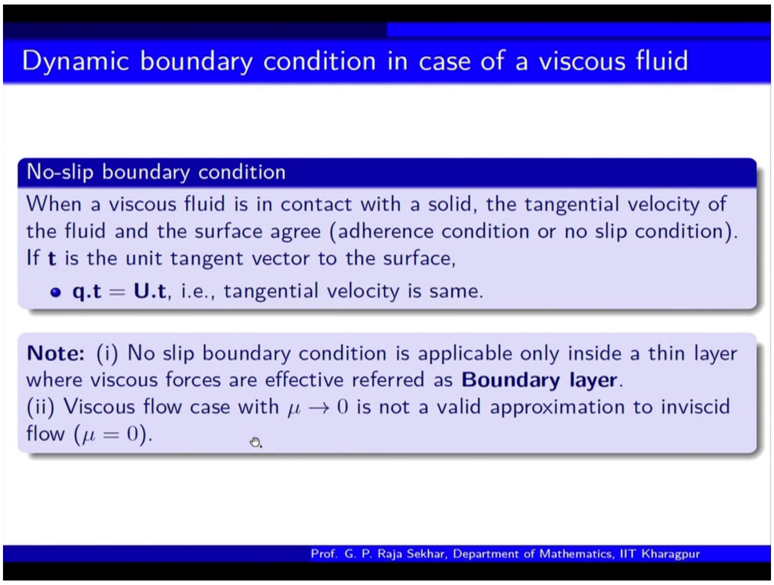

The tangential velocity is same this is the call typically, no slip condition now as I already indicated the no slip condition provides more in a in a layer, thin layer above a boundary because you do not expect a far away from the from the boundary viscous effects are more prominent so therefore so the no slip condition, whatever we are saying tangential velocity equal to the loss tangential velocity of the surface.

So this is more prevalent within a thin layer moreover, so there is another paradox here you take the viscous limit going to 0 and then take the inviscid case so whether these two agree, so that is a that is a paradox so unfortunately so these two solutions do not agree so this we can discuss a little more later on. But at this stage this remark is a sufficient viscous flow case Mu goes to 0 is not a valid approximation to inviscid flow.

Mu 0 is perfectly inviscid flow, but you take a viscous flow solution Mu goes to 0 will not agree with this okay and this already I have indicated it is a valid in a thin boundary layer okay.

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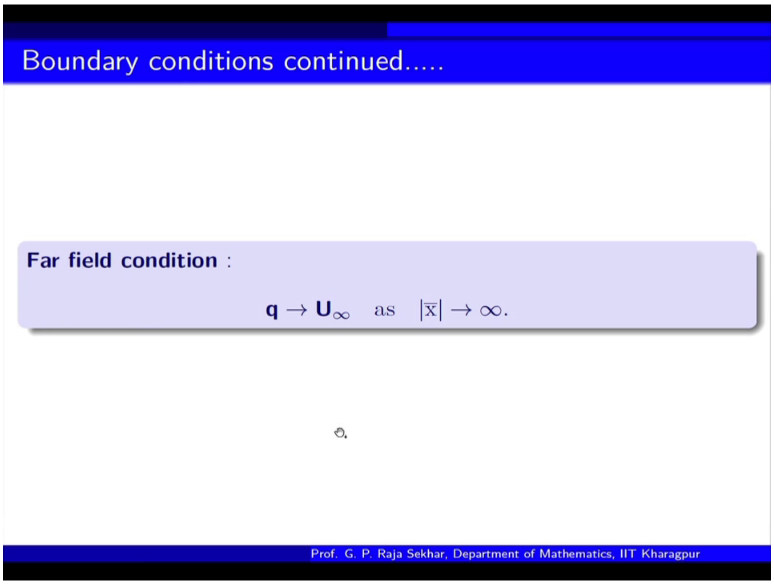

Then typically when we talk about the exterior flows right, so you are talking about the flow past an object then what are the boundary conditions you expect on the boundary of that object, you expect boundary conditions if it is a viscous flow immediately what are the boundary conditions we go for, we go for the kinematic condition and the no slip condition okay one important point here is many times many of the literature they indicate that u bar equals to 0 is no slip okay.

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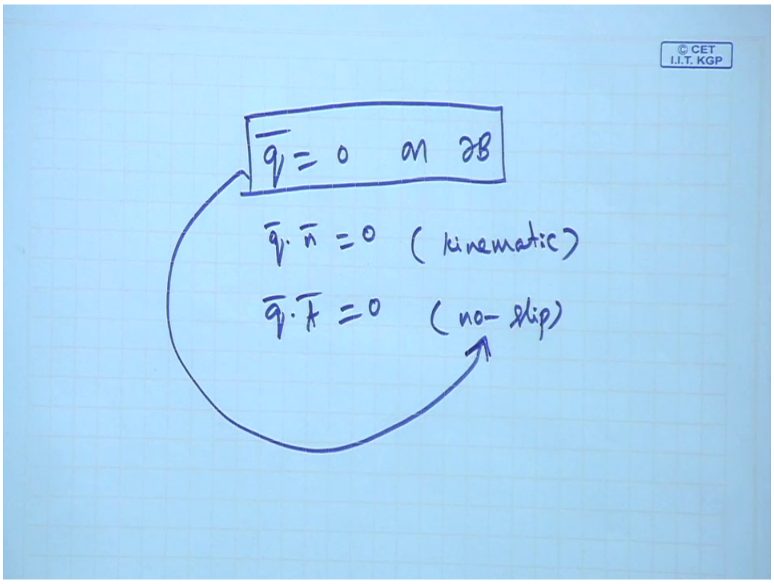

So when I say this is very much I would like to make a point of this, so here we are writing say q bar, q bar equals 0 on say some boundary and that they comment that this is no slip condition okay. So this is a slightly misuse of notation because when you say q bar is 0, what do we mean is the normal is 0, tangential is 0, so these are if you are in three dimensions we have three scalar conditions, this is one scalar condition and the two tangential components so two scalar conditions this one okay.

But strictly speaking this is called no slip, whereas this is kinematic okay, but most of the times people misuse they mention in loose sense, no slip means q is 0 okay. So but strictly if you want to categorize normal is called a kinematic condition tangentially is called no slip condition okay. So now let us come back to our exterior flow case, so you have a flow past an object apart from the kinematics, and then the dynamic condition or the slip, no slip condition, what happens at far field.

Like so you have a flow and you have thrown an object do you expect the disturbance will go to a large extent, it is impossible whatever or disturbance with whatever intensity you create at some stage it dies away right. So we expect a far-field condition right either the disturbance is vanishing or something is provided at far field so that is called a far-field condition. If some ambient conditions are given the velocity agrees with some given ambient flow, so that is the far field condition okay.

Now with this, it is done but it is exterior case suppose you are talking about interior case, like let us say flow inside, inside a ball or some cavity completely closed right, then if you solve you may get terms which are singular at a particular point okay. So like you consider terms of order 1 over r in a sphere then what happens at r equal to 0, it blows up so now it is up to the physics of the problem are we allowing some singularities inside or we are ruling out singularities.

So correspondingly we have to take for interior flows the boundedness condition at r goes to 0. So this is the difference between exterior and interior okay. So you have a boundary condition then you have a far field condition then you have a boundedness condition for interior flows that is the complete package okay.

Now let us revisit the corresponding the no slip condition, that we have discussed so we said the no-slip means the tangential velocity is equal to the tangential component of the velocity of the surface so if the surface is stationary then q dot t is 0 right. That is what I have written just now. Now is this enough or there are cases where this has to be reconsidered? That is the question okay.

And what are these cases typically okay, so you consider varying material surfaces right so then when you vary the material surfaces what could happen, so simplest cases or let us say you have taken a glass, then you have taken wood, then you have taken some metal, so then do you expect in all these three the corresponding no slip condition prevails or we have to think of some correction? So that is the question okay.

So in this context let us visualize you might have seen many times there will be some leaves where you have some roughness, on these leaves so what you would have seen some drops of water if you put so they would not be very stationary, so due to the small roughness on the leaves typically lot of leaves you will see the drops are little, they are not completely taken rest and then they are like no slip.

But they are slightly you can see elevated at the corners that means there is something more happening when the surface is slightly rough, so that is the onset of considering slip okay. So what is this slip condition okay?

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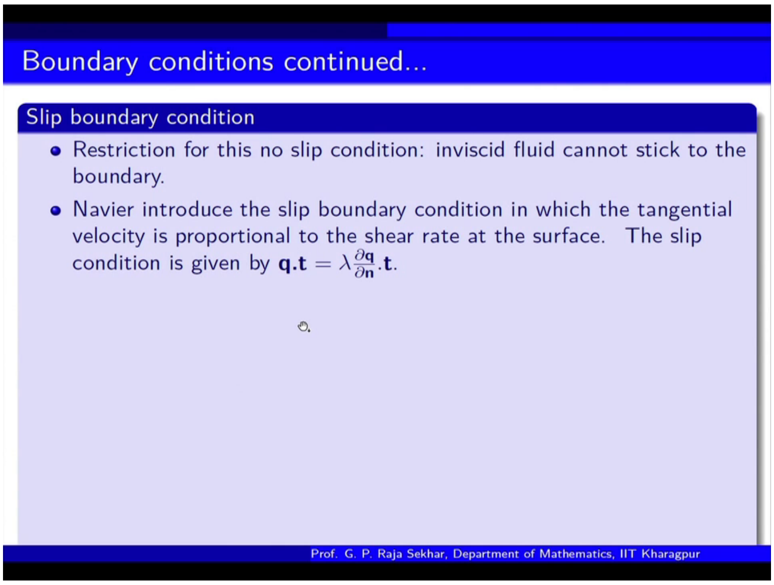

The corresponding tangential velocity is no longer the way we have considered tangential velocity relative velocity is 0, it is not that okay.

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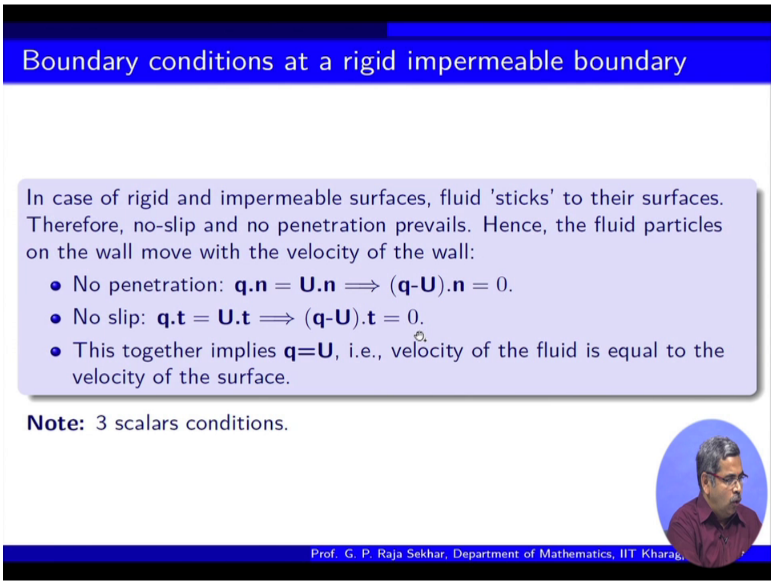

This is kinematic is fine, tangential we are revisiting okay. So what we are revisiting the tangential component is proportional to the shearing of the corresponding tangential component and the proportionality constant is called Lambda, which is the slip condition okay.

And that this there are experiments to validate this particular condition and the experiments will that this particular slip condition is more relevant in microfluidic environment not at a global scale, why what is the general logic? The general logic is, you are considering macro scale so you have a surface here and you are considering flow in a large container so the corresponding dimensionality do not bother the micro level surface roughness okay.

So they do not bother but if you are in a micro level the surface roughness also matters okay. So you will see for example if you in Cartesian, how the corresponding slip can be written?

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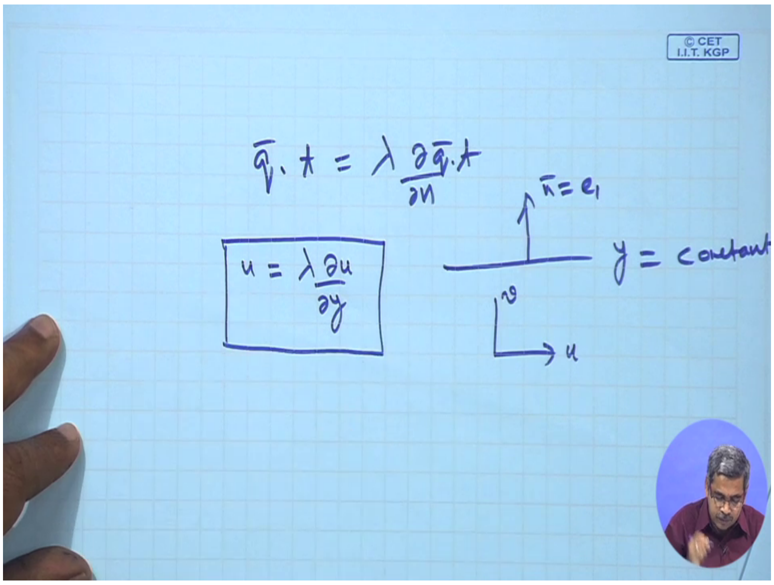

So q dot t, Lambda times, this is normal and the q dot t, suppose our surface is y equal to constant okay. Then what will be the normal, normal will be e1 right. So what will be the corresponding velocities if you go u and then v suppose you decompose then tangential velocity will be corresponding u this equals to Lambda times normal derivative will be along y.

So therefore Dow by Dow y of u so this is slip condition for 2D, if your 3D correspondingly you lose and now the immediate question is what is this Lambda we are talking about? What is this slip coefficient are there any physical in facts to control this? These are the questions, yes there are physical insights to control this, and what are they? So this really characterizes the surface roughness okay.

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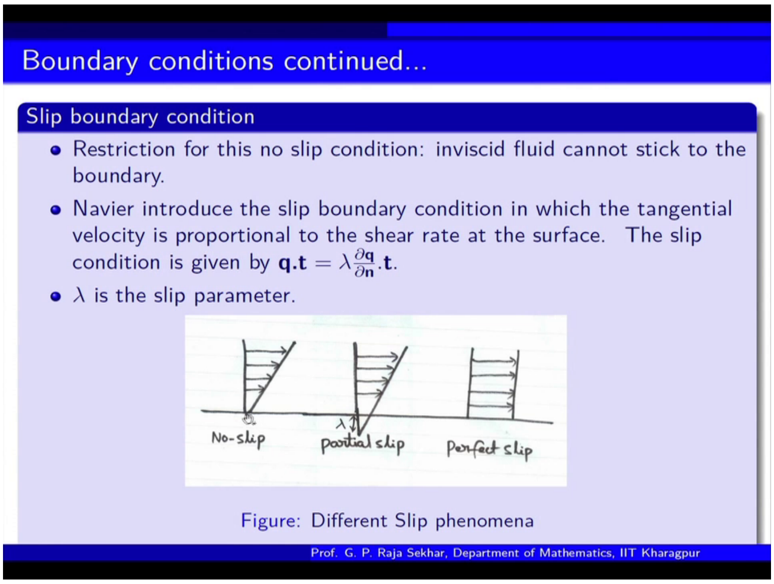

So how you can visualize we expect no slip that means particles a drink to this they are at rest when the surface is rough, if the surface is moving the particles are also moving with that, therefore it is 0. Here but in partial flip where you have Lambda is finite then due to the surface roughness the velocity is not 0, on this is somewhere slightly interior of the surface and this is called the slip length okay.

And this is perfect slip well this is again bit of hypothetical perfect flip means fluid particles are continuously slipping on the surface okay and this Lambda depends on the material that is what I meant if you take glass surface, you take wood, you take some metal surface.

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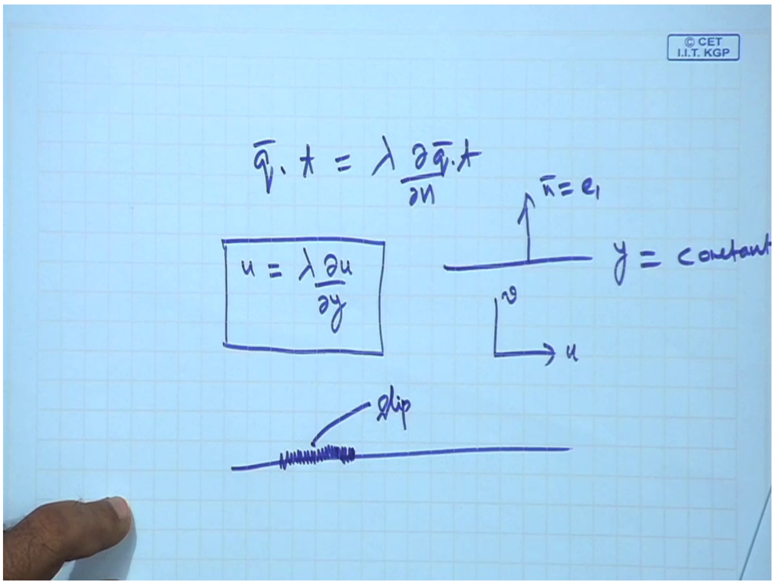

So the surface undulations at micro level it is not crisp boundary, like that the boundaries you have some undulations. So this causes slip okay, so that.

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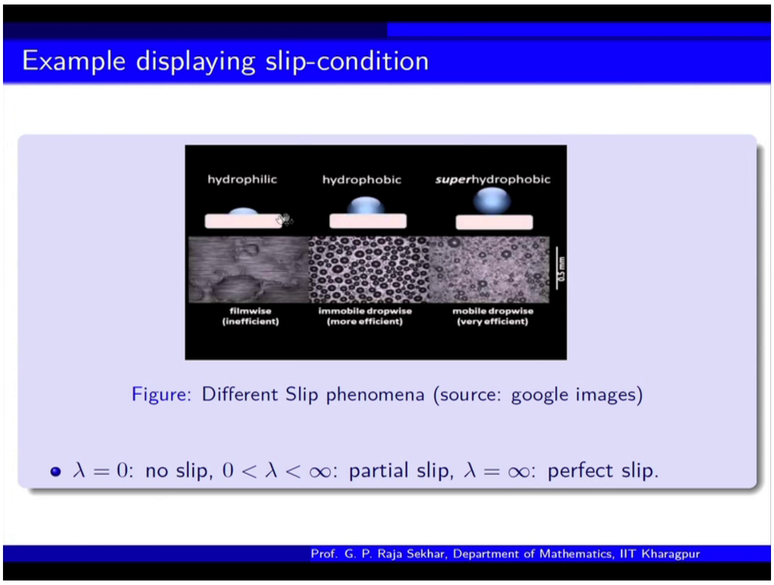

So you can have a better visualization with this, so this is the droplet that I mentioned on a leaves, you see it is it is an almost taken rust, that means the particles are completely taken the velocity of the surface the drop is very much rested, this is nothing but no slip Lambda equal to 0 case okay, if you see here slightly at the edges it is elevated, slightly the edges are elevated okay.

So here the surface roughness is slightly elevated so the surface roughness is causing the velocity is not 0 immediately somewhere interior okay. Then you see it is almost dancing kind of its floating so it is a Lambda goes to infinity and this is Lambda finite okay, and this is Lambda is 0 so correspondingly the surface or characterize hydrophilic, hydrophobic and the super hydrophobic okay.

So you can now see lot of examples to indicating this so these are the various types of boundary conditions and depending on geometry of course we have to resolve the corresponding normal and tangential vectors and depending on the coordinate system one has to adopt and then solve various problems okay. So with this for a simple configuration reduce Navier Stokes you will be ready to solve problems or in the coming lecture we discussed some more about some specific configurations and how to get the solutions thank you.