Modeling Transport Phenomena of Microparticles
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Lecture - 30
(EOF) near heterogeneous surface potential

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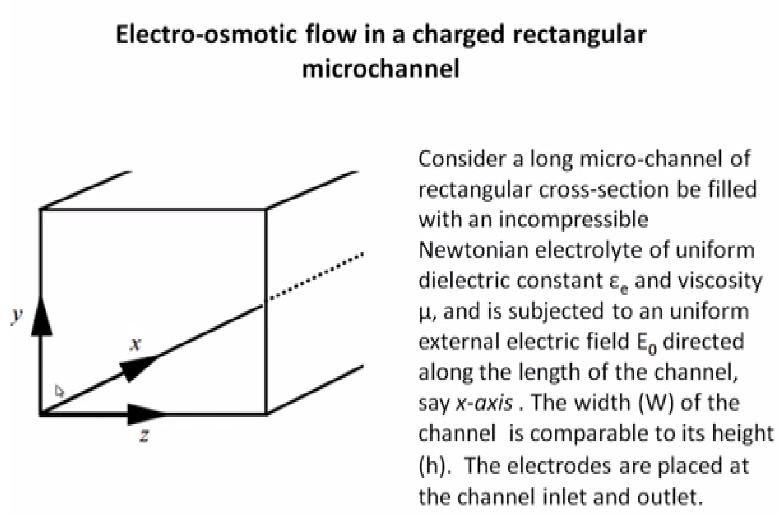

So we have considered flow for a fully developed have a nice slit micro-channel. Where in other words as a two parallel plates, now we may how wave a situation where the channel cross section can be that is the width of the channel maybe of the same order as a height so for example rectangular cross-section channel so we considered a infinitely a long does a rectangular cross-section channel show micro-channel which is the walls are charged and which is filled with Newtonian electrolyte of uniform dielectric constant.

Epsilon e and viscosity Mu and subjected to a uniform external electric field is E0 so this is E0 we take the electric field along the length of the channel show that is means x axis, height means height at the y axis and Z is width wise, so consider the electrons are placed at the inlet and outlet of the channel so it take a cross section this is rectangular section and flow will be symmetry along x axis.

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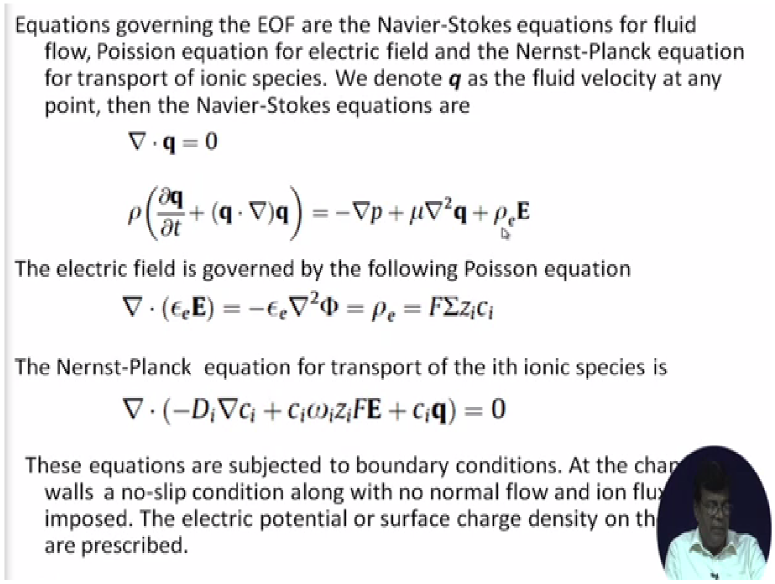

So the equations which governs the with low ion tens concentration and the electric field given by the equation questions. So this is basically the potential Navier-Stokes equation, equation of continuity and year Q with denotes as the of the fluid velocity at any point and this is a Momentum equation and stuff is quotes wants to the electric force you to the non zero charge density row electric body Rho e E is electric field at that point Rho e is the charge density of the electrolyte of the fluid which we are considering at that particular at that point itself.

So this balance of momentum are governed by this way this is the viscous diffusion and this is the pressure gradient and this term is adjective transfer your because of the fluids advection is also referred the initiative and the electric field equation is governed by that Poisson Boltzmann equation now electric field we assume as the superposition of the two type of electric field, one we are imposing from externally and which have a constant electric field uniform electric field E0 and other because of this movement of the ions.

That means the electric field which is induced by the charged surface so that we call as the induced electric field so whatever they will solve the Maxwell equation and we get the Poisson equation for the electric field is governed by this equation show these two equations all supplemented with the Nernst-Planck equation for transport of ith ionic species here we considered a study situation.

So this is the conservation of ion flux so these ion flux has now are governed by the molecular diffusion of the ions if you have a concentration gradient present then and this is the one for the electro-migration that mean the mobility of the ionic species omega i the electric field E and this is advective transport of the ions, due to the fluid flow, so this complete set of equations which governs the Electroosmotic phenomena inside a channel can find symmetry we are considering this can be because we have not specified the form of this gradient operator.

So either way we can conserve the capillary or rectangular micro-channel or a capillary channel or whatever geometry you like to consider because is the basic equation. Now these equations are subject to the boundary conditions imposed channel well as no slip boundary condition along with no normal flow in a viscous fluid if you have been hydro fluid surface so you have no a slit and no flow and ion flux also zero ion flux on the channel all along the direction of the normal as the electric potential or surface charge density I assumed to be given know these equations we skill non dimensionlaised.

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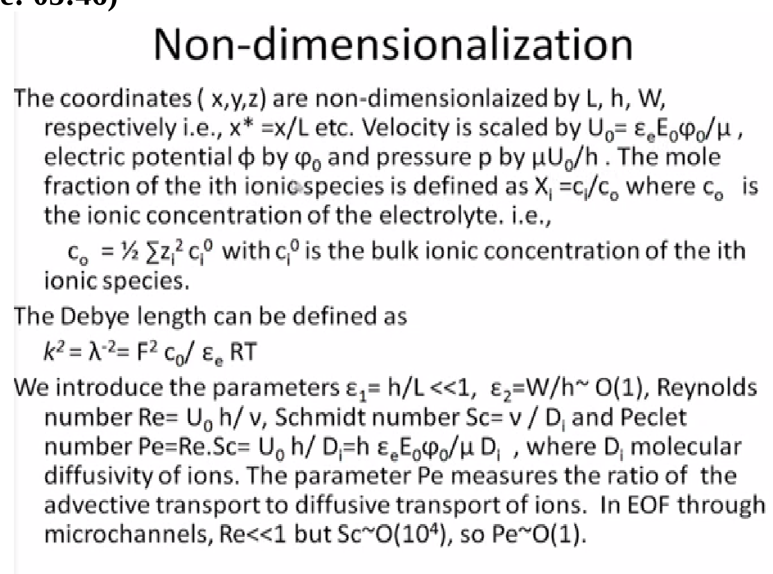

In this manner so we are considering a Cartesian co-ordinate system so this co-ordinate x, y, z, are not analyzed by L the length, height and W width of the channel show the skilled variable if I denote by a star so this is the way now remember that L is quiet larger than H and W and velocity skill by the this U0 is also some can be referred as a Smoluchowski velocity with Zeta as Phi 0 and electric potential Phi Phi0 Phi0 I guess this thermal potential and pressure is split by this Mu Nu 0 by H which measure the non- dimensional which non dimensional is the pressure.

The pressure variable p and we consider the mole fraction Xi ci by c0 the ionic concentration of the electrolyte and so c0 is given by this way ci by c0 super superscript 0 here, implies the bulk ionic concentration at the of ith ionic species which are bulk ionic concentration when you have an equilibrium or electron-neutrality situation should Debye length you define why this manner a Epsilon F Square C0 by Epsilon RT and introduce two parameters.

Epsilon 1 has h/L and Epsilon 2 W by h, so h by L is very, very small where is W by h can be order one what because width we are considering in the same order as the height of the channel and Reynolds number introduced by this manner you are Uo h by Nu and Schmidt number by Nu by dy, the divisibility of the ions now what you find that the Peclet number is governed by this relation now normally these Reynolds number will very small at the Schmidt number are quite high.

So this creates the product that Peclet number of order one now so if we considered a here we have a three-dimensional situations and we have a whole set of equations if I expand in Cartesian coordinates so this gives you three momentum equation this is single equations and this will be giving the ionic species if I consider binary even if I consider symmetry.

So we get it set of equations now also we have taken the length of the channel is quite large compared to the height at all these things now what do you have to do is you have to consider the reduced for of the equations of the governing equation now to do that first let us note that the electric potential.

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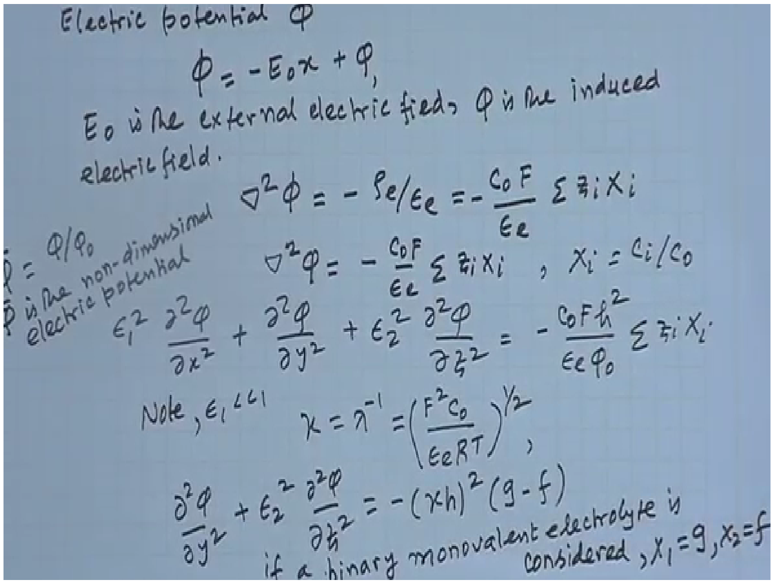

We have define as the electric potential is defined as Phi at any point is just superposition of this 2 case and this is Phi E0X is the external is E0 is the external electric field. External electric field and Phi is the induced electric field now if I see the Poisson equations for the electric field this is covered by Del to Phi = - Rho E by Epsilon e and that is given by we can write as C0 F by Epsilon e Sigma Zi Xi.

Xi is the mole fraction already verified and now Del 2 Phi so this if I substitute this that Phi for this becomes see it but then 2 Phi small that is the induced potential equation so -C0 F by a EF Sigma Zi Xi. If I do this scaling, scaling means how your scale here is Phi scale by Phi0 and the Cartesian co-ordinate sub scale by h and all those things so this is the scaling we have already discussed Phi scaling by the Phi0 somewhere written.

So if we do this scaling get is reduced form with Epsilon 1 square Del 2 Phi Del x2 + Del 2 Phi Del y 2 + Phi to square Del 2Phi Z 2 = -C0 Fh 2, h taken common from this side and Phi was scaled by Phi0 Epsilon e Phi 0 Sigma Zi Xi, Xi C is nothing but Ci by C0 and one thing to remember is that we are denoting the is variable skill variables and the dimensional variable are same notation so basically we should have a different notation.

So if I call this is Phi bar = Phi by Phi0, so Phi bar is the non- dimensional electric potential and we drop the over line over bar; now Epsilon 1 is very, very small. And also note this Epsilon 1 is very small and also this Kappa which is equal to the Lambda inverse is given by F square C0 Epsilon e R T to the power half, now this Phi 0 is nothing but Rt by F so this can be written as so reduced form is Del 2 Phi Del y 2 + Epsilon e square Del 2 Phi Del z 2 = - Kappa h whole square g - f. If I considered, if a binary monovalent electrolyte is considered.

So we are calling X 1 = g, X2 = f, so X1, X2 as g n f because we have a binary monovalent electrolyte this is the reduced form of the Poisson equation. So let us call this equation one and non dimensional, so you drop the maybe somewhere we should write we have dropped the overlay. Now next is the Nernst-Planck equations as we have already discussed.

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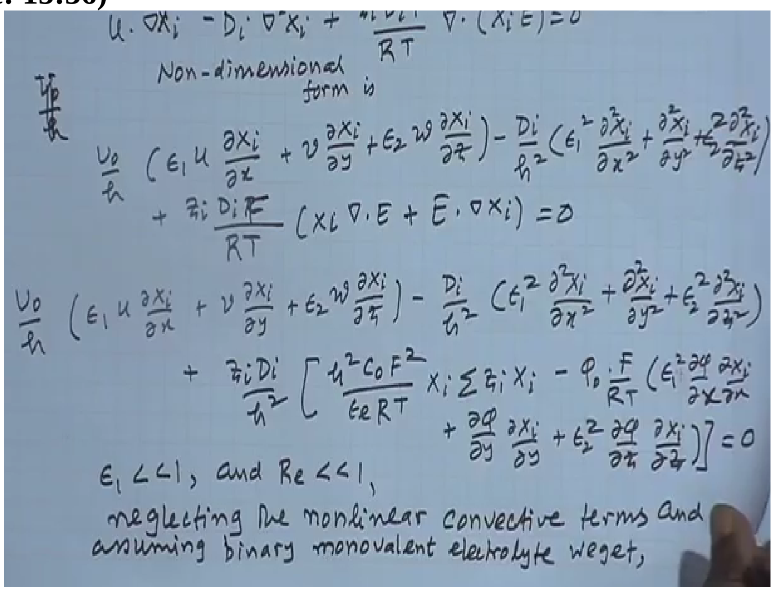

So that gives you Nernst-Planck equation is U dot d Xi, so this Nernst-Planck equation is given by this way so one can write the Nernst-Planck equation is Q of course we have taken q so U dot D Xi we have taken the mole fraction +- Di Del 2 Xi + zi Di f by R T and Xi = 0, so now if we non dimensionalize, so I get is a U0 by h non-dimensional form, u0 by h Epsilon 1 Del Xi by Del X + v Del Xi Del y + Epsilon 2 W Del Xi by Del y - Di h square + Epsilon 1 square Del 2 Xi Del X2 + Del2 Xi by Del y 2 + Epsilon 2 square Del 2 Xi by Del Z 2.

If you see terms + Zi Di F by RT Xi divergence of E + divergence of xi = 0. If I expand that way, so now we know this e = - grad phi and all so now if you do little manipulation what a go get is U by h Epsilon 1 u Del Xi Del X + v del Xi Del y + Epsilon 2 Xi Del Z - di by h square Epsilon

1 square all this term, Del2 Xi by Del X2 + Del Xi by Del y 2 + Epsilon 2 square + Del 2 Xi by Del Z2 + Zi di by h square.

If I take out for this is h square c0F square by Epsilon e RT Xi Sigma Zi Xi that corresponds to the divergence of e then - Phi0 into a F by RT and that is the other one equation at Epsilon square Del Phi Del X + Del Phi Del y Del Xi Del y + Epsilon 2 square Del Phi Del z = 0. So if I introduce these Reynolds number and Peclet numbers so and also note that this is the convective term so Epsilon 1 is very, very small and the Reynolds number is quite small and if we neglect neglecting the nonlinear collective terms.

If we neglect the nonlinear convective terms from this equations and also we assume that this is a binary electrolyte and assuming binary monovalent electrolyte we get it reduced form. We get reduced form of the equation in this manner

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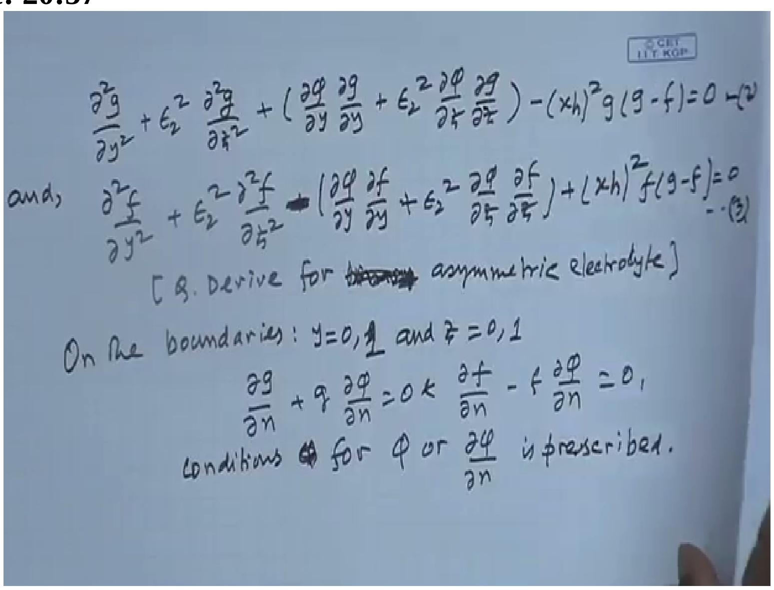

Del 2 g Del y 2 + Epsilon 2 square Del 2 g Del Z 2 + Del Phi Del y Del g Del y + Epsilon 2 square Del Phi Dz Del g Del Z – kh whole square g (g – f) = 0, so this is one equation, so lete us call equation two and similarly for the other equation the binary so we have the other mole fraction equation Del 2 + Epsilon 2 square Del 2 f Del z – Del Phi Del y Del f Del y + Epsilon 2 square Del Phi Del Z Del f Del z + kh whole 2 f into g - f = 0.

This is equation 3, so I suggest derive the equation, question is derived for a binary asymmetric electrolyte for binary asymmetric electrolyte okay, not even binary symmetric electrolyte, so here we of course as always you choose this binary electrolytes for the simplicity, now so this other boundary conditions sorry this other transport equations for the ions g and f, now we can impose the boundaries condition on the boundaries on the boundaries.

We will have no normal flux, so that means you should have on boundaries in this case y = 0,1 and Z = 0,1 we should have the conditions has Del g Del n + g Del Phi Del n = 0 and Del f Del n – f Del Phi Del n = 0, along with condition for Phi or Del Phi Del n is prescribed. Now we need the equation for u, because these two equations does not depend on the u, but depends on Phi, in the equation now u in order to know the Electroosmotic flow.

So the equation of the continuity, the Navier-Stokes equation.

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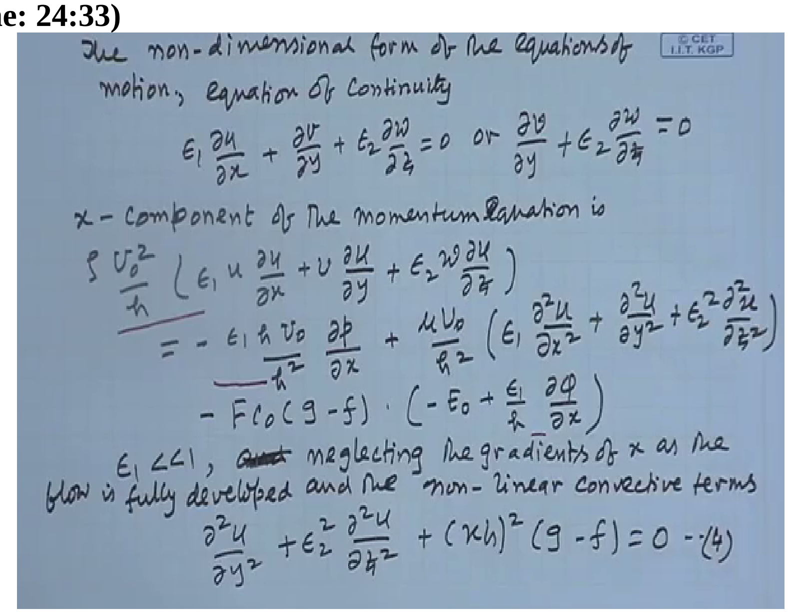

The non-dimensional form the equation of motions that is the next task we, so equation of continuity if I scale, so what I get is Epsilon 1 Del u Del x + Del v Del y + Epsilon 2 Del w Del Z = 0 or Epsilon is very small, so you can have Del v Del y + + Epsilon 2 Del w Del Z = 0. And the x component of the momentum equation is which can be expressed as Rho U0 square by h Epsilon 1 Del u Del x we are choosing a steady state, because it is a electric field, applied electric flied is taken to be steady DC.

So you can have the steady state situation, so this is the and pressure is scaled by Epsilon 1 h U0 by this is the U square by h, this is h square because this you have Del p Del x, so p is scaled and x is scaled by h, Mu U0 by h square this term, diffusion terms there is Mu already there, so Del x 2 + Del 2 U by Del y 2 these are the scaled very above Del 2 u Del z 2, then we have this –F C0 g – f again we are taking has binary.

And inside this Del Phi Del x, so we have term E0, so this capital Phi is –Phi E0 x + Epsilon by h Del Phi Del x, so there are the terms for the momentum equation, now what we can do over here is that if I know Epsilon 1 is very, very small and neglecting the convey two transport terms because the gradients of x as the flow is fully developed, it is independent of the end conditions and neglecting the x gradients the convective terms.

The non-linear convective terms this can also be quite easily neglected, so what I get is this term becoming negligible and this pressure term is becoming negligible, these are the terms which is suspected to be gone, this is gone and so this is also gone and this terms gone, with that this things are gone, so the reduced terms comes out to be Del 2 U Del y square + Epsilon 2 square Del 2 U Del Z 2+ Kappa h whole square g – f = 0, so this is becoming the equation four.

So one, I think that I have that.

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The governing equations for two-dimensional EOF with boundary conditions are

$$\frac{\partial^2 \phi}{\partial y^2} + \epsilon_2^2 \frac{\partial^2 \phi}{\partial z^2} + (\kappa h)^2 (g - f) = 0 \qquad \text{(a)}$$

$$\frac{\partial^2 g}{\partial y^2} + \epsilon_2^2 \frac{\partial^2 g}{\partial z^2} + \left(\frac{\partial \phi}{\partial y}\frac{\partial g}{\partial y} + \epsilon_2^2 \frac{\partial \phi}{\partial z}\frac{\partial g}{\partial z}\right) - (\kappa h)^2 g (g - f) = 0 \qquad \text{(b)}$$

$$\frac{\partial^2 f}{\partial y^2} + \epsilon_2^2 \frac{\partial^2 f}{\partial z^2} + \left(\frac{\partial \phi}{\partial y}\frac{\partial f}{\partial y} + \epsilon_2^2 \frac{\partial \phi}{\partial z}\frac{\partial f}{\partial z}\right) - (\kappa h)^2 f (g - f) = 0 \qquad \text{(c)}$$

$$\frac{\partial^2 u}{\partial y^2} + \epsilon_2^2 \frac{\partial^2 u}{\partial z^2} + (\kappa h)^2 (g - f) = 0 \qquad \text{(d)}$$

If the cross-section is taken to be square then $\varepsilon_2 = 1$.
The above set of equations are subject to the following boundary conditions :

on $y=0,1$ $(0 \leq z \leq 1)$ and $z=0,1$ $(0 \leq y \leq 1)$

$u=0$, $\frac{\partial g}{\partial n} + g\frac{\partial \phi}{\partial n}=0$, $\frac{\partial f}{\partial n} - f\frac{\partial \phi}{\partial n}=0$

$\phi = \zeta$ or $\frac{\partial \phi}{\partial n} = -(\kappa h)\sigma_s$.

The surface charge density is scaled by $\varepsilon_e k \varphi_0$

So this is the reduced from the equations, now one interesting thing from here we do not require to get the solution for u, because this is the most dominating velocity components, so to the solutions for u we do not require the derivation of the other momentum equations. So u is the most dominating compare to other momentum, other velocity component because that is along the axial direction.

So u satisfies this equation, now this equation g and f are involved and g and f are solved by this two transport equation. And again g and f is coupled with the electric potential equation, that should be small Phi so in a way what you find is that, so I write the other two equation the y momentum.

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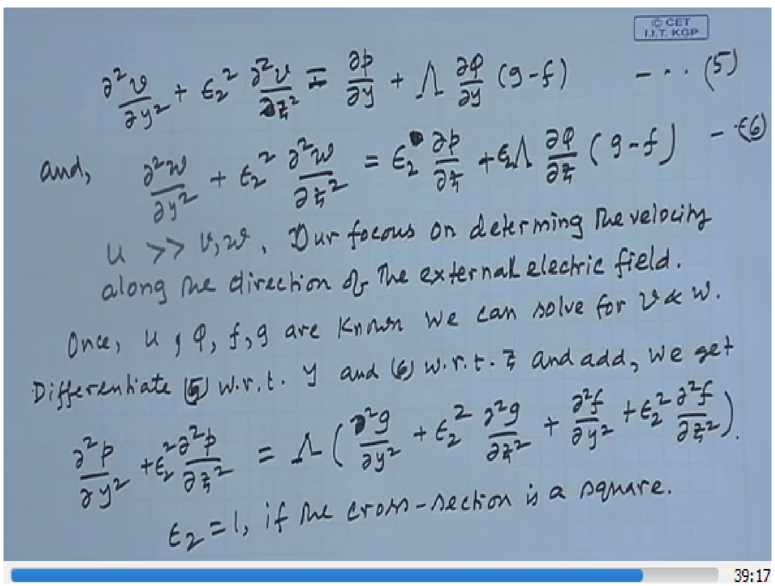

Are that give you Del 2 V Del y 2 + Epsilon 2 square Del 2 v Del Z 2 = Del Phi Del y + Lambda into g – f and we have Del 2 W Del y 2 + Epsilon 2 square Del 2 W Del Z 2 = Epsilon 2 Del p Del Z + Lambda and this will be Epsilon 2 Del Phi Del Z into g – f, here one can verify, in this way one can derive this equation, now you see this u is much bigger that v and w.

Our focus on determining the actual velocity, the velocity along the direction of the applied of the external electric field, so what you find that equation part u is decoupled from here, but if I one can solve this, once u, Phi, g, f are known above we can solve for v and w and also another thing is that from this two equation 5, 6, we can eliminate p and get a equation.

Say if I differentiate Phi with respect to y and 6 with respect to z and add together and do little algebra Del 2 p Del y 2 + Del 2 p Del z 2 and there will be Epsilon 2 square I guess = Lambda into Del 2 g Del y 2 + Epsilon 2 square Del 2 g Del z 2 + Del 2 f Del y 2 + Epsilon 2 square Del 2 f Del z, so this gives pressure distribution and of course pressure does vary at along Del p Del x is 0, so p is a function of y and z.

It does not vary along the axial direction, Now another thing is the Epsilon 2 = 1, if the cross section is taken to be as square, is square cross sections, so that means instead of rectangular we have Epsilon 2 is 1. So to know the axial velocity the streamlined velocity is u, decoupled from the other two components v and w, so as we have described before v and w is comparatively very low. So that can be, is not of much interest because it is a planar flow, so u is the only one important and the boundary conditions we can y = 0, 1.

At the boundary that is the channel walls, y = 0 and 1 and z = 0 and 1, these are the scaled boundary, so the boundary we have the no slip condition non zero flux of ions, these two conditions and constant Zeta potential or constant surface charge density, so obviously these set of equations which cannot be solved in analytical form, straight away because they are coupled and they are non-linear also, but one can solve by boundary volume problem and we get a solution.

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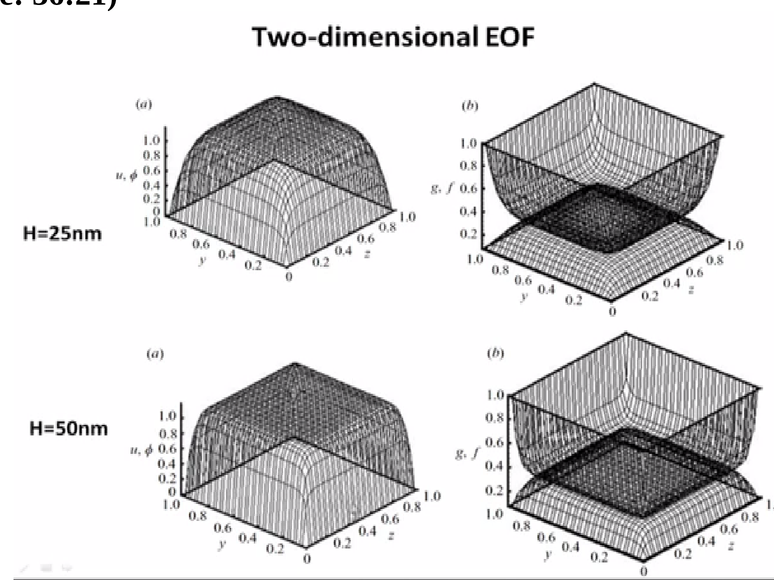

If you do the numerical solution will get it solution of this form now see the interesting part is that when the channel height is 25 NM where the Debye length is channel height are comparable so in that case comparable means the ratio is close to 1 so in that case you have a profile is a kind of little parabolic, but when it is the Debye length is much lower than the channel height what you find is a plug like that is a flat top profile for this u this is a two-dimensional.

That is along the cross section of the channel the velocity profile and another thing is that u and Phi satisfy the same equation and the pattern of the solution is also same and this is the ionic concentration distribution g and f, so what we find that when channel height h as I said is close to the channel Debye length, so you have a non zero value of g and G – f, is what is the charge density.

So if g – f is zero, so that means the fluid is electrically neutral, so what you find that there are reasons g – f can be non zero where is when the Debye length is very smaller than the channel height, what you find that within the bulk region you have g – f is zero, exactly zero so that the flow is governed by the viscous shear stress so that is why we get a constant velocity over in the bulk region where there is electroneutrality of the fluid is established.

Whereas here the flow contribution of the charge density because g – f is non zero is present and if we have the thin Debye length compared to the channel height, so what you get is, it is only governed by the viscous shear stress. And get a profile a linear profile like this so this is interesting phenomena for Electroosmotic floor to have a constant velocity.

So that is why the dispersion effect is small so that is why this Electroosmotic flow is very much applicable in several situations particularly truck delivery or control fluid transport and so far others.