Modeling Transport Phenomena of Microparticles
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Lecture – 07
Stokes Flow Past a Cylinder

Hello, Welcome back. Today we are going to discuss about Stokes flow past a circular cylinder. So when we say stokes flow past a circular cylinder one may be tempted to think so you have a cylinder and flow past. So well this is a different problem. So while one can discuss this cylinder of finite length, but today we are going to discuss about flow past like flow across cylinder. Like you have a circular cylinder and then flow across.

So if you see from top view what we see is just a flow past a circle okay. So this has lot of applications because what we are seeing is flow across cross sections kind of, okay. So if somebody is trying to analyze flow across a 3-dimensional object, so the first attempt would be to get some insights with the help of flow across cross sections. So for that this problem will help and a lot of applications of flow past 2-dimensional objects, okay.

So this is a simplest case that we are going to consider; that is; the stokes flow past a circular cylinder okay. So as we have seen we have introduced stream function, so since this is a circular cylinder we would like to use a stream function in r Theta polar co-ordinates, okay. So let us start from quick look at unsteady stokes equation.

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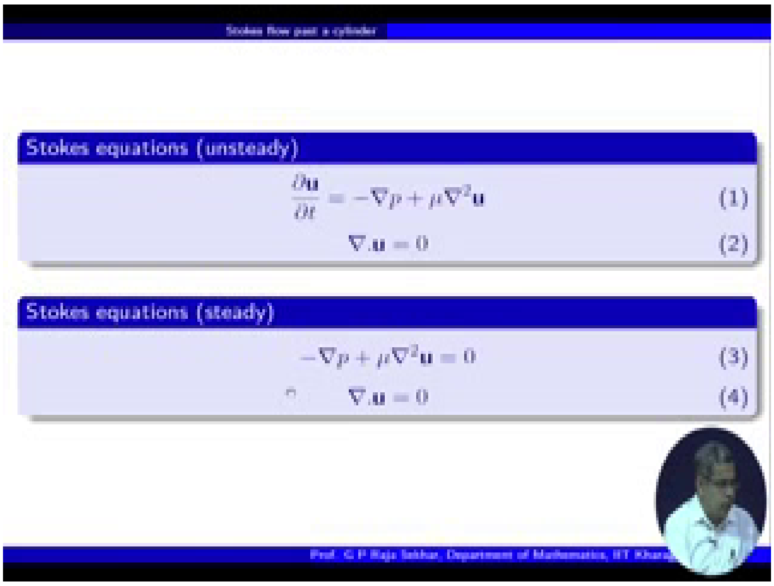

So then we have pressure forces, viscous forces and this mass conservation and this is a unsteady term. So we assume that the flow is steady. So accordingly we have the corresponding linearized study stokes equations, okay.

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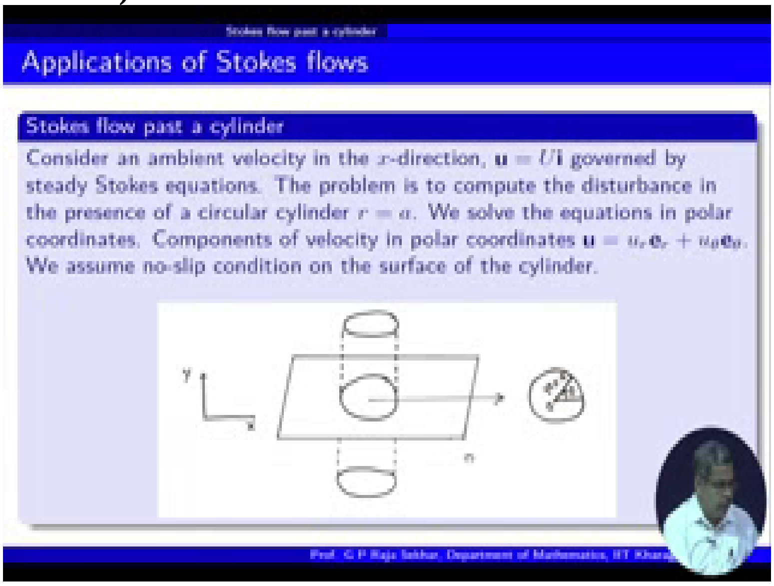

So now what is the problem as I already indicated, we consider an ambient flow that is along x direction where u is constant. Then the problem is to compute the disturbance in presence of the circular cylinder which is of radius r = a. So we would like to compute given this ambient flow we would like to compute the disturbance due to the presence of the circular cylinder and natural question is what are the boundary conditions on the surface?

If you see we are assuming this cross section is impermeable, hence we assume no-slip condition on the surface of the cylinder and since this is circular geometry, we fix r Theta plane polar co-ordinates, r Theta polar co-ordinates okay. So we are interested in computing the disturbance and once we get the disturbance we would like to get some insights. So let us quickly transform the stokes equations and introduce the stream function in terms of polar coordinates.

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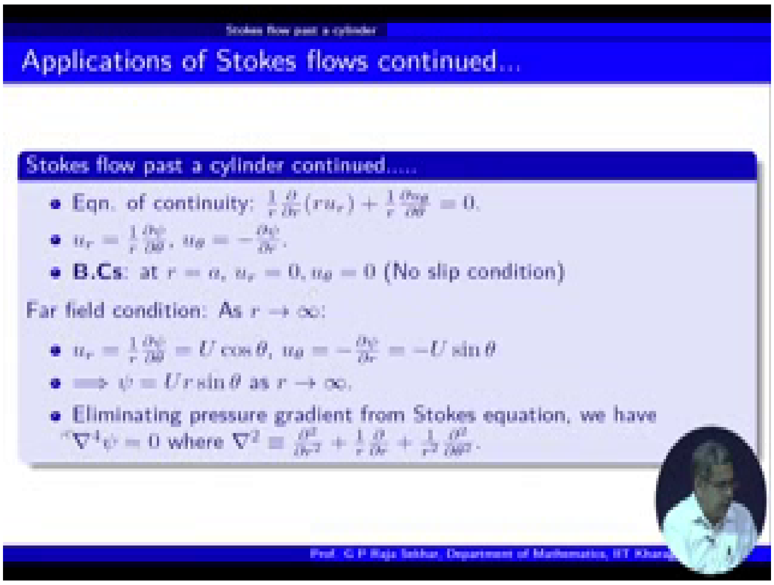

So correspondingly we consider the equation of continuity given by this in r Theta polar co-ordinates. Then we have already seen the structure of this would admit one to introduce stokes stream function like this, okay. So this is a radial component and this is the tangential component. So then we have the boundary conditions. So this is a no-slip, then we have far field condition.

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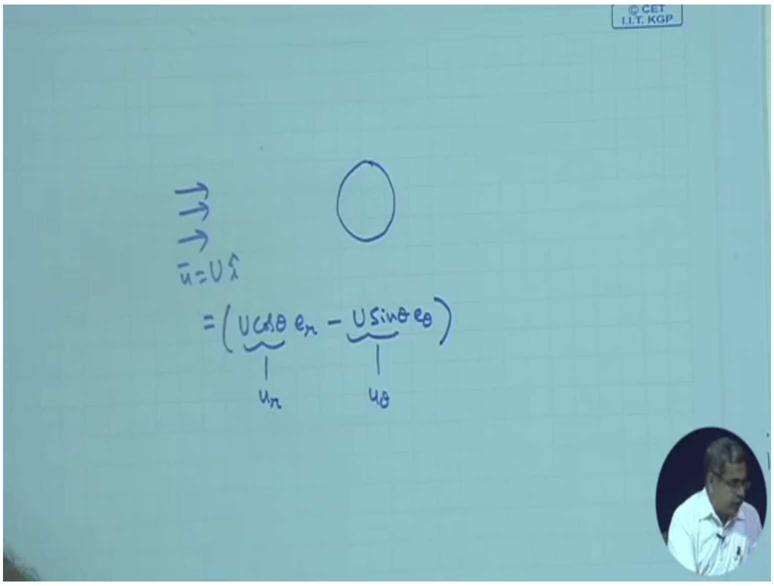

So what is a far field condition? So it is nothing but so we have the cylinder and the far field, so this i can be expressed as Cos Theta r - Sin Theta r. Therefore, we get this correspondingly, this is ur this is u Theta. So therefore we require the corresponding far field condition which is given by this. Now since our aim is to solve the problem in terms of stream function we are interested in getting the corresponding far field stream function.

So one can integrate these two to get the corresponding stream function so this is very straight forward okay, so now we have seen already that we write the corresponding momentum equation in terms of velocity components and then introduce the stream function and then eliminate pressure. So once we eliminate pressure what we get is stream function satisfies bi-harmonic equations okay.

So this we have already seen so correspondingly we consider the bi-harmonic equation so where this is a Laplacian r Theta coordinates. So which means we have to solve this equation subject to this far field condition and these conditions which are written in terms of stream function okay. So let us consider this operator and then try to see what happens. So as you can see this is the linear operator even though it looks little complicated, but the operator is linear.

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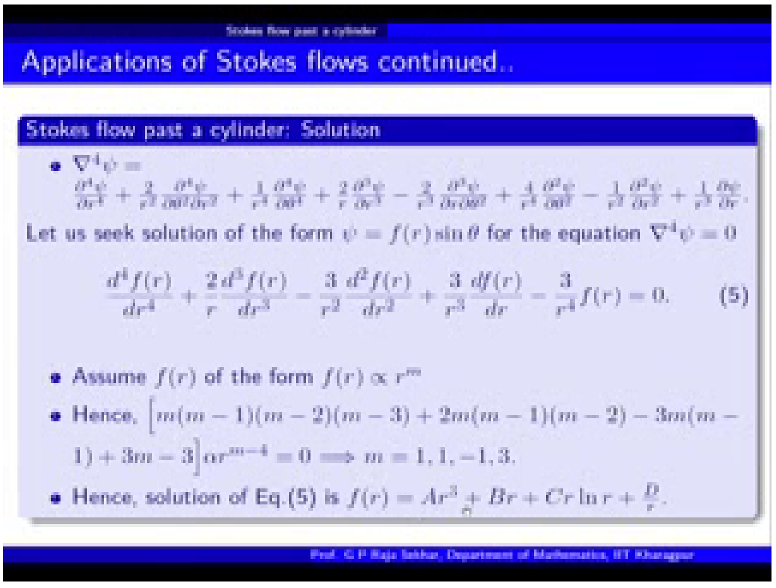

So one would and we have a nice boundary conditions. For example, here we have homogeneous boundary conditions. So therefore one would seek separable solution okay. But we are not looking for general separable solution for reasons which are very obvious because if you look at our far field.

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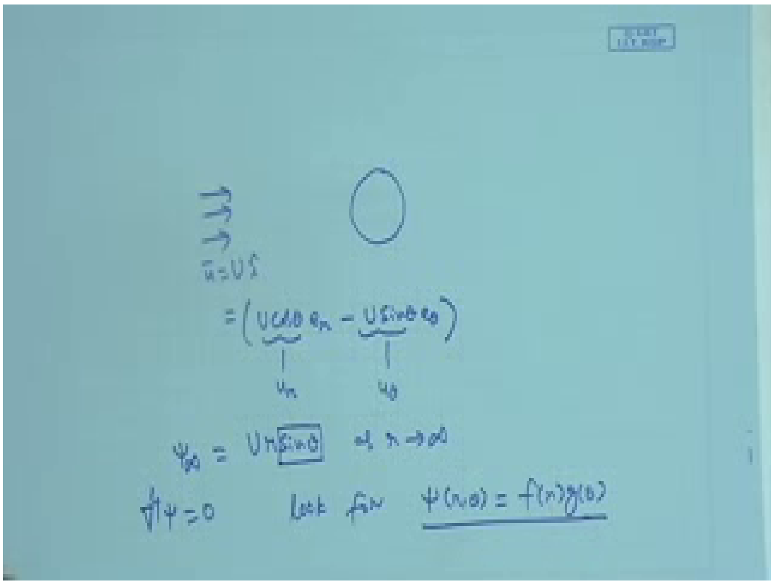

So far field has a structure this correspondingly we have so we have integrated and obtained this one. So therefore we should look for solutions typically we should look for some function of r this okay. So this is typical, separation of variables. But we see the structure of the functional dependency with respect to Theta. So that is admitting only Sin Theta is far field.

So therefore, any functional dependency other than Sin Theta will lead to homogeneous system and a give trivial solution. I mean when you implement the boundary conditions and then calculate the coefficients. So the far field structure indicates that we need not look for the generic separation variable solution. Rather we look for a restricted one as follows okay. So the far field structure of the stream function prompts us to assume the stream function of this form okay. So once we assume this form we substitute in this corresponding PDE okay.

So then we get, this is a ODE. Why because the functional dependency of Theta is fixed, that is Sin Theta. So therefore the operator when it is acted on this quantity you get only the corresponding ODE okay. So at this stage again one might be scared oh we have a 4/3rd ODE how to get it but one can nicely get the solution of this because this is popularly known as Euler-Cauchy type, ordinary differential equation okay.

So how do we get the solution? Well I can tell you the answer first and then try to see. One may look for solutions of the form r power m but natural question is how one is getting this structure okay? So that is natural question right? So let us answer this question so you multiply this equation by r4 and use some shorthand notation.

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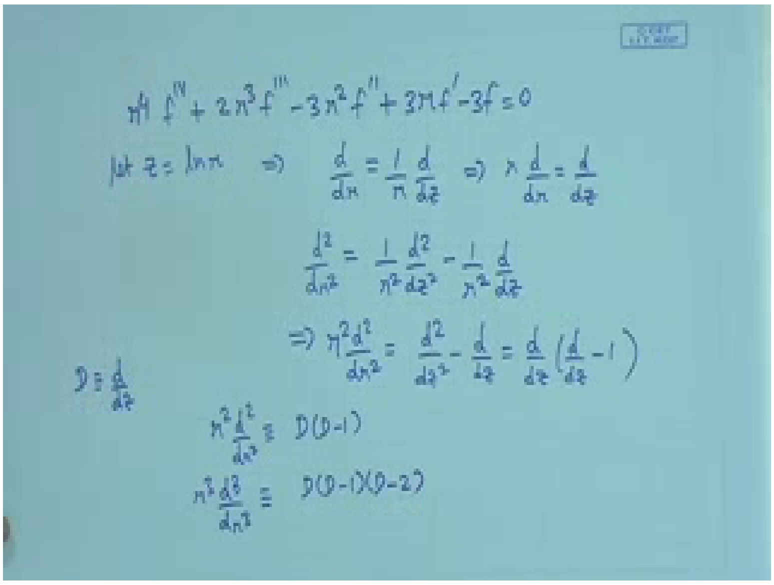

So r4 then so this is a 4 derivatives okay. So then the second term is 3 derivatives then 2 derivatives. The next term is 1 derivative. So what we have done we have written the same equation while multiplying with r power 4. Now here let us take the transformation. So this implies so we have to convert derivatives with respect to r to derivatives with respect to z because we are using a change of variable.

So this is nothing but one can easily check right, so which implies okay. Now we need so again one can differentiate from here. So what we get is the next term is naturally derivative of this. So what we have obtained is that is what we got. So this is just an operator notation. Do not misunderstand that we are differentiating 1 okay. So, correspondingly if you introduce D as d/dz so what we got is r power 2 okay.

So similarly one can verify so this will be and then.

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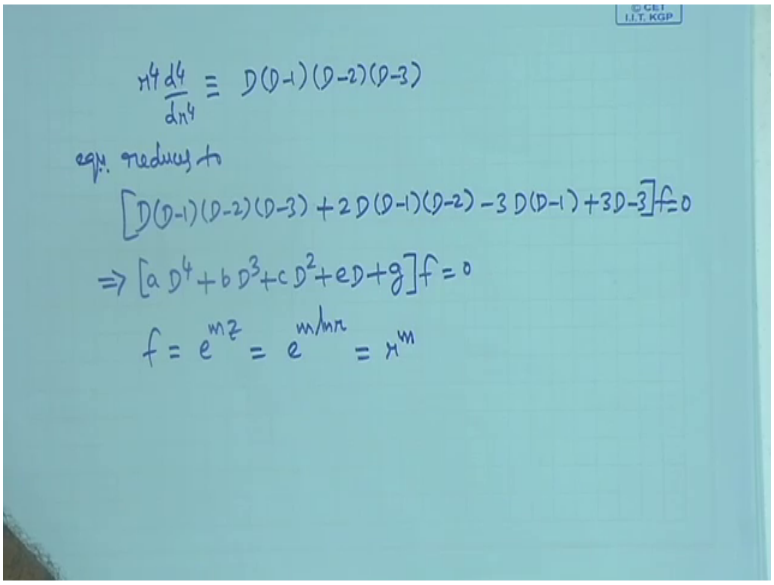

So this is very straight forward to verify. So with this notation the original equation reduces to. What is the first term? First term is this and second term you can see so this is the second term. So r power 3 this is a 3 derivatives with respect r that we have obtained. So therefore only factor 2 is left, then – 3, then plus then. So this whole thing operating on f = 0 okay.

So the equation reduced to this and this can be expressed in the following form for some coefficients okay. So this is the linear homogeneous with constant coefficients. So therefore, we look for f is solution. We know the Euler-Cauchy type which is reduced to this. So therefore, okay. So this prompts us to have the solution of this form okay. So, correspondingly we substitute in this then we get the values of m. So here f is proportional I have taken.

But while fixing it in this we take some multiple so we have taken Alpha and correspondingly these are the values of m okay. So which means we have the solution now. So general solution can be written as corresponding to each of the power. So we can write the solution okay.

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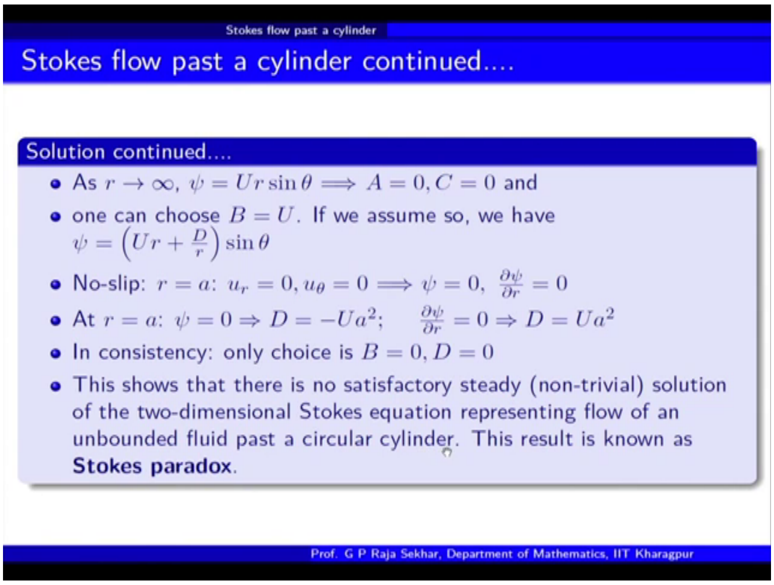

So now let us force the boundary conditions. So first thing is our far field condition, so far field it is of this form okay. So if you see this times Sin Theta so that must agree with the far field. So that is ur Sin Theta okay. So that will give you if you see this is going to vanish at far field okay. So then so we have to compare with r Sin Theta okay.

So here we will be taking the coefficient corresponding to r Sin Theta and then this is anyway going to vanish okay, r goes to infinity. So these are the two terms which are not agreeing with the required structure. So therefore that will give you A 0 C 0 okay. So now one can choose B = u. Why? So let me explain.

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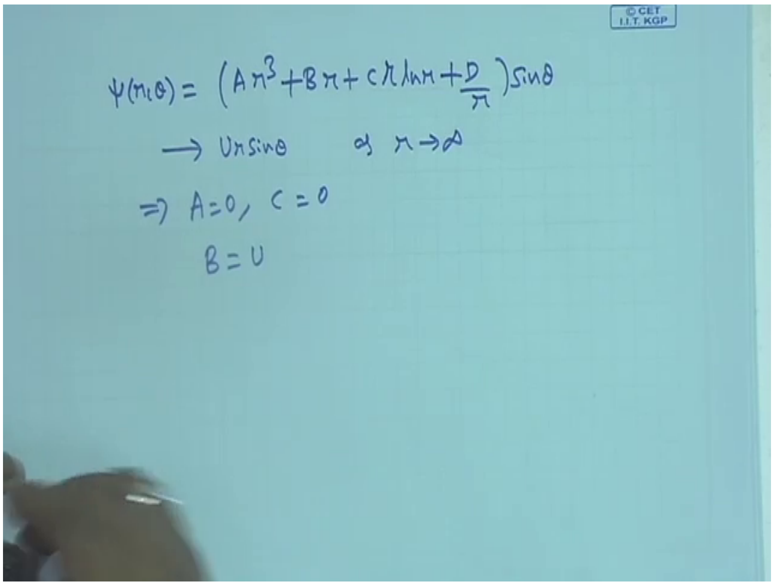

So actually we start with our Psi which is given by so first condition we are looking for. So as I as so I have indicated r goes to infinity this entire thing should agree with this. So this is going to vanish and one can see this term is going to give this term. So therefore we have already forced okay. Because as r goes to infinity these are the singular terms okay. Because we require only of order this.

So this is more regular so that coefficient is zero and this is a singular coefficient is 0 and immediate conclusion is B = u so that this agrees with ur Sin Theta. So that is what we have written. So one can choose B = u. So correspondingly if B is u, we get this because remaining constant is D. Now we have no-slip condition okay. So no-slip in terms of stream function is Psi 0, Dow Psi by Dow r 0.

So correspondingly if we impose these two conditions we get from one condition we are getting this for an another condition we are getting this. So one can conclude immediately that there is an inconsistency because you see this is –Ua Power 2 square and this is Ua Power 2. So inconsistency so therefore only choice is B 0 and hence D 0.

So this sounds little unfortunate because see, physically stokes flow past a circular cylinder and then if you insert a cylinder you have a disturbance so we expect one can compute it. But, mathematically when we are computed we are having some difficulty okay. So what is the difficulty that we have got? So when we superpose a uniform velocity at far field, then try to determine the disturbance.

What we realize we are having some inconsistency and we are getting trivial solution or rather no non-trivial solution. So this is called Stokes paradox okay. So one require additional insights to understand this okay, so this shows that there is no satisfactory studying non-trivial solution of the 2-dimensional stokes equation representing flow of an unbounded fluid past a circular cylinder.

As I indicated this result is known as Stokes paradox okay, so why this is happening? We have a this is one explanation because we have got a system of equations and then try to see the solution is non-trivial or not then for non-trivial solution we are having inconsistency. So, therefore, we have only the trivial solution. So there is a alternate explanation. So let us see that.

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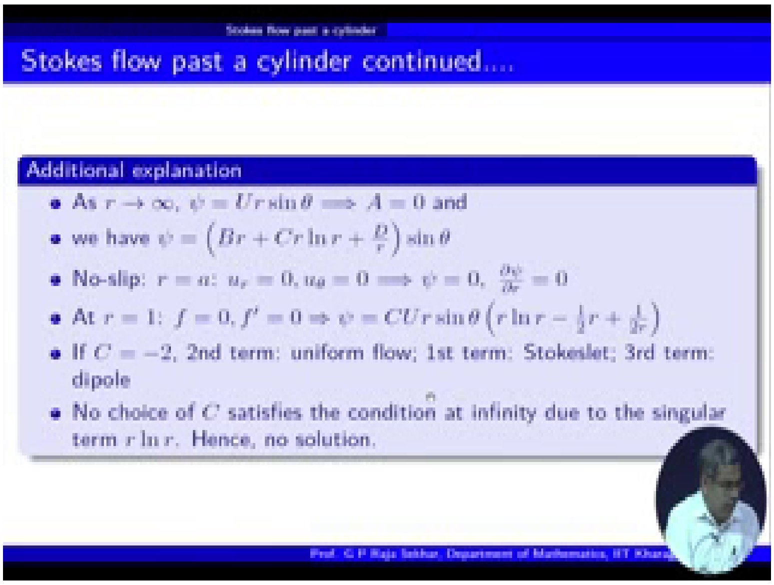

So as r goes to infinity this must happen so that will force us let us say simply that means we are considering alternate explanation. So from this what we are considering? Let us say this is more regular we restrict only is of order r if we leave the functional dependence of Theta aside because that is Sin Theta. So we expect far field should behave only in the order of r okay. So this is more regular.

So let us say only we are forcing A = 0. So then we get this okay. Then if you force no-slip okay. So then one can determine two constants B and D and in terms of the third constant C, so we will be left with this structure okay. So if you see closely if C is - 2 second term is uniform flow first term is Stokeslet okay and this is Stokeslet, third term is a dipole okay. So I apologize because at this stage you do not have more information about this okay.

So I will give insights about these. These are called a singular solutions of stokes equation okay. So in the coming lectures we learn more details about the similar solutions of stokes equation. Then you would understand what is the Stokeslet and then dipole okay. So for the time being you just take it as the terms indicate okay. So what we have done is we have forced far field alone then we are left with three arbitrary coefficients.

Then if we force no-slip condition which we have two conditions so that will fix two constants in terms of the third one. So this is the complete expression at this stage, so if C is -2 this is exactly you can see -2 so it cancels then this will be the ur Sin Theta okay. So that will be the uniform flow and then this is a Stokeslet and the third term is a dipole okay. So no choice of C satisfies the condition at infinity okay.

So the condition at infinity is, whatever choice this is not going to satisfy the conditioner at infinity. So this is another explanation okay. So now why this is a happening okay? So what is the remedy? So the remedy is: so what we have considered is a flow past a cylinder and then we have assumed stokes flow where we have neglected the inertial terms and then far field ambient velocity and we try to compute the solution.

But then we have realized that we have some inconsistency okay. So this is called this phenomena is called stokes paradox okay. So that means physically we expect solution but mathematically we see that there is a issue. So, therefore, this is termed as a very popularly termed as a strokes paradox okay. So how to resolve this okay? So one would think we have neglected the non-linear terms okay.

So I think that is a major concern when it comes to flow past a circular cylinder. So one would expect that you have an object then we have viscous flow.

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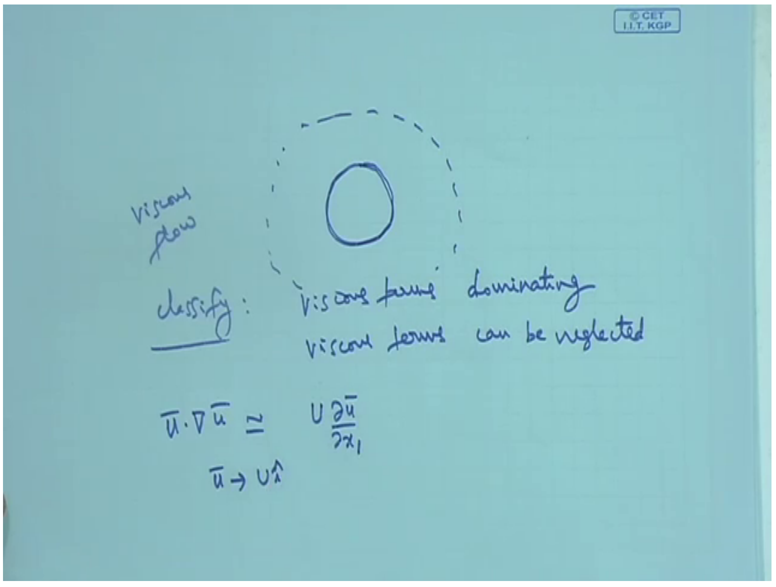

So viscous effects are dominating near the boundary up to some region okay that is expected, so that means one may need to classify okay viscous terms dominating, viscous terms can be neglected. So this is a natural intuition. That means one would expect within some region viscous terms may dominate the inertial terms beyond that viscous terms because there is no boundary. Boundary effects are not dominant beyond certain distance.

So there the non-linear terms are expected to dominate.

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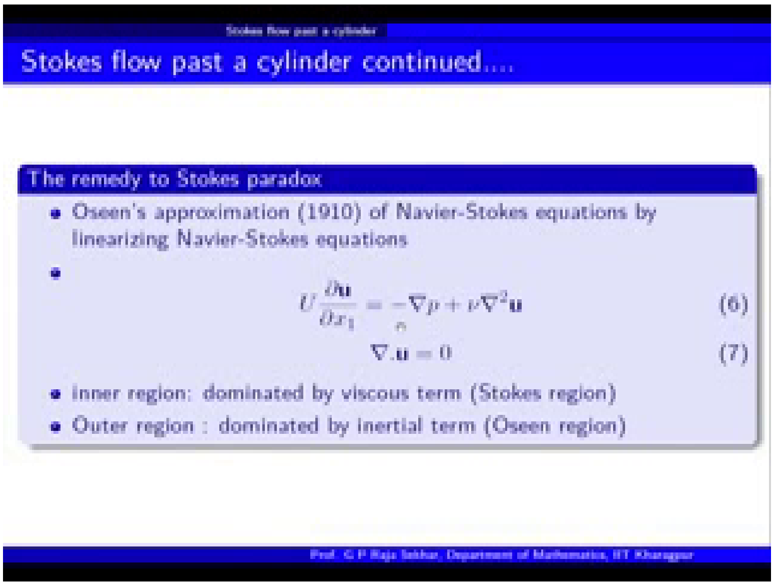

So correspondingly Oseen in 1910 has indicated that one would consider the non-linear terms but in a linearized sense. So what is the proposal? The proposal is we have uniform flow along x direction. So consider the inertial term. So this is linearized with our because u is. So this is a linearization proposed. So correspondingly since it is a x direction so with x1, x2, x3, notation, we are using this.

So this is inertial term then Stokes. Then classify the zones as inner region where viscous terms are dominant, outer region where the inertial terms are dominant okay. So correspondingly one has to solve the solution within the inner region stokes equations and the within our region one has to solves the Oseen equations and then do some analysis to get the combined solution okay.

So this is a very popular paradox and then the resolution is one has to consider Oseen equations which are linearized so that within a particular distance viscous terms are dominant and you call it Stokes region and then beyond it is called Oseen region where linearized version of Navier stokes is used and then naturally there is something called a matching technique so that these two solutions agree at the interface region okay.

So this is a stokes flow past a circular cylinder and this gives us some insights for understanding 2-dimensional objects. Thank you.