Modeling Transport Phenomena of Microparticles
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Lecture – 08
Stokes Flow Past a Sphere

Hello in the previous class we discussed about the stokes flow past a circular cylinder. So now we move on to stokes flow past a sphere okay. So since the title of the course is transport of microparticles so most of the micro particles are spherical. For example if you talk about microorganisms or colloids' so they are spherical so this is going to be very helpful for us okay. So stokes flow past a sphere.

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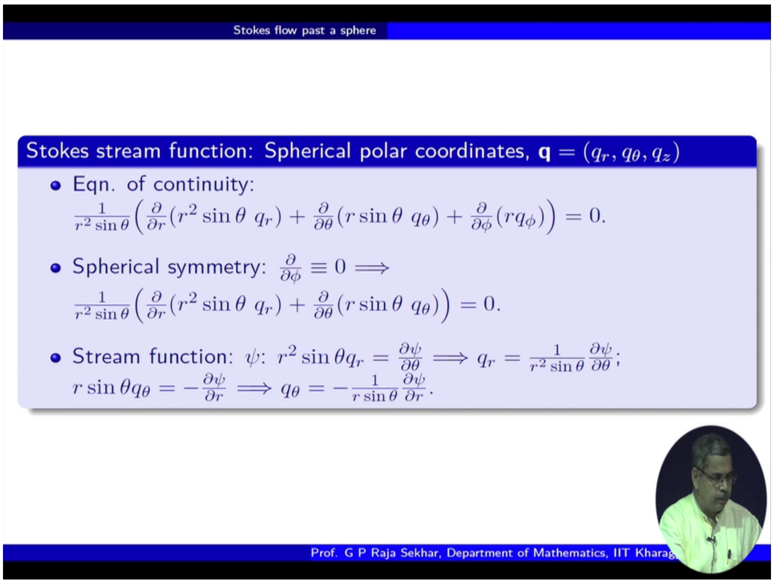

Again we are considering axisymmetric case so that we can introduce stream function. So what we do we assume axisymmetry and via equation of continuity we introduce the stream function. So this already we have seen. So the velocities are defined like this in terms of the stream function so make a note in terms of 3-D stream function it is typically known as the stroke stream function okay. So we have stock stream function now we are going to use this to solve stokes flow past a sphere.

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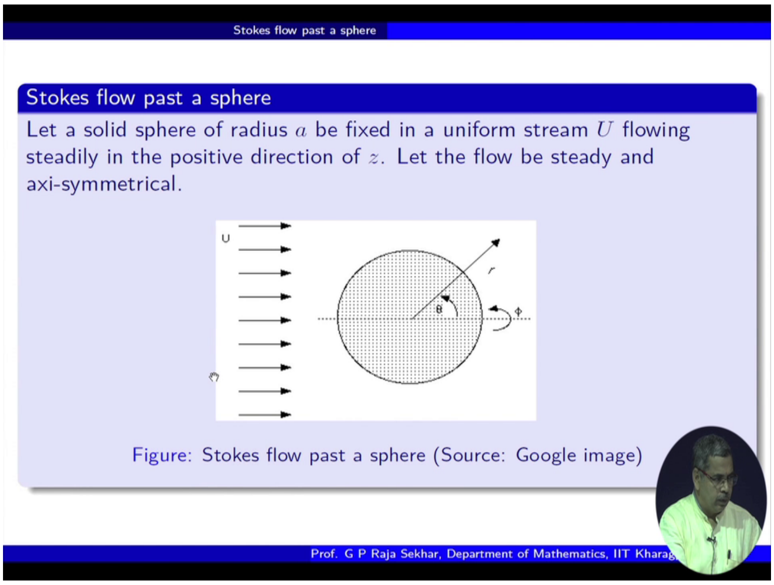

So let us consider the problem is similar to cylinder problem. So we are considering a far-field velocity uniform velocity along z direction and flow is assumed to be axisymmetric and we assume no-slip on the boundary of the sphere. So this is a rigid impermeable circle okay.

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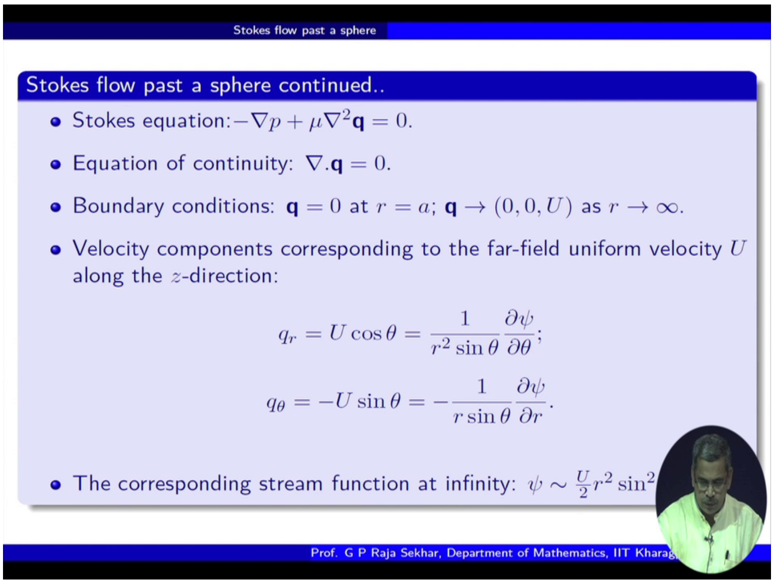

So once we have we have to get the corresponding equation for stream function that also we have discussed. So you introduce stream function and then we have a bi-Laplacian in terms of stream function right? But in this case it will be slightly variant so we have to drive okay. So the corresponding no-slip condition is again q = 0 and this is a far field we have u along z okay. And this is the definition of the stream function.

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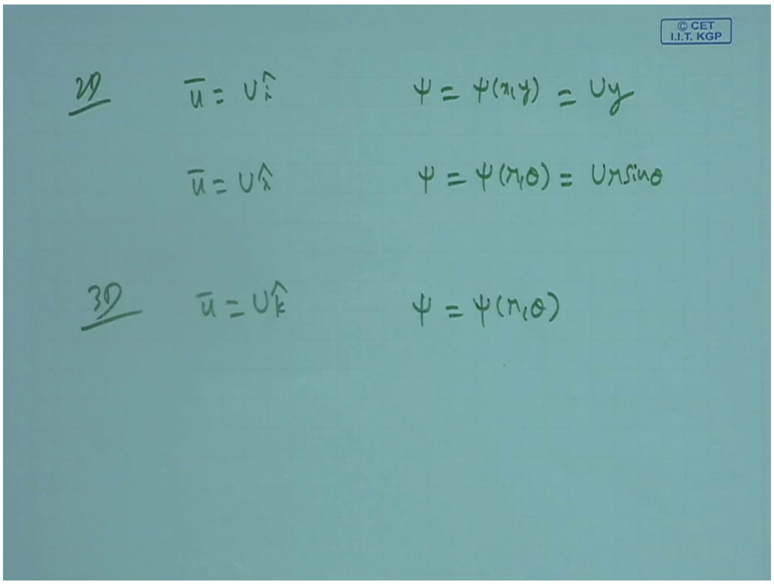

Now I have shown you two examples how to derive stream function corresponding to a given velocity. So what are the examples that we have? In 2-D we have discussed u bar is ui. Then corresponding to this what is Psi, Psi(x, y). So we have seen already this is Uy. And similarly corresponding to this Psi is Psi(r, Theta). So this also we have discussed okay. So therefore what we need is a similarly using the using stock stream function if u is Uk, then what will be Psi(r, Theta).

So I think since we have already discussed this should be a good exercise for you okay. So you can do it. So once you do you will get the corresponding stream function okay. So this is the far field for us now in terms of stream function. Why we have to reduce because we are solving the problem in terms of stream function.

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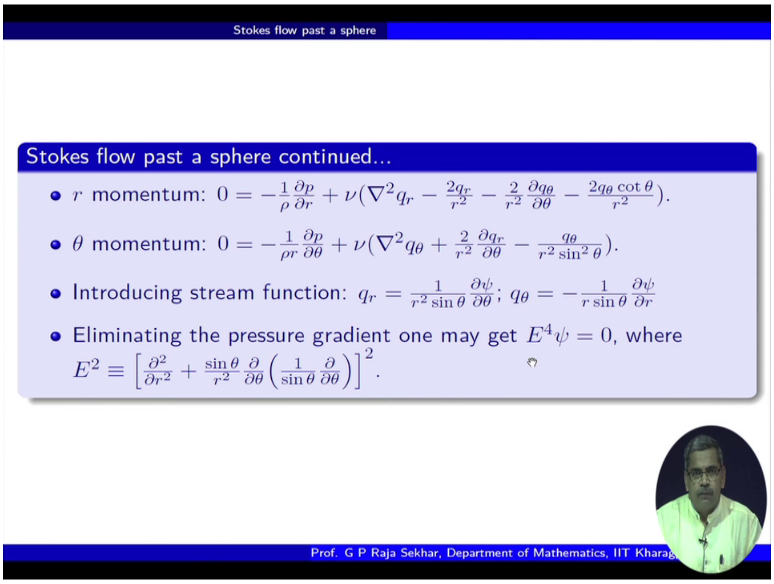

Now consider the r momentum equation and Theta momentum equations because we are talking about the axisymmetry. Now introduce stream function. The procedure is same. What did we do to get the corresponding governing equation? We have considered the component form, then we have eliminated pressure and we have substituted the definition of the stream function in velocity. So more or less similar process we are following.

So in order to do that we consider r momentum and Theta momentum and we have the stream function. So we differentiate this with respect to Theta and this before we differentiate with r first you multiply with r throughout so that this factor r is gone. Then differentiate with r and subtract okay. So if we do this we get okay. Now if you do typically using component form, the algebra is very complicated okay.

Because you have the Laplacian involved on the right hand side so that will give some challenges right. So maybe let us understand the problem first assuming that we get this equation so then towards the end I give some hints on alternative approach how to derive this equation and maybe more details you can do it as exercise okay.

So for now what we are assuming from these momentum equations by eliminating pressure and using this definition we get E power 4 Psi is 0. Remember in polar coordinates or Cartesian we got Del power 4 Psi is 0. So here we get E power 4 Psi 0 where we have the corresponding definition of E power 2 okay.

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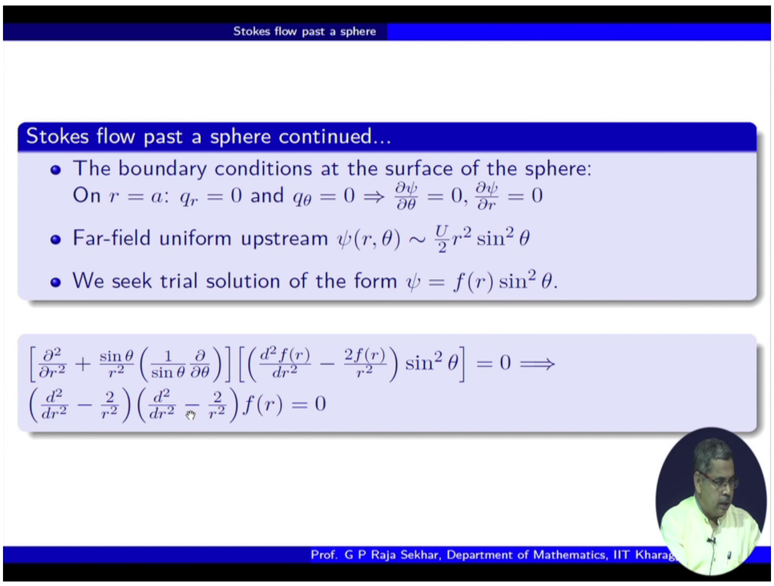

So once we have this operator we try to convert the problem in terms of stream function okay so this is a no-slip then far field we have. Look at the far field structure okay.

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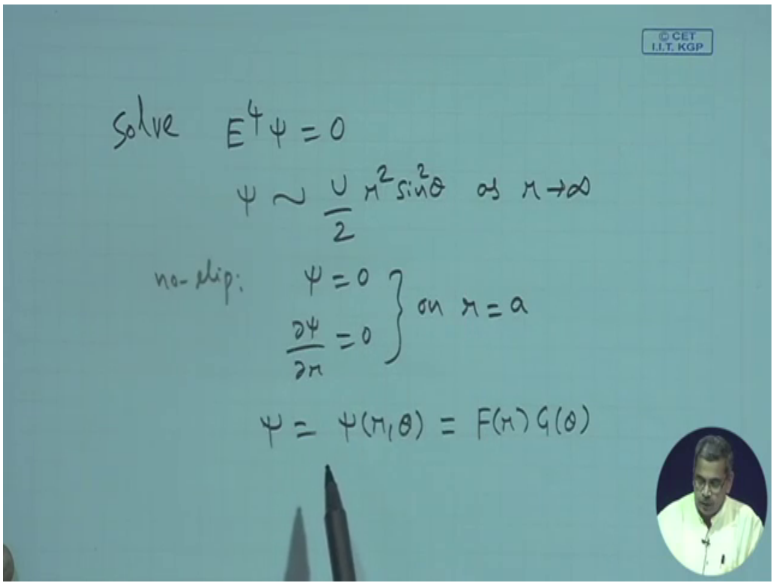

So the problem at hand is now solve E power 4 Psi = 0 such that Psi goes like as r goes to infinity. Then you have no-slip that is nothing but Psi = 0, Dow Psi is 0 on r = a. So this is the problem it is reduced to this. Now this is a partial differential operator. So now Psi is function of r and Theta and anybody who has done first course on PDE so they will be attempting for something like a function of r function of Theta. So this is a separation of variable solution okay

But we need not do for a generic case here. So the reason is look at the far field behaviour. The functional form of the far field behaviour is r power 2 times Sin power 2 Theta okay. So in particular the functional form of Theta is Sin power 2 Theta. Which means any other functional form towards G do not contribute because we have homogenous boundary conditions.

So once you force you will get a simple algebraic system corresponding coefficient of Sin power 2 Theta will produce a non-trivial solution and the other than Sin power 2Theta any functional form you have we get a trivial solution. So therefore what we do is we seek trial solution of the form function of r times Sin power 2 Theta. Again I am repeating this is due to the structure of the stream function okay.

So therefore we have E power 4Psi 0 okay. E power 2 is this and E power 2 is this so E power 4 Psi 0 implies simplified form okay. Because this is valid for only this functional form. So once you operate okay you get this. So now it is a differential operator and you will see one can get the solution very easily.

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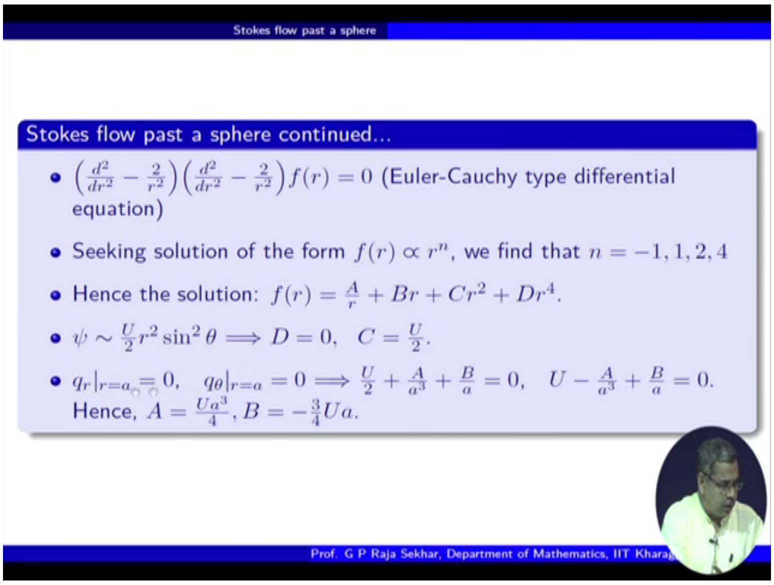

It is a Euler-Cauchy type so therefore one can get the solution very easily. So once it is Euler Euler-Cauchy type what we do is typically we look for a polynomial solution as powers of r okay. So that is what we do. We seek solution of the form f proportional to r power n okay. So let us say you take f equal to some constant times r power n and substitute so then we find these are the powers which lead to non trivial solution okay.

Corresponding to n -1, 1, 2 and 4. So therefore our solution structure is A/r, Br, Cr power 2, Dr power 4 okay. So this is the structure of the solution and a correspondingly now we have to take the far field condition and no-slip conditions and then determine the arbitrary coefficients.

So here once we have f(r) is A/r Br Cr power 2 Dr power 4 we have the corresponding Psi as because our Psi is nothing but f(r) into Sin2Theta okay.

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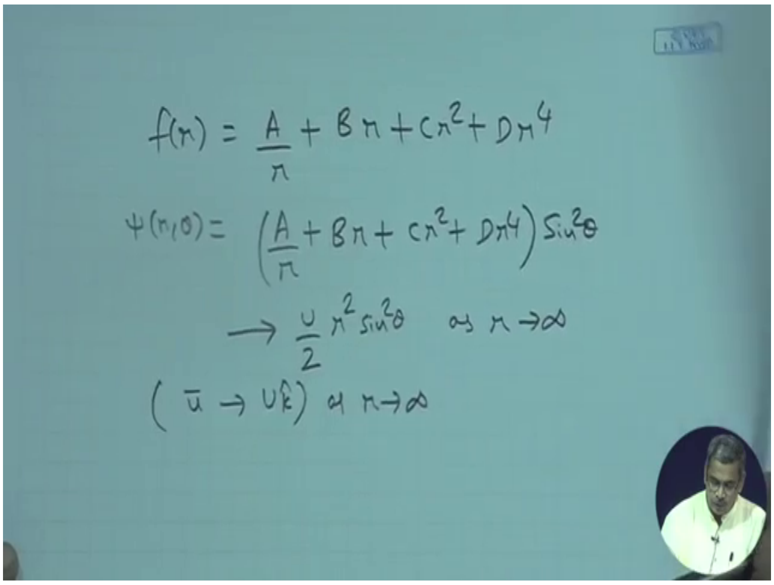

So now our condition is this should behave like we should behave u/2 r power 2 Sin power 2 Theta as R goes to infinity okay. r in terms of velocity ū should behave like Uk okay as r goes to infinity. So either of them can be u booster so let us see the Psi notation.

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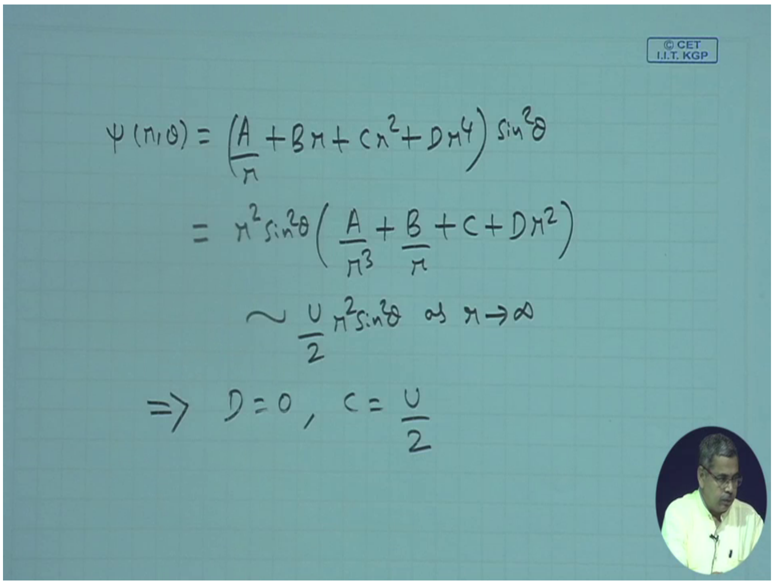

So we have Psi given by so this is Psi we have and we need r power 2 Sin power 2 Theta okay. So this will be A over so we are taking r power 2 common okay plus plus C plus D r power 2. Now we need this should go like u/2 r power 2 Sin power 2 Theta as r goes to infinity. So this will imply so these going to r goes to infinity this goes to 0 and we get.

So this is a more regular than r power 2 so which we do not want at infinity because at infinity the maximum regularity we have is r power 2. So therefore D is 0 and C must be U/2 okay. So that is what we got here. Now once we have this structure remaining coefficients are we have A and B. Now we have no-slip qr is 0 q Theta 0.

Either we can use in terms of the corresponding stream function or in terms of velocity we can use and then we get the corresponding algebraic equations which determine A and B like this okay. So I am not giving you the, this simple algebra so you can try so that really you learn okay. So once we have these coefficients we have the stream function given by this. So you can see the far field structure is visible.

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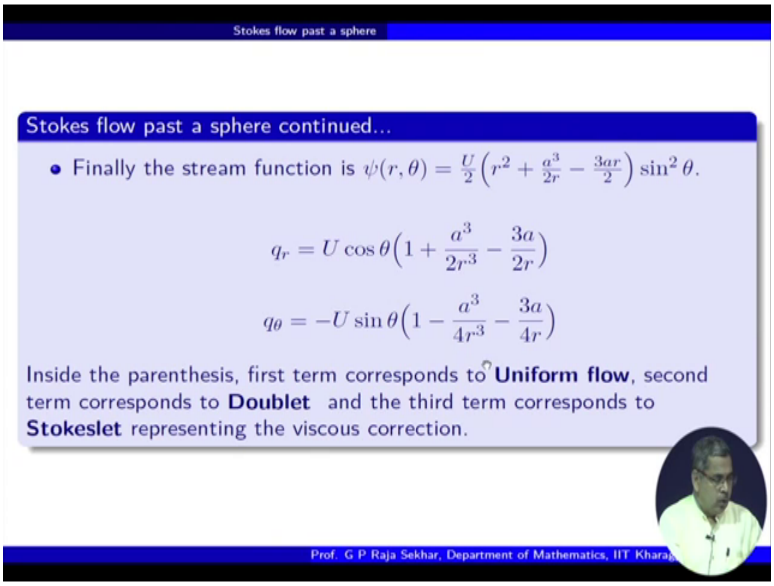

Far field is U/2 r power 2 Sin power 2 Theta. So that is why we have taken the corresponding functional form as a function of r Sin power 2 Theta and we got some two additional things.

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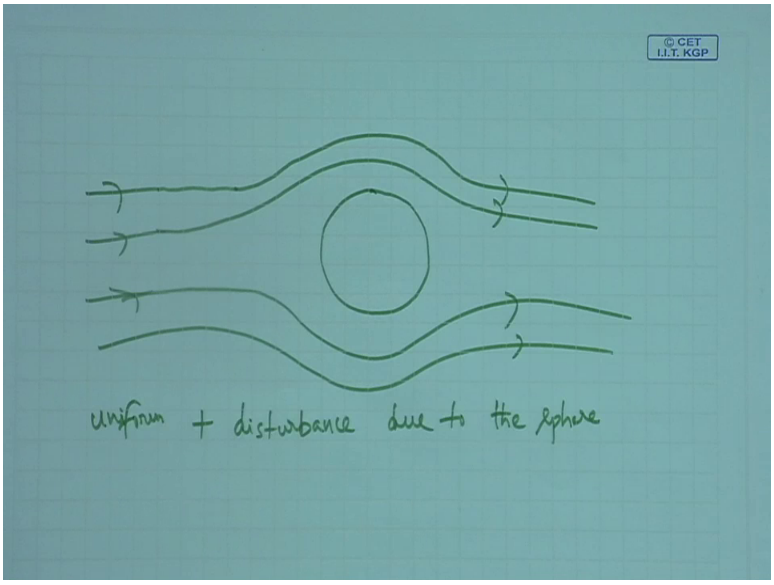

So what are these additional terms okay? So you have a sphere and you have uniform velocity. What will happen? A streamline coming close to the particle realizes that there is an obstruction so it gets deviated okay. So we have a symmetry so there we have such symmetric steam lines. Now what we have computed is far field uniform plus disturbance due to the sphere. So that is what we have computed and what are they?

So this is the contribution 1 and contribution 2 due to the disturbance. The corresponding velocities can be nicely represented. So this is in vector notation if you see in component form or vector notation this is simply Uk or in component form it is U along k direction and these are the disturbances okay. Now these disturbances can be represented okay. So this is uniform flow as I explained then second term corresponds to a Doublet and third term corresponds to a Stokeslet right.

So you will be confused what are these Stokeslet and Doublet. So these are called singular solutions of stokes equation and we discussed more details in coming lectures. But for now I will just mention briefly and you assume that and in details so we will learn little later. So what do you mean by singular solution? You take homogeneous stokes equation. Whatever solution so one can get their regular solutions.

But instead of homogeneous you take right hand side a point force that is like a source or a sink. So then if you solve the corresponding equation whatever we get there is a singular solutions. That means they are due to source or a sink similarity. So if you take at some point r = some A okay, singularity and solve the stokes equations the corresponding solution is

Stokeslet and higher-order singularities also can be classified. So we discussed in coming lecture.

So this is a Stokeslet and this is a Doublet okay. So for the time being this much information is enough. So we have obtained the corresponding velocity components. Now what is the next job is? You have a sphere, so flow is coming from far field and it gets disturbed. So in many of the physical problems very important physical quantity of interest in particular flow past a particle is the corresponding drag okay. So that will give lot of physics.

So the next task for us is to compute the drag force acting on the sphere okay.

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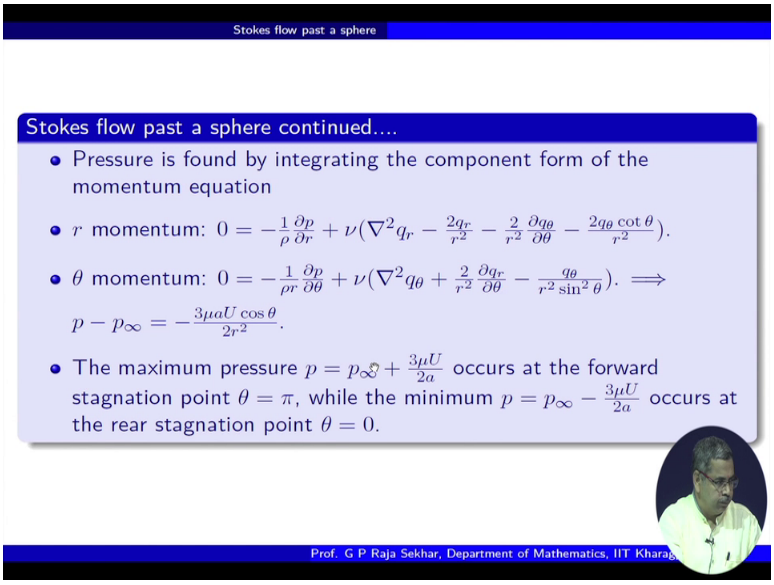

So in order to do that so we need the pressure okay. So pressure is found by integrating the momentum equations. Again this Laplacian over these two components they bring in a lot of algebra. But one can handle it is not very difficult. So how do we integrate the pressure? We know now the velocities so we substitute and then integrate this. Then you will get a function of Theta and then use this and get another function of Theta.

One can determine the pressure up to a constant so that constant is p infinity okay. So p is given by this comes this side p infinity minus this okay. So why we are interested in pressure? Because in order to compute the normal stress we require pressure and then to compute the drag forces we require the normal stress okay. So therefore what I give a caution is the velocity components that we have obtained they should be used here and integrated for pressure okay. So this again could be a nice exercise so please try.

Now one can compute the maximum pressure okay which occurs at the forward stagnation point Theta = Pi while the minimum occurs at the rear. So what do you mean by the stagnation points?

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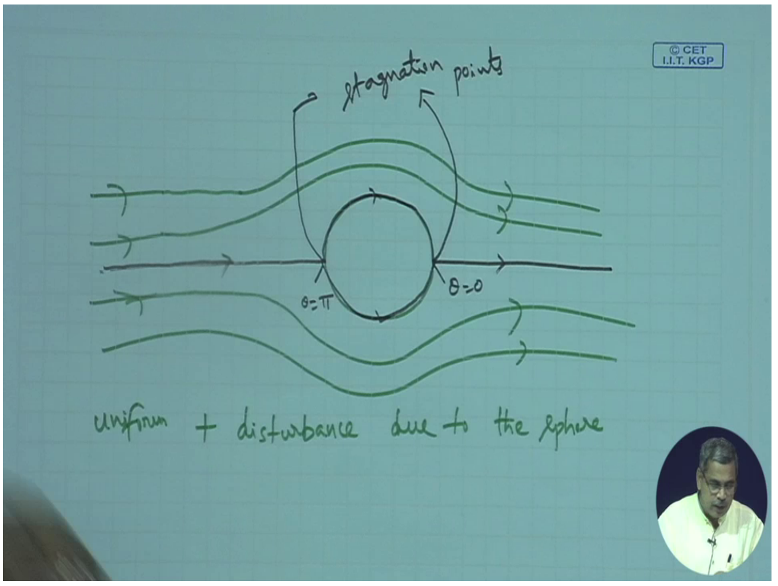

I mentioned this is a streamline. It will realize the deflection presence of the particle and gets deflected. But if you take this streamline so which will hit the particle due to the symmetry and this is called the dividing stream line okay. So it divides so that means the surface itself is a streamline but however if one can measure the velocities they will be 0 here. So these points are called stagnation points okay.

So this point and this point, so stagnation point means no flow is taking place, velocity is 0 right. So correspondingly at Theta = π the maximum occurs and the minimum occurs at the rear stagnation point Theta = 0. So that means we are talking about this and this Theta = 0 and Theta = Pi okay. So this is a very useful insight.

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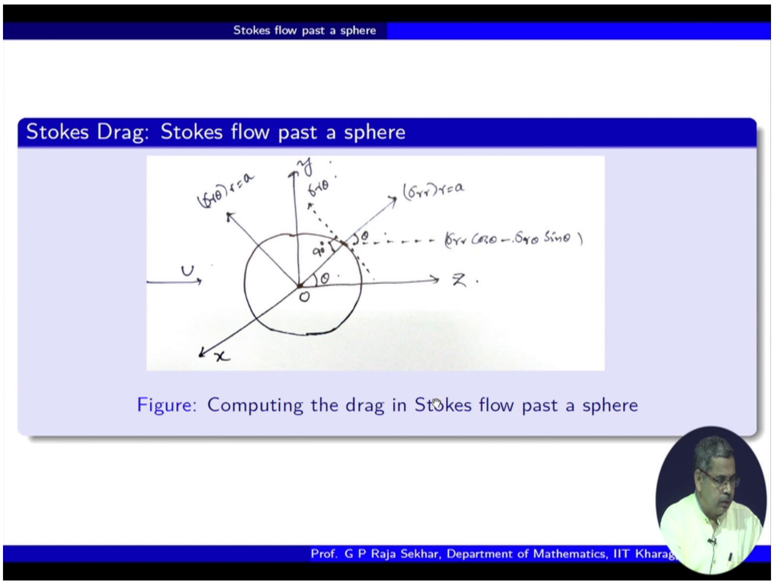

So now we proceed to compute the drag okay. So again let us recall the geometry. So we have a uniform flow along z axis then we have computed the difference. Then correspondingly pressure is obtained and then we got the corresponding stress components. Once you have pressure and velocity one can compute the stress components okay. So we would like to use these stress components and compute the drag force.

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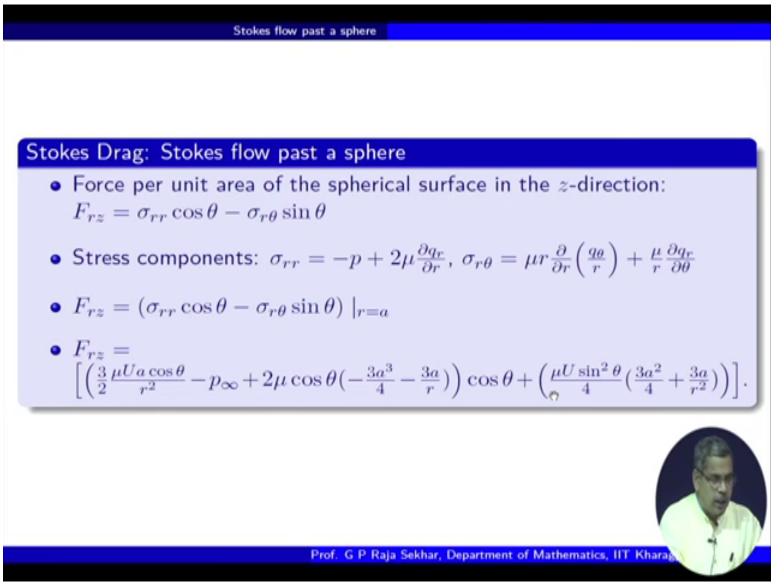

Now what is the definition? So definition is force per unit area of the spherical surface in the z-direction. That is a force per unit area of the spherical surface in the z-direction that will be the drag in the z-direction. So why because we have u equals Uk. So therefore it exerts force. This is constant so due to this the uniform flow exerts drag force okay in the z-direction. That means if the flow is in this direction so corresponding opposite direction we get the reaction okay. So that we are going to compute.

So it is a given by the normal stress multiplied by Cos Theta minus the corresponding shear stress multiplied by Sin Theta. So this is again a exercise the stress tensor can be resolved in terms of from three directions r Theta, rTheta Phi. So since here we are interested in z-direction, so we are indicating this along z-direction Cartesian notation. So we are getting this. So now normal stress is given by this okay and then the shear stress is given by this.

So this just indication to know that this is along z-direction. So normal stress and shear stress we have all the ingredients available so we can compute Frz at r = a okay. So normal stress and then shear stress combination by virtue of the z-direction we are getting this Cos Theta and Sin Theta and we can compute. So Frz in detailed so again this is not especially looking very complicated but it is very straight forward.

We have all the ingredients and simply substitute and use some partial differentiation to get this okay. Now how do we get the total force by integrating.

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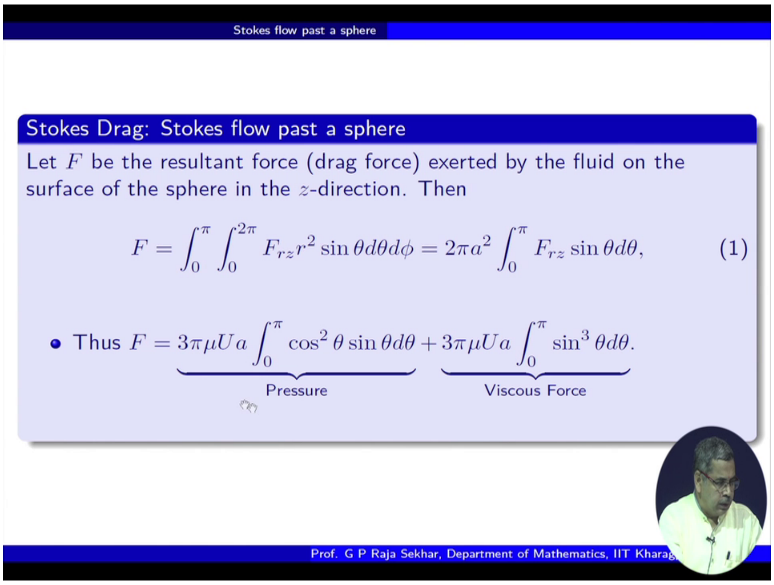

So this Frz that we have computed that is the force along z-direction, then we are integrating with the corresponding surface element okay. So that is what we are doing. So surface element for the sphere is r power 2 SinTheta dTheta d Phi. And since we are on the surface of the sphere, so r is a and these are due to access symmetry this is independent of Phi. So I have pulled r power 2 as a power 2 and Phi is coming to be variations of Phi is 2Pi. So we have this and this can be integrated.

But before we do that so we can split this by substituting Frz. One can split this and there is a reason for splitting this okay. What did we do? See what we have we have this Frz and some part this is coming due to the pressure and this is due to the viscous force. So correspondingly what we have done we have decomposed part due to pressure and the part due to the viscous force okay. Then you will realize total stokes drag. This is called stokes drag is 6Pi MuUa.

But portion is coming due to the pressure and portion is coming due to the viscous force.

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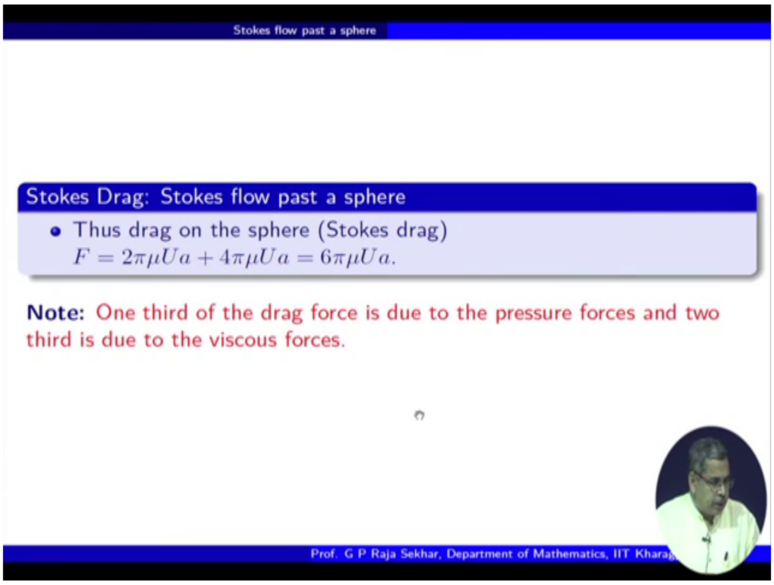

So to be more precise we have the stocks drag like this. So 1/3rd is due to the pressure forces and the 2/3rd is due to the viscous force. So this is a very useful relation. So for interpreting various physical quantities this will be helping us a lot okay. So this is a drag force.

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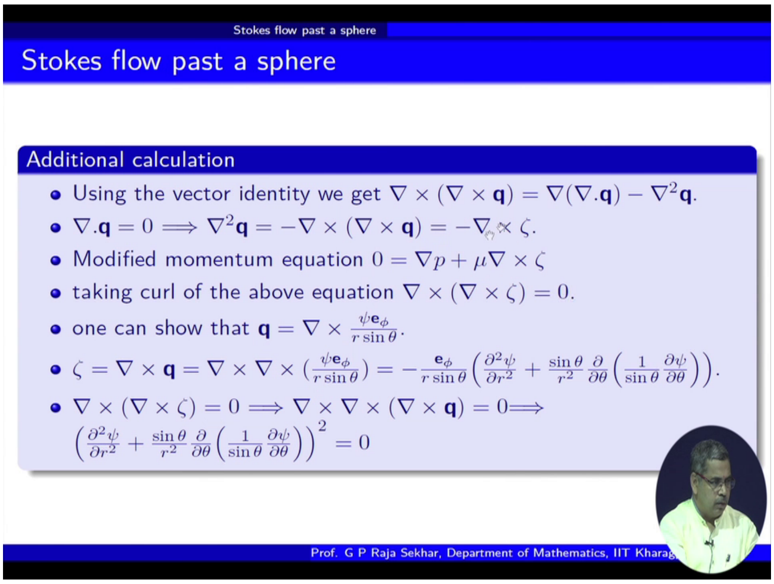

Now how do we compute the drag force okay? We have used I promised you getting the equation in terms of stream function, yeah. So if you look at it so one approach is consider the momentum in component form then eliminate the pressure then substitute the velocity in terms of stream function okay. But this looks slightly involved so let us see in brief okay. I may not be able to give you full details but in brief let us see an alternative approach okay using completely vector notation.

The idea is reducing stokes equations in terms of vector notation okay. So this is the vector identity Curl Curl vector = Grad divergence –Del power 2. But for stokes equation we have divergence q is 0. Therefore, from these two since this is 0 Del power 2 q is negative of this and this is the vorticity vector. This is a vorticity vector. So stokes equation reduces to now Del power 2q is now this. Therefore stokes equation reduces to this okay. It is nothing but minus velocity plus Mu Del power 2 right.

Now taking curl of the above equation we get curl curl vorticity is 0. That means this is the modified stokes equation in terms of vorticity. One can represent velocity in terms of stream function using this. This is nothing but ur whatever definition ur u Theta we have, exactly we get this okay. We will see if we expand what you get. But for the time being you assume that u can be written like this and therefore vorticity simple curl curl if you take we get.

Already you are seeing whatever e 2 we have that the E power 2 for the stokes operator we are getting that. But what is the stokes equation? It is two times curl. So therefore we take another curl and equate to 0 exactly we are getting this is e power 2 and square of that so E power 4 Psi is 0 okay. So this is a very simple approach via vector equation okay. And this is anybody can see you are exactly getting the stream function okay.

So these are some simple things I thought in compact form we will show you but you can verify okay. So similarly curl q also one can calculate and you are almost getting E power 2 okay. So this will give you stokes flow past a sphere. So what we have done is we have integrated stokes equations and obtained one scalar equation in terms of stream function then as an application stokes flow past a sphere has been discussed and for physical insights we have computed the stokes drag that is force acting on the surface of the sphere. Thank you!