Introduction to Abstract and Linear Algebra
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Lecture - 32
Rank of a Matrix (Contd.)

So, we are talking about elementary row operation, and will see how that will be useful for finding the Rank of a matrix. So, there are basically 3 types of elementary operation; either on row or column.

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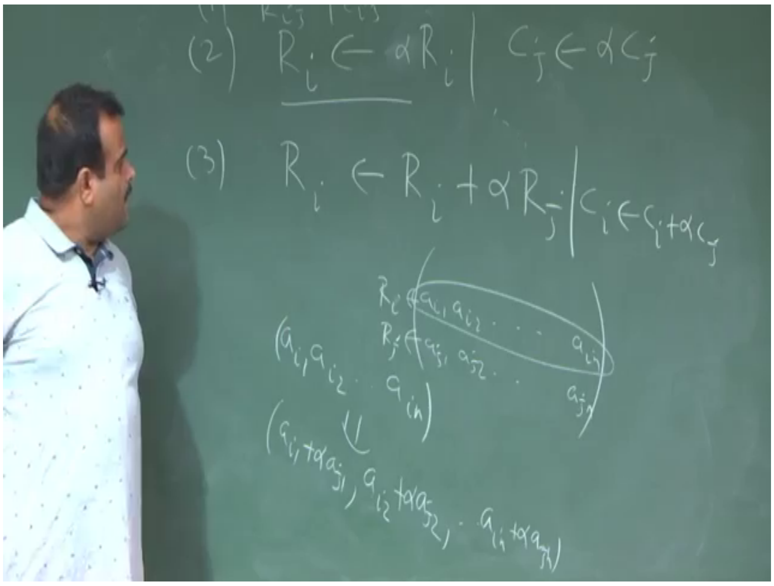

So, first one is to exchange 2 rows or 2 columns. Like, i th row is going to exchange with the j th row, and the i th column is going to exchange with the j th column. And the second operation is we have a row i th row and that we are multiplying with some scalar some real number l and m are field element. And that will be replaced by that. And similarly further column.

And we have another elementary operation is likewe are replacing the i th row by it h row plus some constant, some scalar into the another row. So, that is the another elementary operation, and that will be similar for the column operation. So, that there are 3 types of elementary operation we are considering, and this will be useful for finding the rank of a matrix. So, just will see how it will help us to find the rank of a matrix.

So, let us consider an example where we will apply this row operations and column operations to get the rank of a matrix ok. So, actually we will take some matrix.

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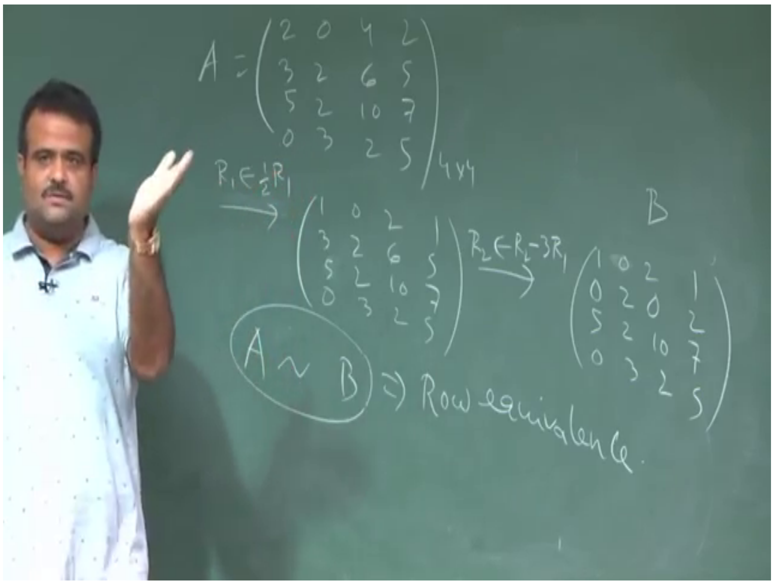

So, suppose you have a 4 by 4 matrix 2, 0, 4, 2, 3, 2, 6, 5, 5, 2, 10, 7, 0, 3, 2, 5 ; suppose we have given this matrix and we have asked to find the rank of this matrix, ok. So, one to find the rank of the matrix to check the all possible minors.

We can check the determinant of this that is the minor of order 4, and if it is non-0 then rank is 4 and it is non-singular matrix, now suppose it is 0. In fact, it will be 0 I think. So, it is 0 then, we have to go for checking the whether rank is 3 or not. So, for that we need to check all possible 3 cross 3 sub matrix and take their minors take that determinant value that is the minor of order 3. And then so, this is the way, but we do not want to go into that way, because that will take some time to get the all possible minors.

So, on the other hand we will just do the row operation and column operation person to reduce this matrix to a row reduced echelon form what you say. So, we will just apply row operation and will try to get some identity kind of matrix over here, some 0 and 1, that is the normal form of the, that is called row reduced echelon form. So, will say will come to that. So, now if we apply this, will try to apply some row operation on this matrix, and we try to get the matrix which is contain 0 and 1.

So, we will just do R 1 is replaced by half into R 1. This is the one row operation. So, then this will be reduced to this matrix. So, we just divided by half. So, this is 1, this is 0, this is 2, this is 1, this is 3, 2, 6, 5, 5, 2, 10, 7, 0, 3, 2, 5 ok. Now if we subtract this with if you multiply 3 with this and subtract with this then this may get 0. So, we want to find we want to get all 0's other than this. So, we want to basically the idea is to get the kind of identity matrix.

So, that is the standard normal form, row reduced echelon form. So, if you do that so, if you do this R 2 is going to this is another elementary operation, R 2 minus 3 R 1. So, we are going to replace R 2 by we are multiplying 3 with this and then subtract with this. So, then it will be so, only R 2 is changing. So, row 1 will be remain same, row 2 we are multi. So, this will be 0, then and then we have this is 2, this will be remain unchanged and this is 3. So, we are multiplying 3 with this. So, this will be again 0, and you are multiplying so, this will be 2 and remaining things are will be unchanged, 2, 10, 7, 0, 3, 2, 5 ok.

So now, this matrix we got by applying the row operation elementary row operation on this. So, that they are called row equivalence. So, 2 matrix A B will be the row equivalence, if we get A to B by applying the elementary row operations. So, this is our A matrix, this is our B matrix, now we are getting B by applying elementary row operations. So, then we say they are row equivalence, equivalence matrix or row reduced matrix.

So, the idea is we are going to replace this matrix, we are going to apply keep on apply the row operation on this matrix. And we will try to get the similar to identity matrix on this ok. We want to have all these 0 and 1 and we want to try to feed the one in the diagonal element and the remaining element all that will be 0. If it is; so, that we are going to do ok. So, similarly we can just apply the again we apply the row operation on this. So now, this is this we are going to change so, we multiply 5 with this.

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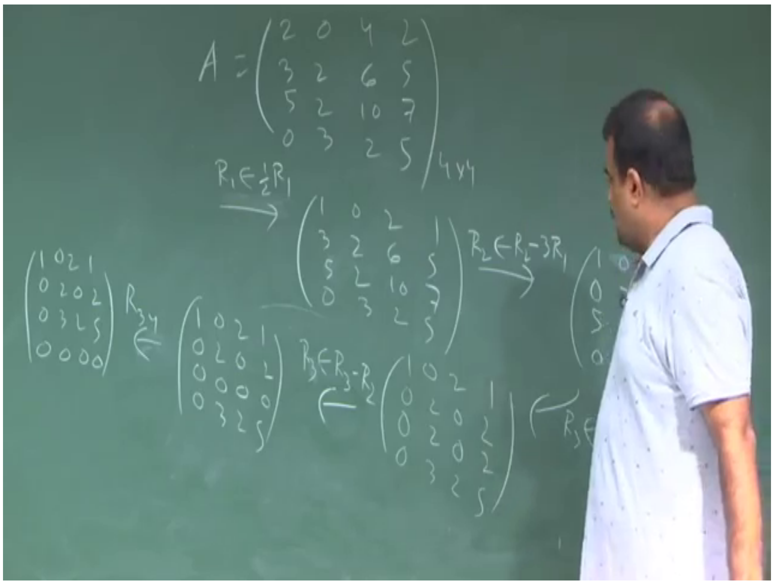

So, this is R 3 so, if you do the R 3 is going to replaced by R 3 minus 5 into R 1, this another elementary row operations ok. So, these if we do then it is becoming so, only R 3 is changing. So, we multi we take the 5 then this becomes 0, and then we take this is 2, and we multiplied by 5 with this. So, this is again 0 and 5 with this is again 2 good. So now, this will be remain unchanged. So, this is again we apply the another row operations on this.

So, this will be so, these 2 if you subtract these 2 row then 1 will be 0. So, what we do? R 2 is going to replace or R 3 any one of this, R 3 is going to replace by R 3 minus R 2. So, if you do that, then this will be 0 because these 2 rows identical 0, 2, 0, 0 then R 3 we are replacing 2, then these are all 0's, fine. Now we want to keep this 0's row in the end. So, that for that what we do? We exchange this rows, we exchange R 3 and R 4. So, if you do that, we will get 0 0 2 0, then 0 2 0 2 then 2 3 2 5, then 0 0 0 0.

We are going to exchange the R 3 and R 4 with this. So, with is to there, and if we do the further row operation on this matrix so, these will be eventually like this matrix.

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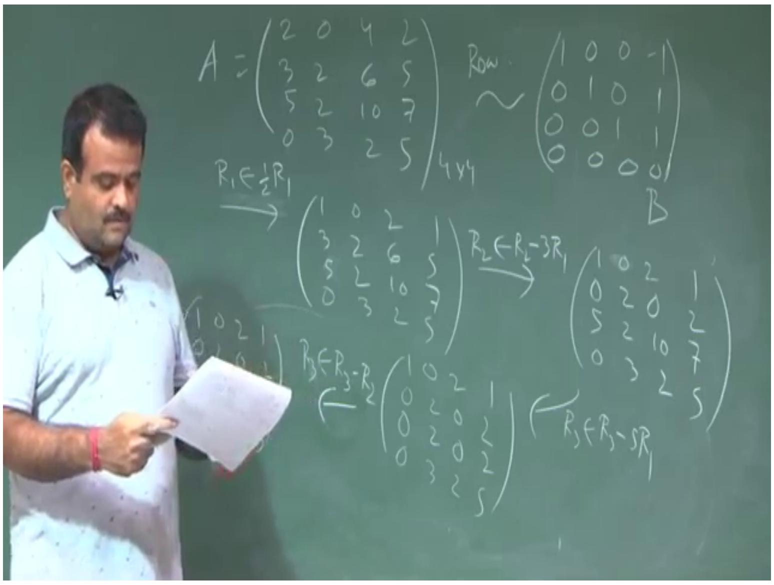

All the row operation we are doing 1 0 0 minus 1 0 1 0 1 0 0 1 1 0 0 0 0 ok. So, again after this we can replace this row by we can divide it by half. We can multiply with half with this row. So, this will be like this, then we can multiply with this. So, this will become this will vanish like this we can do.

Then eventually this will be of this form. So, this is the row equivalence, and this form is called row equivalence echelon form. And from here we know that if we have a matrix A, and if we have a row reduced matrix, then their rank is same. Why it is? Will see that. If we apply row operation, this is similar for column operation also if we apply row operation on the matrix we will get another matrix. Now this rank of these 2 matrix are same.

So, will discuss why it is so? So, if the rank of these 2 matrix is same, we can easily verify rank of this matrix is 3, because this is a minor of order 3; which is non zero. And this is minor of order 4, this only 1 minor of order 4. So, rank of this B matrix is 3. And this is the row equivalence with a matrix. So, rank of A is also 3, but the question is why they are same. Why if we have a matrix and we are getting another matrix by applying the row operation or series of row operation, then the question is why their rank is same that we have to do that.

So, for that we need to bring the concept of elementary matrix. Elementary matrix means so, this is this is the row operation we are doing on this is nothing but if I apply the same row operation on the corresponding identity matrix, and we are multiplying that corresponding that is called elementary matrix, we are multiplying that corresponding elementary matrix in the prefix of that. And that is so will come to that. So, like, let us take these 2 row operations. So, I will come back to this. So now, let us define what do you mean by elementary matrixes.

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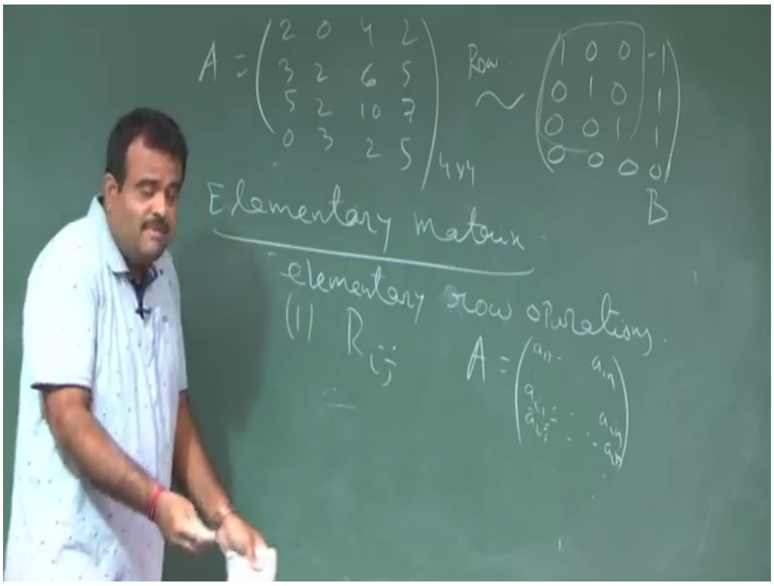

So, what is a elementary matrix?

Now, we have 3 types of elementary operations. One is we exchanging the rows or columns ok. First we talk about rows on rows, we are considering the elementary row operations ok. So, what is the elementary row operation? There are 3 types of elementary first operation is like exchanging the it h row and j th column. We have a matrix A and this is a 1 1 a 1 n. And we have a i 1 a i n dot dot dot a j 1 a j n like this. Now these operation so, these operation means we are exchanging the i th row and j th column.

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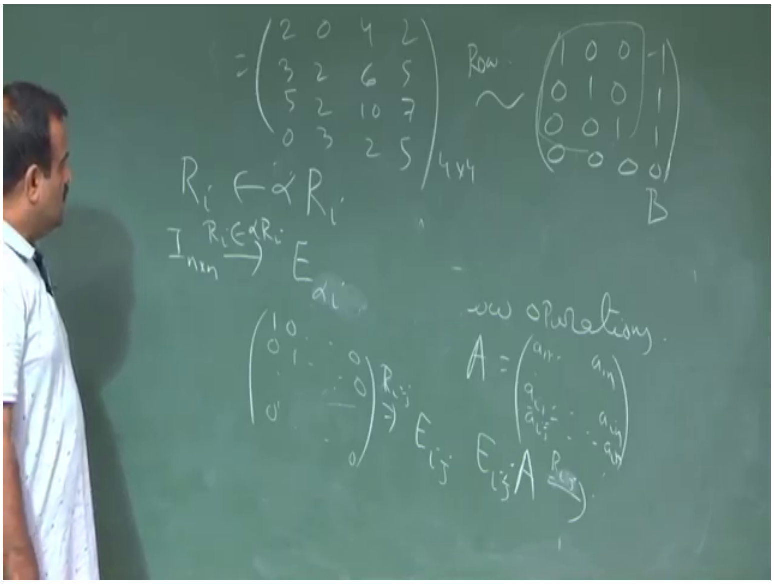

Now, this is same as if we have a identity matrix corresponding identity matrix 1 0 0 0 1 0 like this 0 0. So, for i th so, we are doing the same operation on this. So, R ij so, this is converting this is denoted by E ij. So, E ij means we take a identity matrix of the order is same as the order of the row. And then if we exchange the i th row and j th row on this that will become E ij. Now doing this operation is nothing but multiplying E ij with A. So, this will give us this is nothing but R ij.

So, this you can easily verify this. So, we just instead of doing the row operation on row operation elementary row operation, we just take the identity matrix. We apply the corresponding row operation on this identity matrix and then we multiply prefix. We multiply with a that will give us the corresponding row operation elementary row operation ok. Similarly, for other 2 row operation is also same. If we want to multiply say if we are going to replace R ij by some constant alpha R i. What we will do? We take the identity matrix E of size n by n.

Now, we apply this row operation R i is going to alpha R i on this identity matrix, and we got a this E matrix. This is the elementary matrix, this elementary matrix is denoted by i plus alpha, I sorry I alpha i we can say. We are multiplying alpha with this. So now, doing this operation on A is same as we just multiply this, we just multiply this identity element this is called elementary matrix under this corresponding elementary operation.

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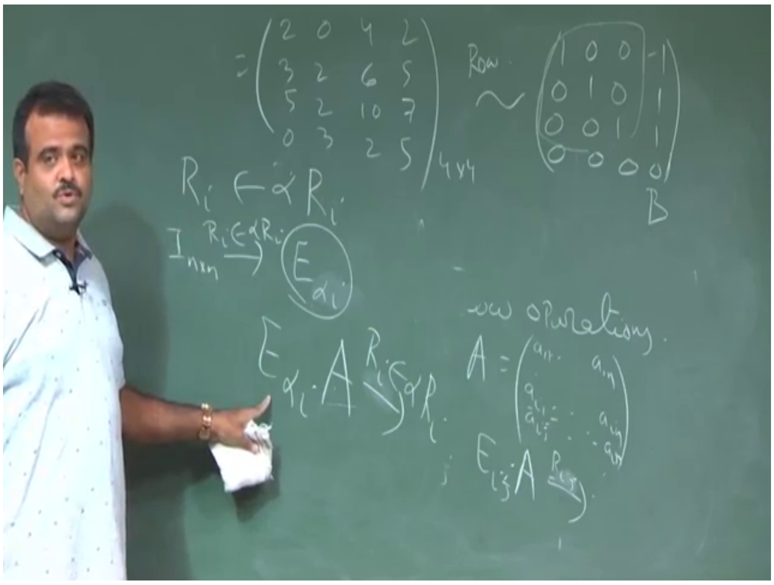

So, we just multiply this with A, and that will give us this elementary operation.

That R i is going to alpha. So, the doing elementary operation, elementary row operation is nothing but multiplying this elementary matrix with the matrix a prefix way. Now since this is elementary matrix, an elementary matrix is non-singular matrix. So, that means, that that means their rank will be same. So, if we have a matrix A and if we are doing the row operation on that, we are getting another matrix B, that is nothing but multiplying the elementary matrix with A.

So, then the rank is will be same so, that is why the rank of this is same. So, if we have matrix A and we are getting another matrix B by doing the elementary row operations so, that this is nothing but we have EA, we have some elementary operation E 1 E 2 E l say n times we do not know, this is our this is becoming B, and these are all elementary operations ok.

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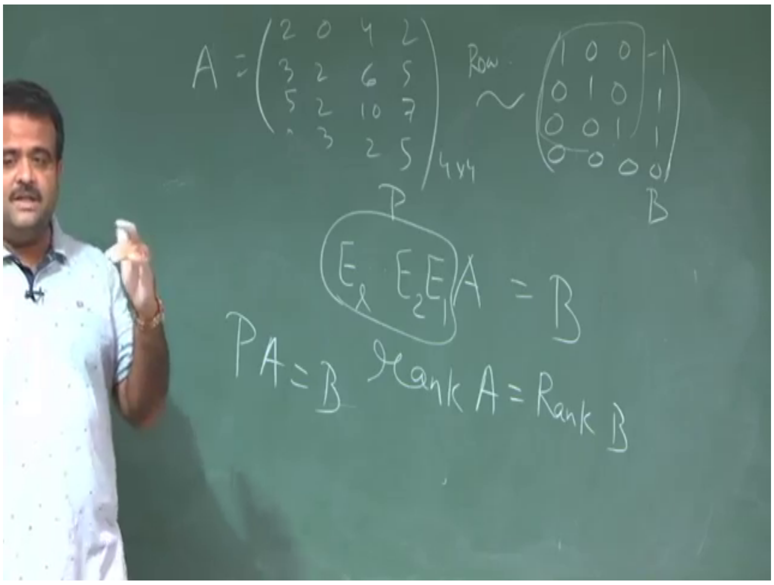

And these are all non-singular matrix. So, that is why ranks so, rank of A is same as rank of B. And this we can denote by P, P matrix. So, PA is equal to B where P is a non-singular matrix. And this is called row reduced echelon form.

So, P of P is this, now we can apply the corresponding column operation also. Then it will be giving us the column reduced echelon form. So, but the column operation is corresponding elementary matrix we need to multiply the after A. So, for column operation so, for elementary column operation so, what elementary column operations have?

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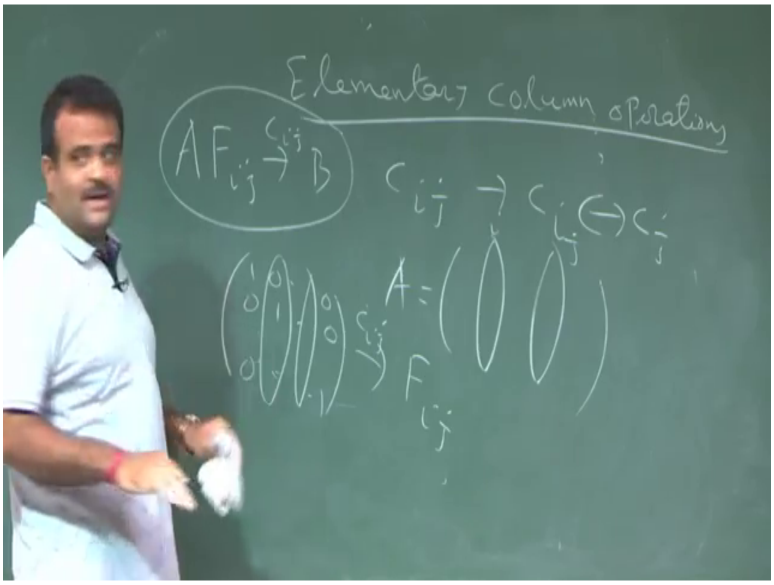

First operation is C ij that means, we are exchanging C i with C j. We have a column, this is i th column, this is j th column we are exchanging these 2.

This is our A, now this is same as if we get the corresponding elementary matrix that; that means, we take the identity matrix 0 0 0 1 0. And size of this matrix is the number of columns over here. So, if we just exchange the ith column and j th column so, if we do the C ij on this. So, we get corresponding E ij ok. Or with the column we can denote the F ij. Now if you multiply a with F ij, this will give us this operations. This is, but C ij we are getting another matrix B.

So, this is this for column operation this multiplication is from the right side, and for row operation this multiplication in the from the left side. So, if we apply only column operations on this matrix A. So, eventually will get a equivalence column equivalence echelon form.

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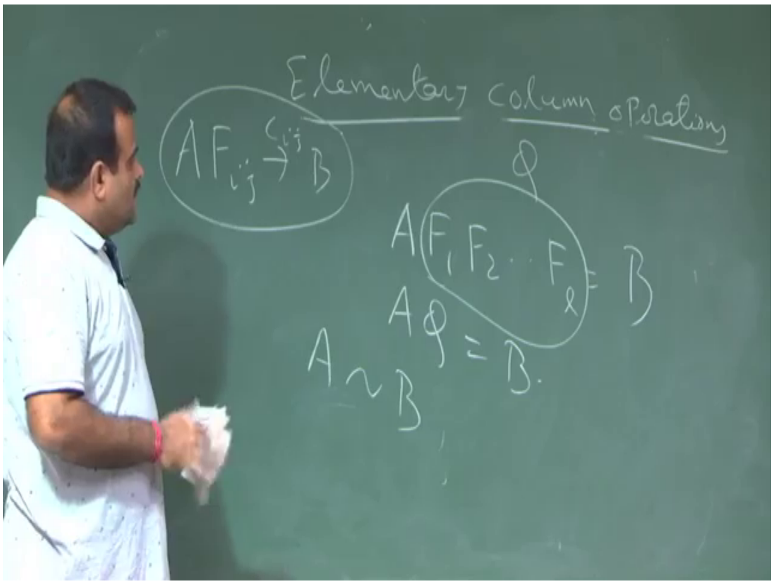

So, we have a matrix we have keep on applying F 1 F 2 F l say; so, will be getting this. So, these are basically the elementary column operation on this is so, this is if you denote this by Q. So, A Q is equal to B so; that means so, this is the column equivalence because we are applying elementary column operation on this.

So, applying elementary column operation is nothing but applying the multiplying the corresponding elementary matrix from the right side like this ok. So now, now we have we can have a normal form of this. So, if we are if we are doing both row and column operations, suppose we have a matrix A whose rank is a m by n matrix, or for simplicity we can take n by n, suppose rank of this matrix is or for simplicity we can take this.

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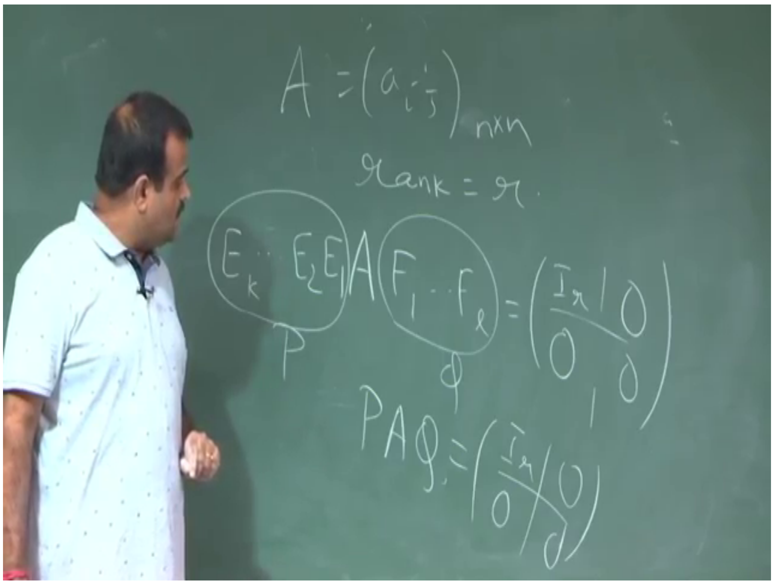

Rank of this matrix A, then if we apply the row column operations, if we apply row operation it will reduce to a row reduce echelon form.

Then again if we apply the column operation, it will reduce to a column reduce echelon form. So, this way so, applying row operation means we multiply the elementary row elementary matrix row matrix in the prefix, and we apply column was elementary column. So, basically we will have a so, this is the elementary row operation we are doing n; say, k times and this is elementary column operation we are doing say l times. So, these give us the normal form of this is the 0 0, this is the 0 matrix. So, this is our P, this P these are all non-singular matrix, because these are elementary matrix.

So, we have P A Q is equal to this form, these are all sub matrices ok. This P is nothing but the product of the elementary row operations, elementary row elementary matrices. And Q is nothing but the product of the elementary column matrices, and they are non-singular ok. So, this way we can proceed. Now suppose A is non square, A is a square matrix and suppose A is non-singular. So, that means, rank of A will be n, so that means, we have a 2 matrix P Q. So, similarly. So, these will be nothing but identity matrix fully I n, ok.

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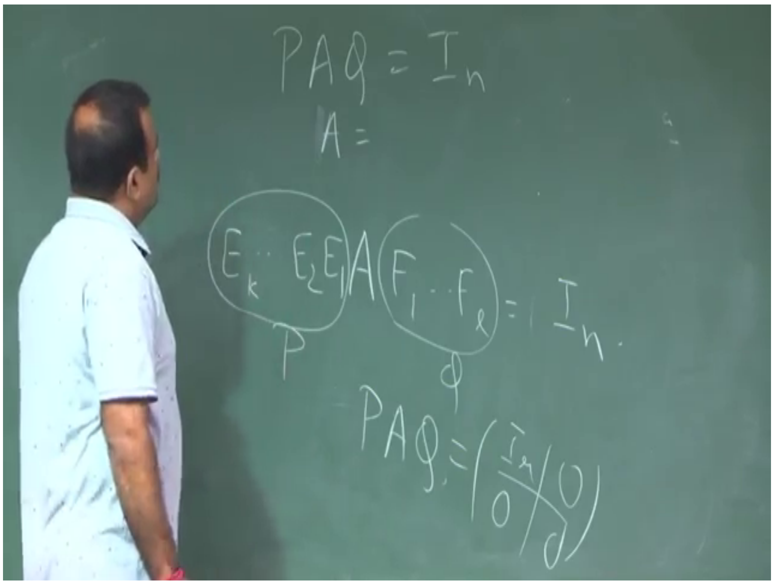

So now we got PA Q is equal to identity matrix. Because this is the row operation we will do there that column operation we will do and eventually we will get the form. So, what is then A so, we will just do the inverse of this. So, P is a non-singular matrix Q is a non-singular matrix. So, A so, A is so, so, actually I want to get the inverse of this. Inverse of inverse A inverse. So, a inverse is nothing but if you multiply the P inverse so for this side.

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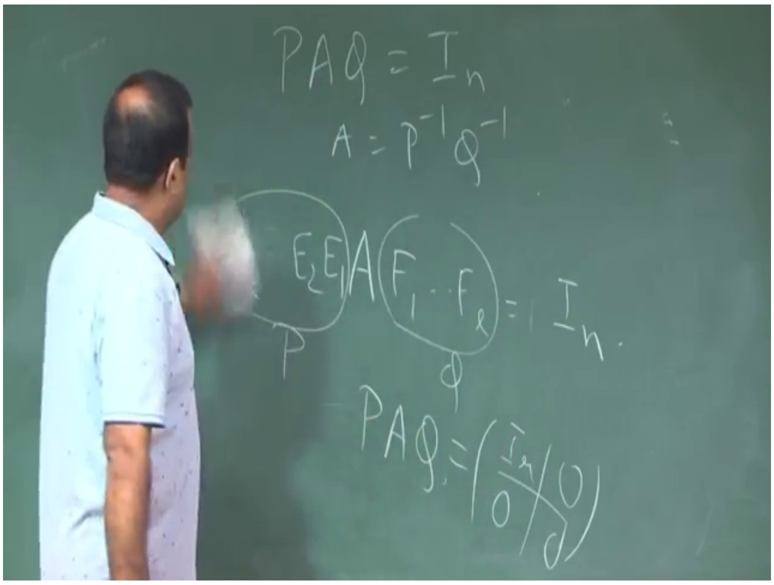

So, A is nothing but P inverse Q inverse like this.

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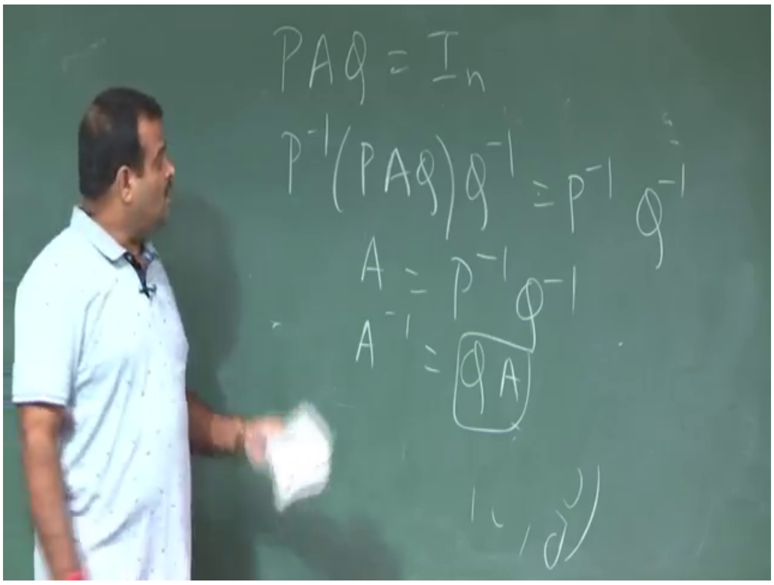

Because if we multiply both side by P inverse PA Q Q inverse this is P inverse Q inverse, because that is the identity matrix. So, this will give us A, A is equal to P inverse Q inverse. Now if you take the inverse of this so, this is nothing but inverse of this, this is Q A. So, this is the inverse of a non-singular matrix. So, that is why how to get the inverse of a matrix, this technique we used to do.

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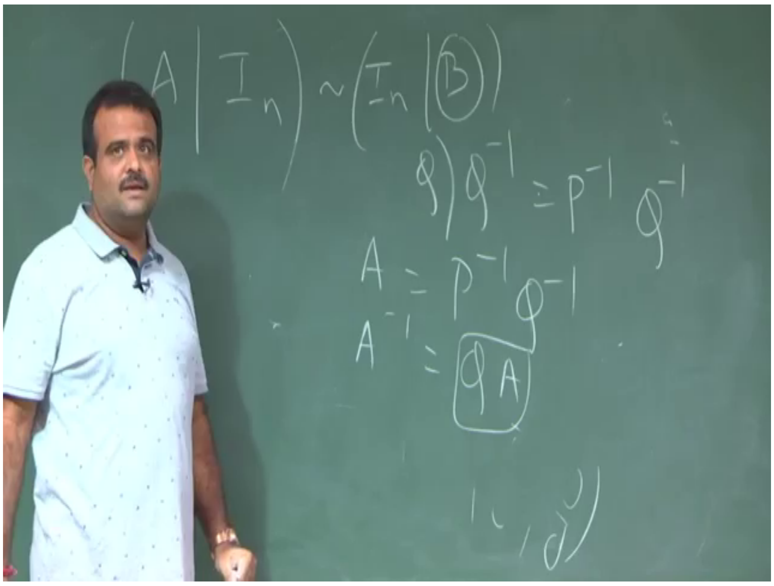

So, we take a matrix a and we take a identity matrix. Now we apply the row operation and column operation on this. So, that will give us basically this P Q's.

And once this became non-singular matrix and this is B, then this is the inverse. So, we keep on apply row operation column operation on this until this becomes a identity matrix, and this became this B B, B is the inverse of this. So, then these should be inverse. So, this is nothing but we are doing the multiplying the P Q. So, that is why this will work ok. So, this is the way we find the inverse of a matrix. So now, just to give an a quick example, we can let me check with a example of now. So, I want to take this normal form.

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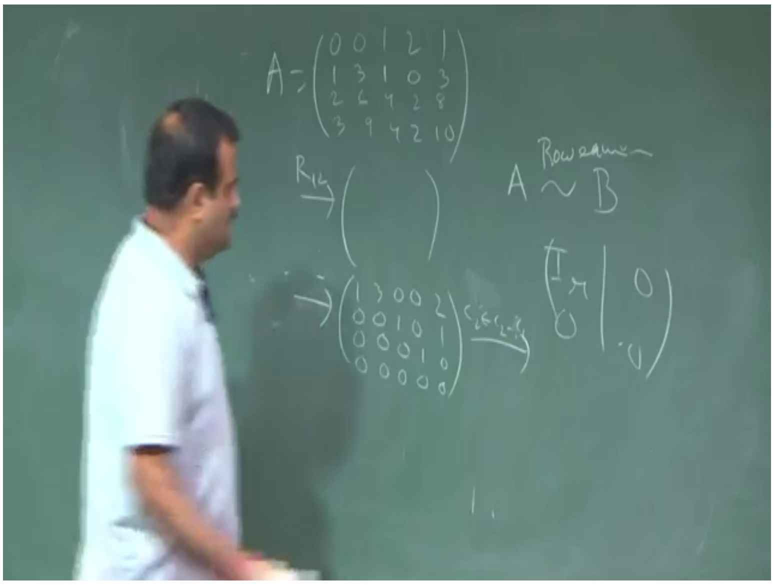

So, suppose we have a matrix 0 0 1 2 0 1 3 1 0 3 2 6 4 2 3, 4 2 8 3 9 4 2 10. So, we want to find the rank of this matrix, and in particular we want to get this normal form of this matrix. If the rank is R we want to write this in I R 0 0 0 kind of things ok. So, that is the fully reduced normal form. So, we have to do the row column operations. So, I am just I am not going to the details of this. First we will do this R 1 2 so, we are exchanging these row and these row.

So, that is same as we are multiplying the elementary row operation on this; that means, we take a identity matrix. So, we will do the same exchange on that identity matrix that will give us elementary matrix. So, doing the elementary row operations means multiplying the elementary matrix before that A prefix. So, these if we do then eventually will be getting. So, will be getting a matrix, then again we have to do this some row operations. So, will first try to do the row operation. So, if you continue this eventually, I am not going to details of this, we can just work out the example at home.

So, already doing row operation we can reach to this level. 1 3 0 0 2 0 0 1 0 1 0 0 0 1 0 0 0 0 0 0 ok. This is the row reduced echelon form. And these we are getting all the by row operation this is A B so, A is row equivalence to B, row of equivalence. So, we so, this is the row this is row reduced echelon form, but we want fully normal form. So, it is time to so, up to these we achieve by only the row operations. So, that means, up to this achieve by multiplying the elementary matrix prefix way.

Now, we have to go for column operations also. So now, we have to apply the column operation. How will do that? So, if we just take these columns. So, idea is to convert into this type of matrix, equivalence matrix ok. So now, we can take this column and this column these 2 column are similar. So, we can just make it 0 so, C 2 we can replace by C 2 minus C 1, if we do that then these are all 0’s. So, we can just write that.

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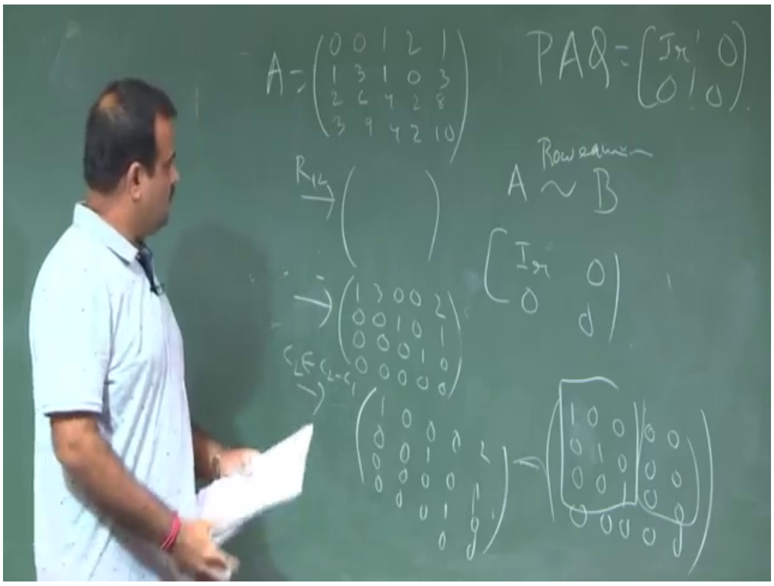

So, from here we can just do C 2 is going to C 2 minus C 1.

Then this first column will be remain same, this will be all 0's, then 0 1 0 0 0 0 1 0 2 0 0 0. Now we can exchange these 2 column. In fact, we can exchange these 2 column, then we multiply these with this. So, by doing another column operation we can reach to this I

3; I 3 means, 1 0 0 0 1 0 0 0 1. This will get 0 0 0 0 0 0 and then we have another 0 0 here.

So, this is our I 3 and this is 0, this is 0. So, this is the fully reduced normal form. And this we can achieve by applying another column operation over this, ok. So, any matrix can be reduced to their fully normal form of this. If the rank is here rank is 3, if the rank is R, it will reduce to I r 0 0 0 by applying both the row and column operations elementary. So, both the row and column operation means, P is coming from row operation, Q is coming from column operation, ok.

So, that is the; thank you.