Introduction to Abstract and Linear Algebra
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Lecture - 36
Eigen Vector

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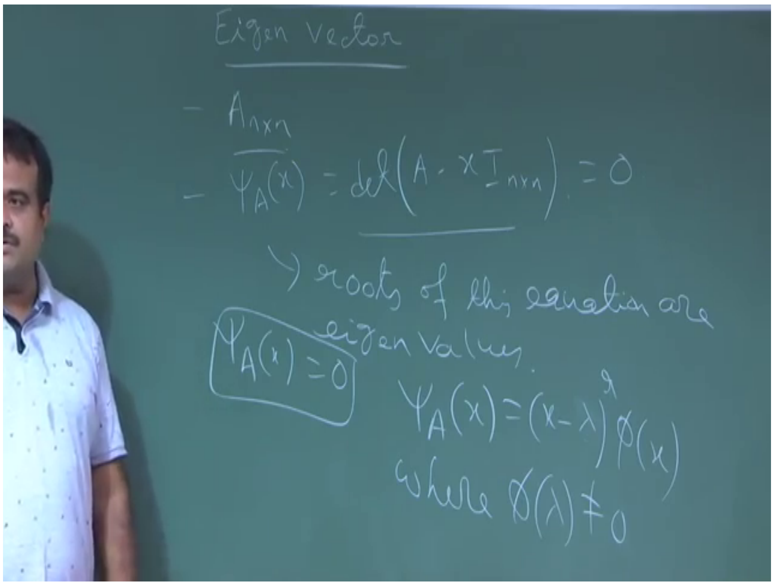

So we are talking about eigenvalue. Today we will start the Eigen Vector. So, just to recap, so we have a square matrix A is a square matrix n by n matrix. And we are taking the characteristics polynomial of this matrix which is defined as determinant of A minus this is also n by n matrix. So, this is a characteristic polynomial. And if we take the equation, if we get to 0, then the this will give as characteristics equations. And the roots of these equations are the eigenvalues.

So, this is a n degree n degree polynomials. So, it has n roots of these equation are eigenvalues. So, if lambda is a eigen value that means so this is the equation. If lambda is a root, then x minus lambda is a factor of this polynomial. Now, if the lambda is a root of r times root that means A root of multiplicity r, then it must write this as x minus lambda to the power r and some other polynomial, where lambda phi lambda is not equal to 0.

Then it could be a another multiple then lambda is a root with multiplicity r, and this is called algebraic multiplicity. So, lambda is coming r times, it has a n roots among them r roots are lambda repetition roots. So, this is called algebraic multiplicity of eigenvalue.

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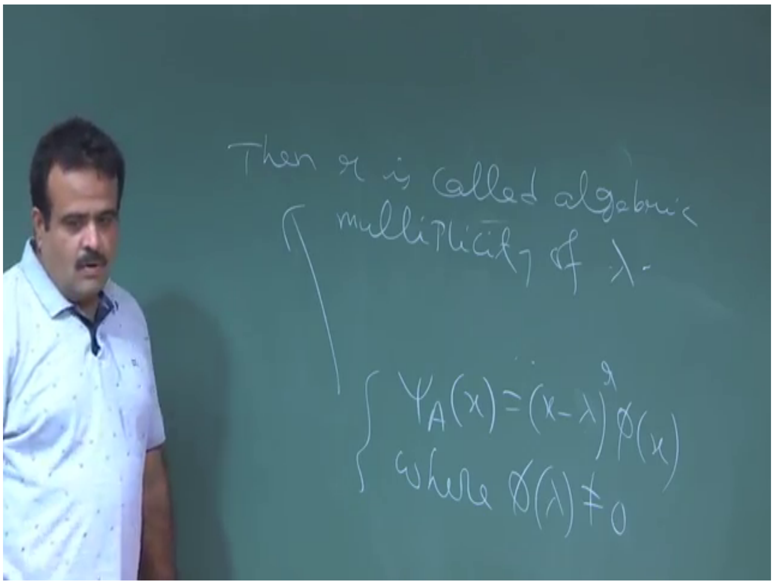

So, then if this is happened, then r is called algebraic multiplicity of lambda. Later on will define the geometric multiplicity, which is based on the eigenvector. So, this is the thing, and we also have seen that this lambda may not be from the same field.

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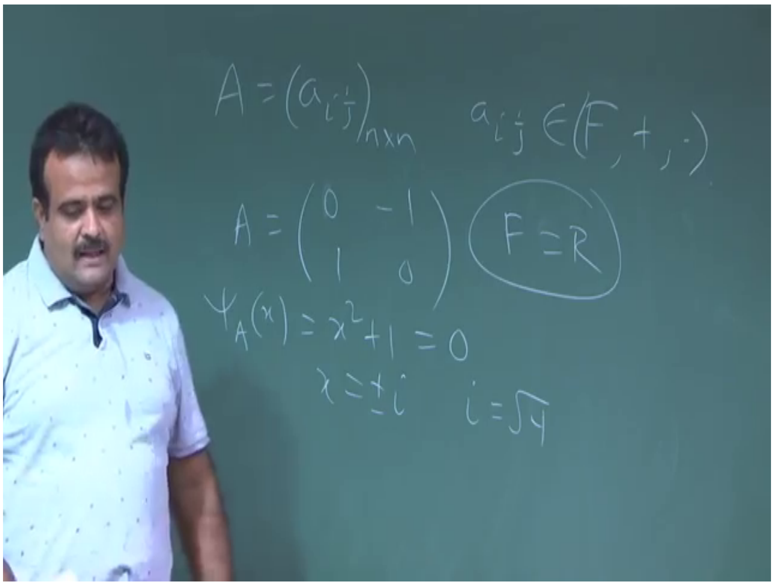

So, this you have a matrix A which is a square matrix where a i j is coming from the field, which is a field on that this operation. Now the a i j it could be real number or anything. So, it is coming from the field. So, in the last class we have seen, if you take the matrix like this (Refer Time: 03:27) real 1 sorry 0 minus 1, 1 0, then the corresponding characteristics equation is x square plus 1 characteristics equation is this. So, x is plus minus i, i is root to the power of minus 1. So, this is a complex number, but our field is a real field. So, here field is real field. So, so eigenvalues are not from this field here. So, this is one example.

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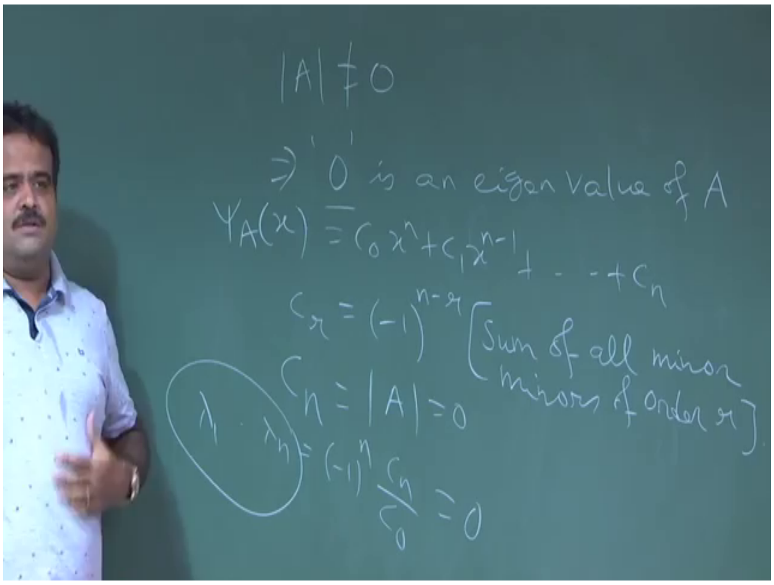

And in the last class we have also seen that if the if we have a non-singular matrix, then the there has to be a if you have a singular matrix, if the determinant of this matrix is 0, then there has to be eigenvalue with 0 0 is the eigenvalue is an eigenvalue of A. this is because we have seen from the characteristics equation this product of the eigenvalues.

So, the characteristics this we discussed in the last class just to recap. Characteristics polynomial can be written as C 0 x to the power n plus C 1, x to the power n minus 1, this is ending the polynomial then C n, where C r is the minus 1 to the power n minus r and the sum of the all principle minor of order r, sum of all principle minors of order r, so that means, if you put C n, C n is nothing but determinant of A, because principle minor of order n is the determinant.

And we know that this is a polynomial of degree n. So, we know the product of the eigenvalues. If there are n eigenvalues lambda 1, lambda 2, lambda n, then the product of the eigenvalue will be written as this coefficients C n by C 0.

Now, if the determinant is 0, then this is 0. So, product of the eigenvalues are 0 so the that means one of the eigenvalue must be 0. And on the other hand if we have a non-singular matrix, if this is no not 0, then the all the there should not be a 0 eigenvalues. So, this is the result we have seen. Now, we will talk about something more on the eigenvalues suppose we have a singular matrix A. And if lambda is a eigenvalue, then the 1 by lambda is eigenvalue of a inverse.

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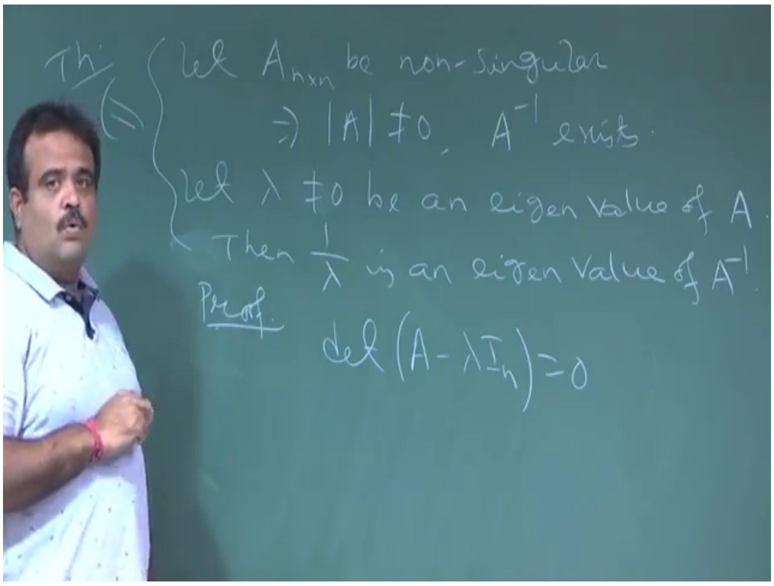

So, let A be non-singular so that means it has a inverse. This implies determinant of A is not equal to 0 that means A has inverse A inverse exists. So, non-singular means there is no 0 eigenvalues. So, let lambda which is not 0, because 0 cannot be eigenvalue of this let lambda be an eigenvalue of A. Then this theorem is telling this is a theorem there we have to show 1 by lambda, which is exists, because lambda is not 0, 1 by lambda is a eigenvalue of A inverse, is an eigenvalue of A inverse. It is a theorem (Refer Time: 07:50).

So, how to prove this theorem? So, to prove this theorem we need to take a a A is a eigenvalue so that means sorry lambda is a eigenvalue of A that means determinant of A minus lambda I n this is 0. So, lambda is a root of the characteristics equations. Now, we need to check whether A in lambda inverse is also an eigenvalue of this so, for that so will use this.

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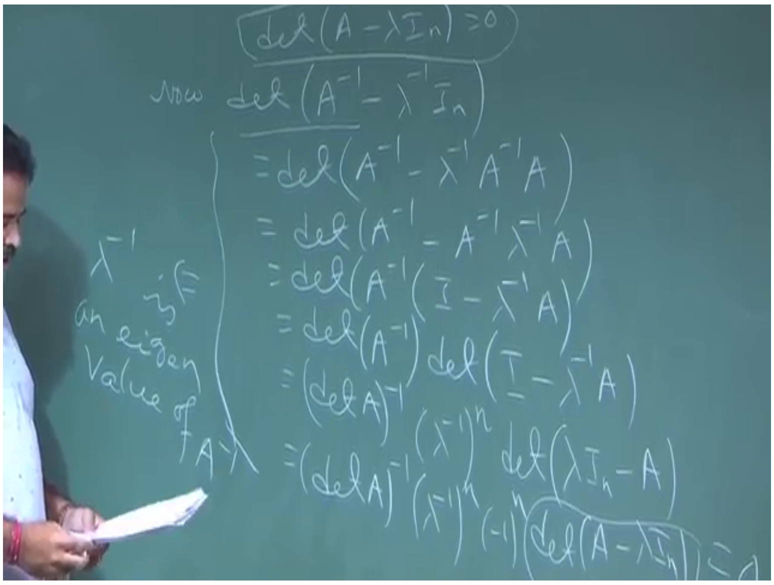

So, prove we are doing. So, det of A minus lambda I n is equal to 0. Now, we take the det of determinant of A inverse minus lambda inverse I n. If we can show, this is 0, then lambda is inverse is an eigenvalue of this. So, this we have to show to be 0.

So, what is this? This is nothing but det of A inverse (Refer Time:09:02) is and here will write lambda inverse is there, now A inverse A which is basically the identity. So, now, this will write as det of A inverse minus A inverse lambda is a constant lambda inverse is A constant we can write this as this.

Now, we take the A inverse common from this I minus lambda inverse A. So, this is nothing but det of A inverse det of identity matrix I minus lambda inverse A. So, this is nothing, but det of A then the inverse and we can take this lambda common n times. So, this is lambda inverse. So, this is lambda inverse n, and determinant of lambda I n minus A.

Now, again we can take minus common over here. So, this will (Refer Time: 10:26) det of A which is none of this is 0. So, this is we can take minus, and this det of A minus lambda I n. So, this is this value is 0, because lambda is eigenvalue of this. So, this whole thing is 0 so that means det of A inverse.

So, this is the characteristics equation of the A inverse. So, it has a root lambda inverse. So, lambda if this implies, lambda inverse. This implies lambda is inverse is an eigenvalue of A inverse. So, this is the proof lambda is the inverse is the eigenvalue of A inverse. Now, we will see another result another theorem. So, if we have a matrix A, and if we have a square matrix sorry singular all are square matrix. So, this is another theorem.

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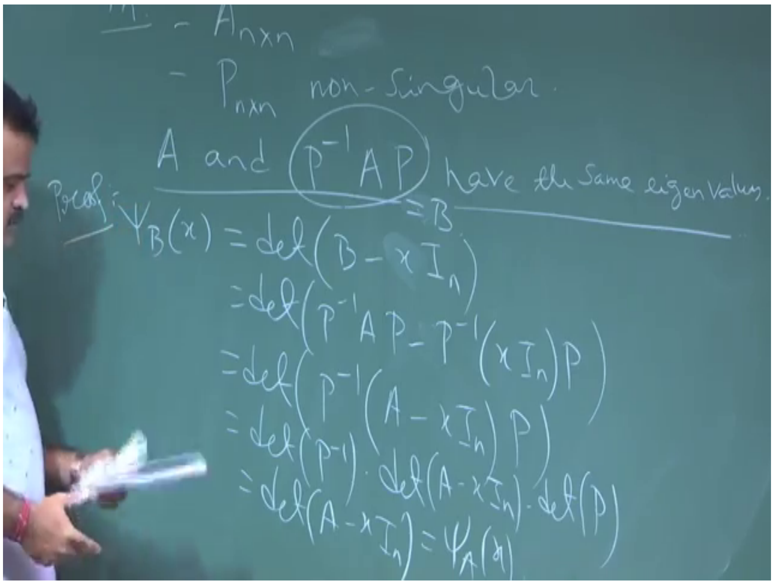

So, I have a square matrix A, which is could be singular and non-singular we have no we do not know anything about it is on singular, and non-singular, but and this is one matrix, and we have a matrix P, which is non-singular matrix, which is non-singular matrix. Then this theorem is telling that P inverse A P, and A has same eigenvalue.

A and P inverse A P have the same eigenvalue; have the same eigenvalues so that so how we can show that 2 matrix are having same eigenvalue, the if their characteristic polynomial are same so that we are going to achieve. So, these we have referred as sum matrix, this is we have B matrix. So, we have to show that characteristics polynomial of A is same as characteristics polynomial of B.

So, what is the characteristic polynomial of B, characteristics polynomial of B is basically determinant of B minus lambda x I n. So, let us write B; B is P inverse A P minus x I n. So, this x I n we can write as P P inverse form. So, P inverse x I n into P, because P into P P inverse will give the identity.

So, this we can write as determinant of we can take P inverse common over here A minus x I n into P. So, product is the determinant of A into B same as determinant of A into determinant of B. So, if we apply that, it will give us determinant of P inverse into det of this, into det of P. So, this and this will give us one and this is nothing but det of A minus x I n, which is the characteristics polynomial of A.

So, characteristics polynomial of the; characteristics polynomial of polynomial of these, and these are same so that means it has these two has these two has the same eigenvalues. So, this is the proof, these two has the same eigenvalues. So, this is the proof of this theorem. So, if we have a square matrix, then we will be getting this the same eigenvalue of this.

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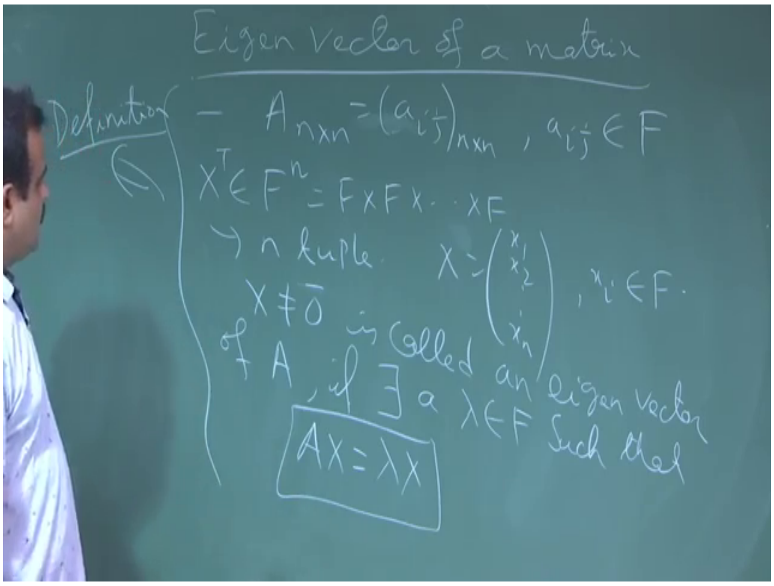

So, now we define eigenvector of a matrix. So, we define eigenvector of a square matrix. So, first let A is a square matrix n by n matrix. We define the eigenvector is. So, this square matrix over A, so this matrix is coming from elements are coming from the field. So, a i j is coming from the field under the operations plus and we have two operations in the field. So, this matrix is over field, this field could be the real field complex field set of real numbers set of complex number, but in general this is a field.

Now, a non-zero vector X, which is a non-zero vector, X is coming from F to the power n. So, F to the power n nothing but F cross F cross n time so, this is a n tuple. X can be written as x1 x 2 x n, I mean the transpose of this is a n tuple. So, it is coming from a so

X transpose is this we can say, or yeah this is X, and X transpose is (Refer Time: 16:26) anyway. So, this is coming from the, this is a tuple, either you write in row way or column way, this is a n tuple and elements are coming from the field all of these x i's are from the field.

And X is non-zero vector. A non-zero vector X is called a Eigen vector called and eigenvector vector of A, if there exist a lambda scalar, lambda eigen from the field such that A X equal to lambda X, such that AX equal to lambda. This is the definition of this is the definition of eigenvector, is the definition.

So, what is your definition? That if X is a the a non-zero vector, X is called eigenvector of a matrix A, if and only if there is a scalar of the same field there is a scalar, such that A X equal to lambda X, it is satisfying this equation A X equal to lambda X. Then X is called a X is called a eigenvector of the matrix. A lambda is the corresponding eigenvalue. So, we will come to that, so what do you mean by A X equal to lambda X.

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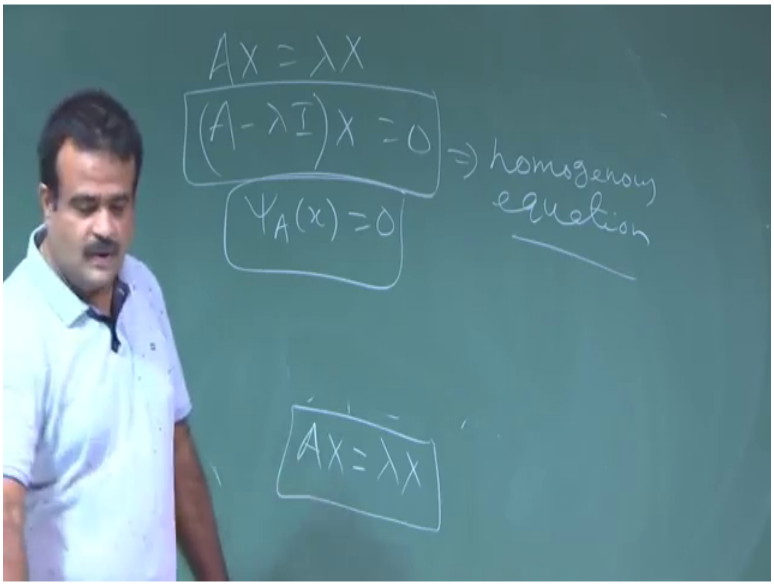

So, once we have A X equal to lambda X that means, A of lambda I X equal to 0, you got this equations. Now, we are looking for a this equation we are familiar with, so this is nothing but the characteristics equations. So, this is a homogeneous equation homogeneous equations and so this is a case.

So, the how we get the eigenvalues, eigenvalues we are getting by the roots of these equations, so that means, this X is the non-zero solutions, non-zero solutions of this homogeneous equations are called eigenvector. And this lambda is the corresponding eigenvalue. So, for each eigenvector, there is a there for each eigenvector, there is a eigenvalues. But, the question is the one eigenvector can corresponding to eigenvalues, so that we have to check. So, this will give us the eigenvalues and the eigenvalues will give us so, let us check that.

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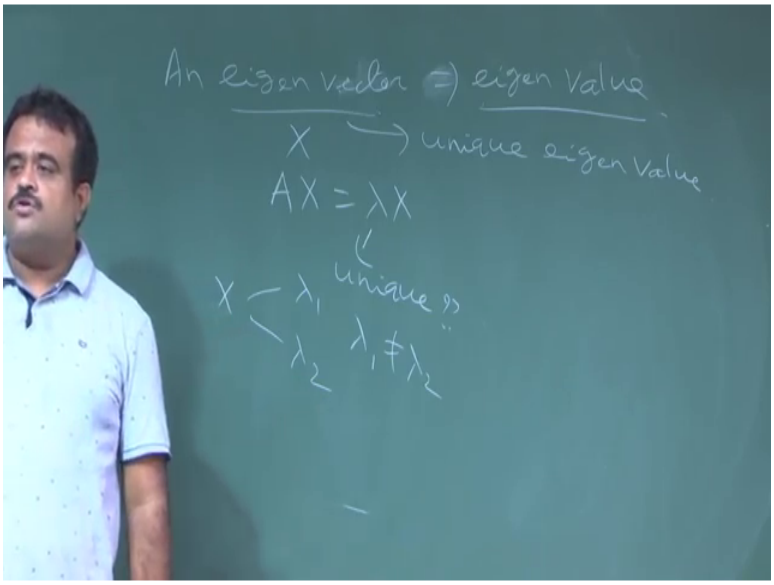

So, if you have a eigenvector, so the so an eigenvector eigenvector sorry vector gives us a corresponding to a eigenvalue. Now, the that means, if X is the eigenvector that means, there is a lambda such that this. Now, the question is this unique, this lambda? The question is if we can you get two eigenvalue corresponding to a eigenvector? Answer is no. Why? So this eigenvector will correspond to the unique eigenvalue, so that means, corresponding to a eigenvector, we cannot have two eigenvalues lambda 1, lambda 2, where they are not same. This is not possible. Why, how to justify that?

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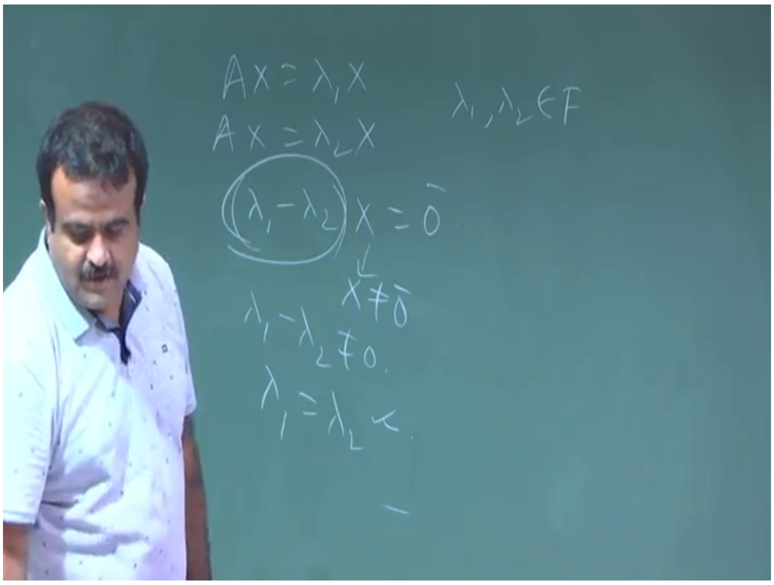

Suppose, it is possible, suppose for a given eigenvector, we have two eigenvalues, now one is lambda 1, and other one is lambda 2 that means, what that means, we have two equations A x equal to lambda 1 X, and A X equal to lambda 2 X. So, from here what we can do, we can just and well lambda 1 and lambda 2 are coming from the field at the matrix elements are there.

Now, what we do here, now we can just subtract these two, and we get lambda 1 minus lambda 2 X equal to 0 0 vector. So, now this is not possible, because X is non-zero, X is a non-zero vector. And also this is a non-zero scalar, because we. So, to happen, this we need to a lambda 1 equal to lambda 2, so that is the uniqueness. So, for a given eigenvector, we have a unique eigenvalue. So, one eigenvector cannot corresponding to two different eigenvalues. So, in a other word, the eigenvector is giving the unique eigenvalues. So, this is one observation. But, other way suppose, we have given a eigenvalue, then how many eigenvectors are there.

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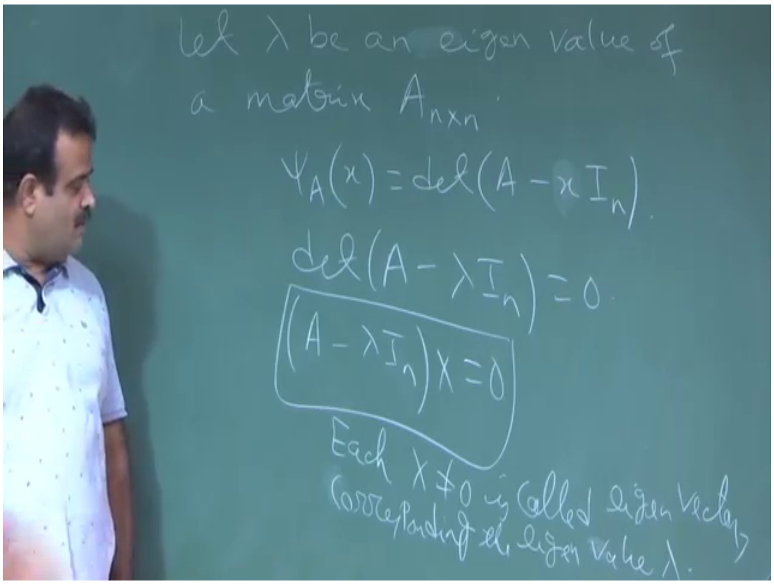

Let lambda be an eigenvalue of a matrix A, which is n by n matrix, then it will leads at to a some eigenvectors, so at least one eigenvector, so that means, this determine this characteristics. So, lambda is the root of this characteristics equation. So, this is a characteristics equation.

So, lambda is a root of this characteristic equation, so the that means, determinant of A minus lambda n is equal to 0, so that means, if we consider this homogeneous equation, so A minus lambda I n equal to X. If we consider this homogeneous system of equation, then it has because this determinant is 0. If determination non-zero, then it has only one root, which is 0 or only one a solution, which is 0, but determinant is equal to 0, so the that means, it has many solutions non-zero solutions infinite number of solutions.

And each of this is a eigenvector, each of this non-zero solution is corresponding to the eigenvector is eigenvector corresponding to this eigenvalue lambda. So, each X not equal to 0 solution of this is called eigenvectors corresponding to the eigenvalue lambda, is in corresponding to the eigenvalue lambda.

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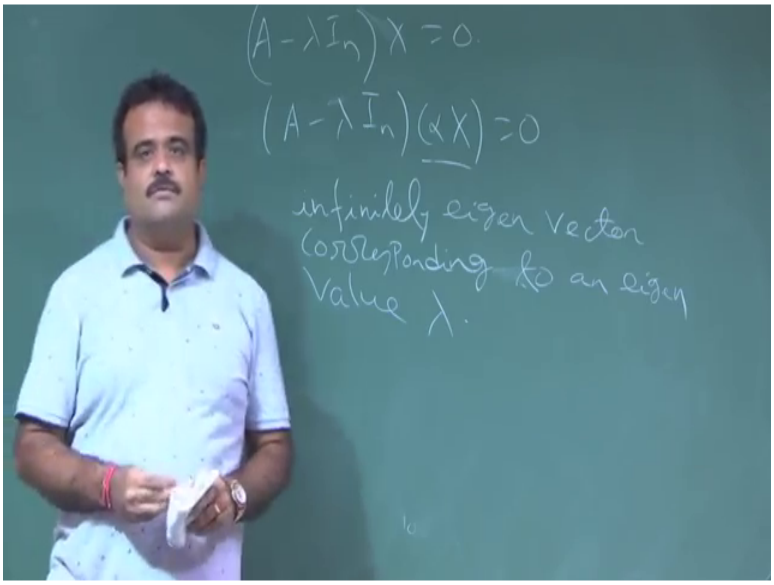

So, an if X is eigenvalue of if X is a eigenvector, suppose X is a eigenvector, so the that means, sorry so, if X is a eigenvector, so A minus lambda I n X is 0. Then alpha X is also is a eigenvector, because X minus lambda I n alpha X, alpha is a scalar. This is also an eigenvector; X is non-zero. So, we have infinitely many eigenvectors corresponding to a eigenvalue.

And if we consider this set, so we have infinitely eigenvector corresponding to an eigenvalue lambda. So, given a eigenvalue we have infinitely many eigenvector, because that those are the non-zero solution non-zero solution of this system of homogeneous equations. So, we will continue this in the next class.

Thank you.