Introduction to Abstract and Linear Algebra
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Lecture – 08
Permutation

Ok. So, we talk about Permutation today. So, basically you have seen some mapping. So, now permutation is a mapping it is basically bijective mapping.

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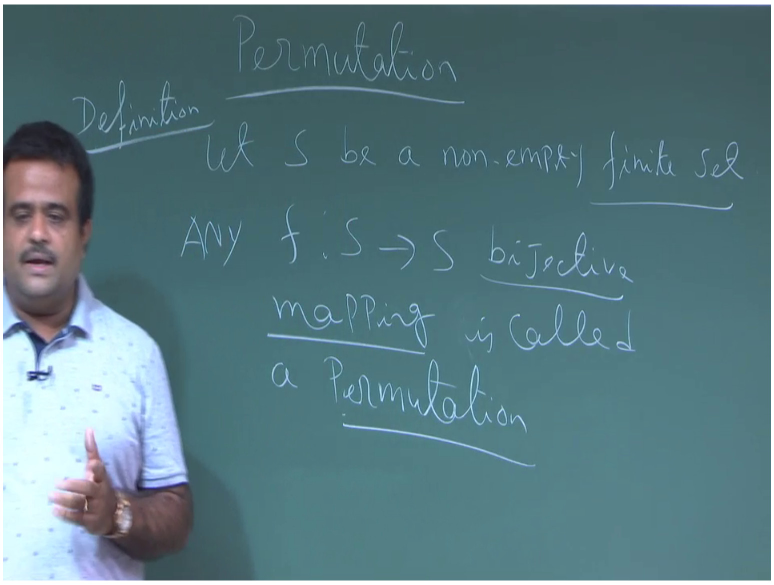

So, let S be a non-empty set non empty finite set finite set of element then any bijective mapping from S to S; any bijective mapping or function on S to S is called a Permutation ok. It is basically a bijective mapping from the set to itself.

So, any such mapping any such function is called a permutation. So, for this is the definition of permutation and this is defined on a finite set of elements ok.

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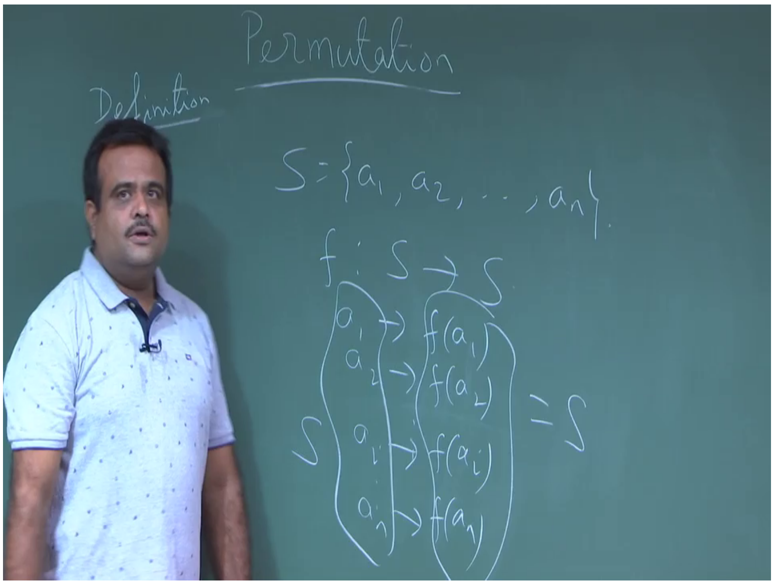

Now, for example suppose we have n elements Suppose S is S is finite. So, suppose these are the elements suppose there are n elements in the set and suppose you have a bijective mapping S to S then; that means, a 1 is going to f of a 1 a 2 is going to f of a 2 like this.

So, we have a i is going to f of a i dot dot dot a n is going to f of a n and this f is a bijective mapping so; that means, this is basically nothing, but this set same set S. So, this connection is basically equal to S and this connection is also S so, S is going to S.

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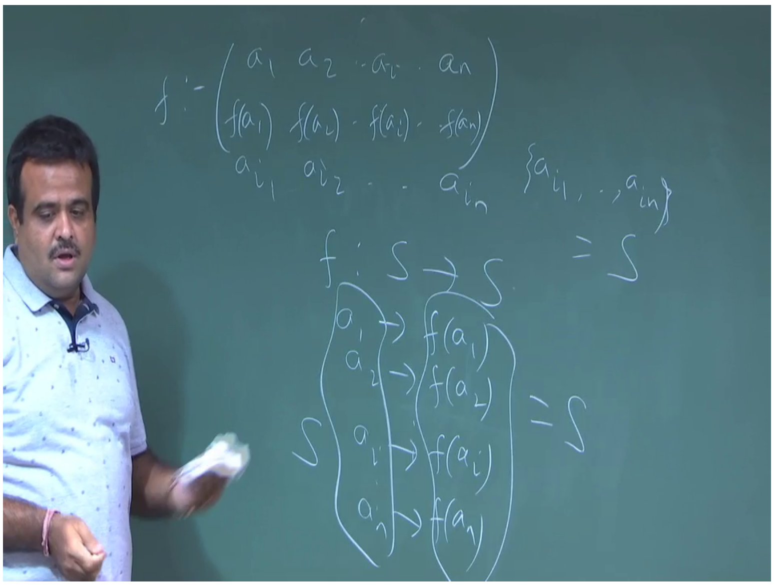

So, this can be written in some other way like in this form a 1 a 2 a n and this is a i say and here f of a 1 f of a 2 dot dot dot f of a i f of an ok. This basically a rearrangement because this is a bijective mapping so, this will be also 1 2. So, this will cover whole set S so, this is our S again this will be our S.

So, any rearrangement is a permutation. So, that will gives us a mapping bijective mapping 1 to 1. So, there will be no repetition in the this so, that is why it is a distinct collect distinct sets distinct points and it is on 2. So, it is covering all the elements. So, this is nothing, but S again. So, it is in some order so, a i 1 a i 2 a i n. So, this is basically rearrangement basically a i 1 this set a i 1 a i n this set is basically S.

So, in some sort of rearrangement. So, this is called a permutation also written as this so, it is basically this. So, any bijective mapping from S to itself will give us a permutation.

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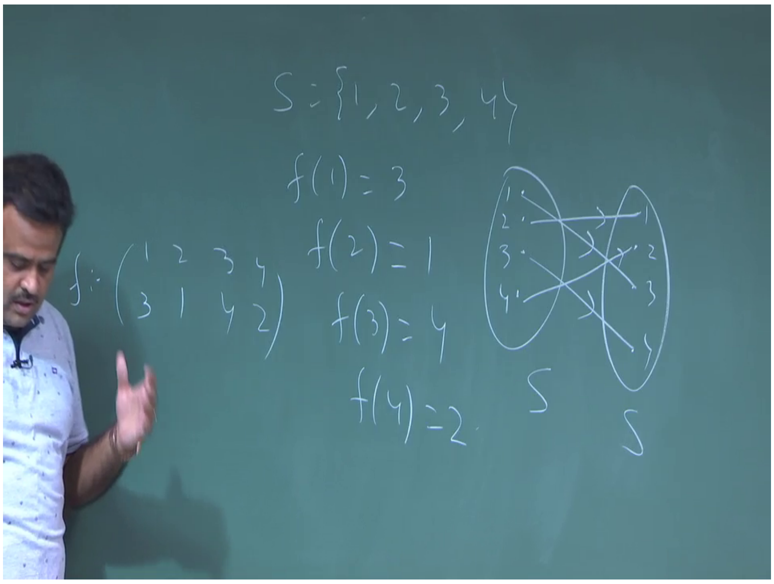

So, we can take some example like if we have 4 elements. So, like if it is S is a 1, 2, 3, 4, and suppose you have a bijective mapping like this f 1 is going to 3 f 2 is going to 1 f 3 is going to 4 and f 1 is going to 2.

So, we have we have 4 points 1, 2, 3, 4 and here we have 4 points 1, 2, 3, 4. So, this is S set, this is S set this is a mapping. So, 1 is going to 3 2 is going to 1 3 is going to 4 and 4 is going to 2. So, this is a 1 to 1 mapping and also it is 1 2. So, this is a bijective mapping, this is a bijection. So, it is a permutation so, how we write this permutation in this that form of this 1 2 3 4.

So, 1 is going to where 1 is going to 3 then 2 is going to 1 and 3 is going to 4 and 4 is going 2. So, this is the permutation ok. So, any bijective mapping from S 2 itself can be written as a permutation. So, then we talk about inverse permutation whether the this is also a inverse of this function is also a permutation or not. So, let us talk about that. So, how many rearrangement is possible if there are n will n elements, how many permutations are possible there factorial n permutations are possible.

So, basically there are factorial n number of bijective mapping from S 2, S 2 itself ok.

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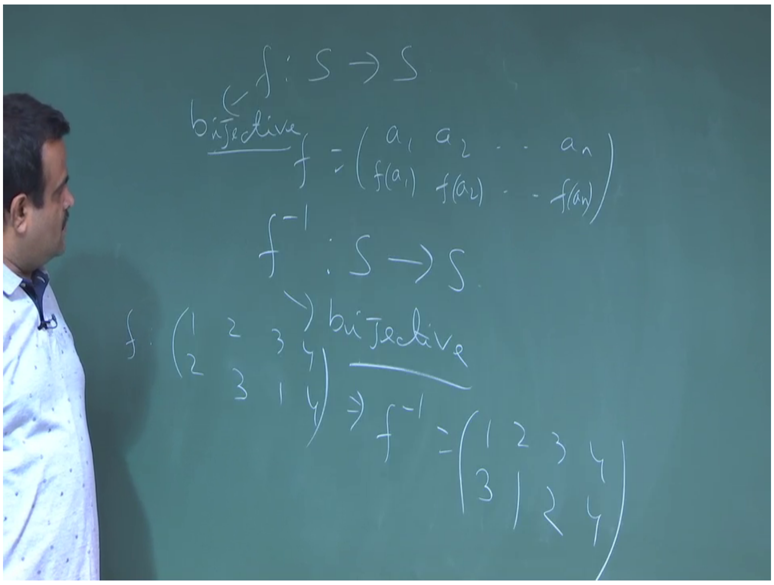

Now, suppose you have a permutation f from S to S so; that means, basically we have a bijective mapping f a 1 f a 2 f a n. Now, we know if f is bijective then f inverse exist f inverse is basically from S to S again and if f is bijective we can show that f inverse is also bijective this say bijective. So, if f is bijective we know the f inverse is also f inverse is also bijective.

Now, now what is how to get f inverse f inverse is basically will follow the reverse way like, if we have say for example, if you have this permutation 1 2 3 4 and if say 1 going to 2 2 going to 1 3 going to 3 4 going to sorry 3 1 going to 3 4 going to say 3 going to 1 and 4 going to 4. Suppose this is our f then what is the f inverse, f inverse basically from this to this like if we apply f compose f compose will defined on the permutation is basically composition of two function.

So; that means so, f inverse is 1 2 3 4. So, basically 1 is going to so, who came from 1 3 came from 1 so, 1 is going to 3 under f inverse then 2 2 is here so, 2 came from 1. So, basically so, 2 is going to 1 3, 3 is here 3 is going to 2 and 4 is remain unchange. So, this is basically our f inverse ok. So, and this is also a bijective from mapping so, this is a inverse permutation.

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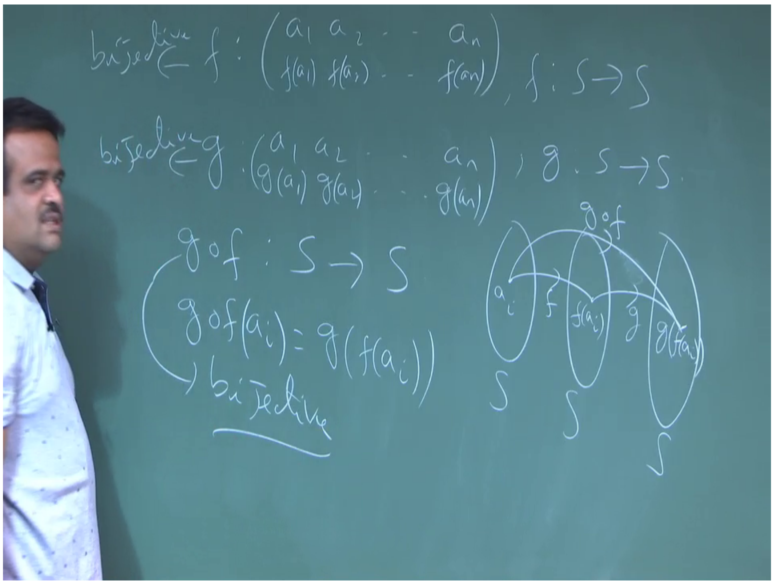

Now, we defined composition of 2 permutation. So, suppose we have a we have 2 permutation f and g. Say f is a a 1 a 2 this is our set f of a 1 f of a 2 these are the images and we have another permutation say g a n g of a 1 g of a 2 g of a n. So, f or g 2 bijective function from S to S f or g are 2 bijective function from S to S.

Now, we under know their composition of 2 permutation so; that means, g compose f say. So, we know composition is a function from S to S whether this is this is our S set so, for this is all the S sets. Now, f is a function form so, this is a i f is a function from here to here this is f so, this is f of a i. Now, on this f of a i we can apply the g. So, this will give us this is g so, this will give us g of f of a i.

Now, if you consider the mapping from a i to this one so, this is basically called composition g compose f because we are applying f first then g. So, this is basically g compose f of a i is nothing, but g of f of a i. This is how we define the composition and we know this if g if f and g are both bijective then this is also bijective. Now, this is bijective and this is bijective then we know this is also bijective.

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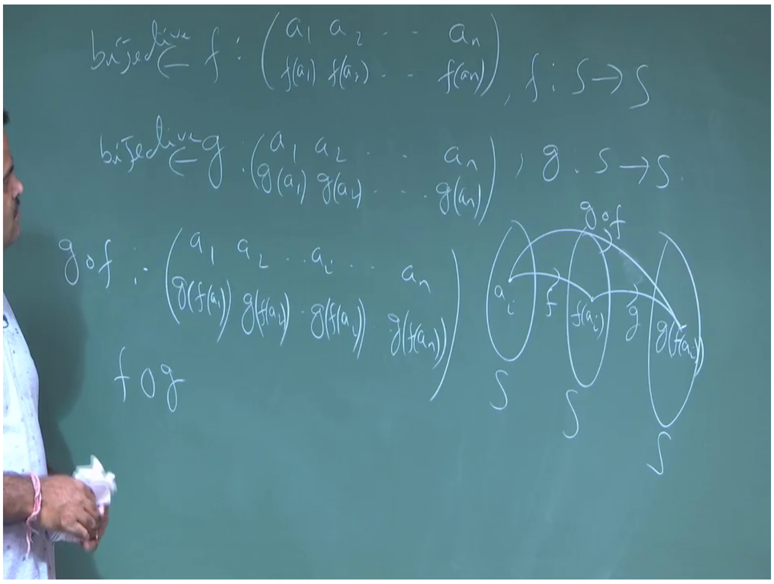

Now, once this is bijective this is called a permutation and how we define this in this way this is basically a 1 a 2 say a i a n; this is g compose f permutation. Now, this is g of f of a 1 g of f of a 2 dot dot dot g of f of a i this is the i-th g of f of a n. This is the composition and if it is f dot g then we first operate g then f a this is the convention we use.

So, let us take one example on composition on the set S; suppose you have a set is say 4 element 1, 2, 3, 4.

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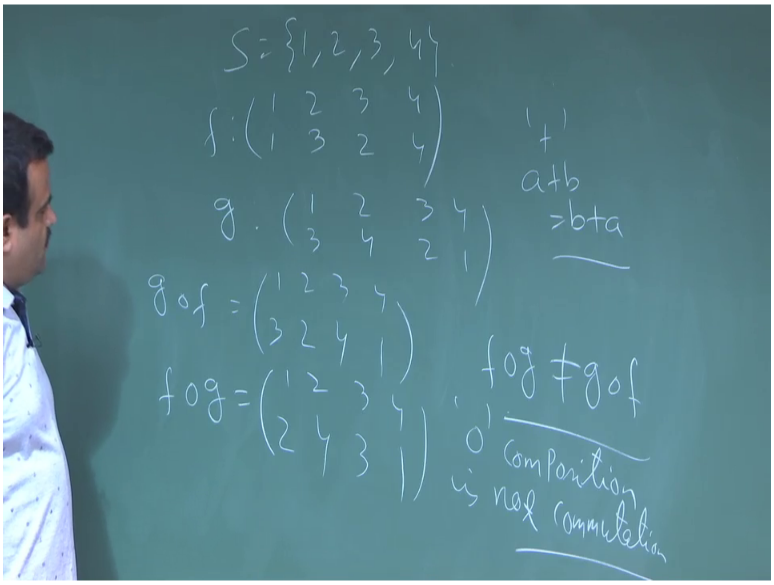

Now, suppose we define two function bijective function basically two composition 2 permutation; say 1 is going to 1, 2 is going to 3, 3 is going to 2, 4 is going to 4. This is g basically 1 2 3 4, say 1 is going to 3, 3 is going to 4, 3 to a sorry 1 is going to 3, 2 is going to 4. So, any rearrangement will give us a permutation that is a bijective mapping and this is say 2 1 ok.

Now, we want to know the g compose f. So, first we apply f and then we apply g. So, this is basically first let us writing g 3 4 2 1 or maybe we can directly do that 1 2 3 4. So, how we do we first apply f 1 1 f 1 1 will be 1 then we apply g of this so, this will be 3 then 2 we first applied f 1 so, will go to the 3 then 3 will go to that 2. Then 3 so, we apply f it will go to 2 then 2 will go to 4 then 4, 4 will go to 4 and 4 will go to 1. So, this is the scenario.

Now, now we want to know the this f compose g for curiosity. So, what is f compose g f compose g is first we apply f then we apply sorry first we apply g then we apply f ok. So, if we apply g on 1 so, we will go to 3 and 3 is going to 2 then 2 2 is going to 4, 4 is going to 4 then 3 3 is going to 2, 2 is going to 3 then 4, 4 is going to 1, 1 is going to 1. So, these 2 permutation is not same.

So, that is why this composition of function is not commutative; that means, f dot g is not same as g dot f. So, this dot operation this composition composition operation is not will define the commutativeness commutative; like plus is commutative real number plus because a plus b we can write b plus a plus is commutative. But this operation is not commutative ok.

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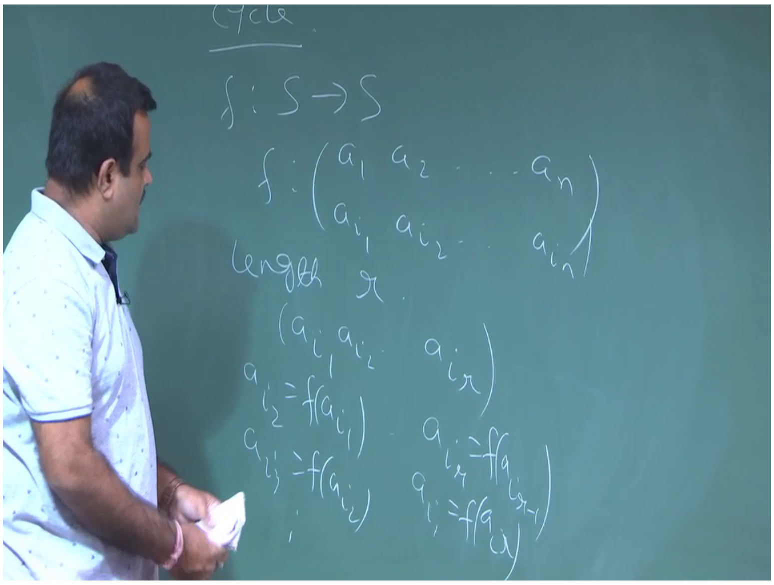

So, now we define cycle of a permutation. So, we define cycle a we define cycle; suppose you have a permutation if from S 2 S and S is basically a 1 a 2. So, a i 1 yeah some a a some rearrangement so, say a i 1 a i 2 a i n some rearrangement on this. Now, we call this is this represent a cycle ah said to be a cycle of length r, if we have this property say we have some a i 1 up to a i r say length r cycle such that the next one that a i 2 is nothing, but f of a i 1.

So, this one is basically f of this one so, this one is also f of this one. So, a i 3 is basically f of a i 2 like this dot dot dot a i r is basically f of a i r minus 1 and the last one is this is again going to this. So, a i 1 these basically f of a i r ok. So, this way if we can represent this then it is a basically cycle of length r; it is basically coming back to same.

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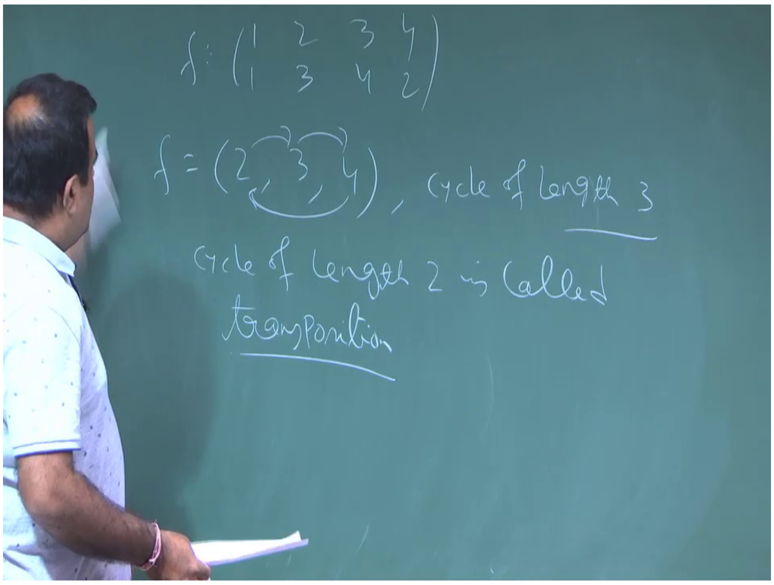

Well let us take an example of cycle; let us take an example suppose we have a permutation like this 2 3 4 there are 4 elements and say 1 is my 1 3 4 2 suppose this is a permutation. Now, what is the corresponding cycle of it this is also a cycle. So, because 2 is going to 3 and 3 is going to 4 and again 4 is going to 2.

So, this is even writing in this way we can write a permutation also in cycle way ok. The cycle means 1 is not there so, 1 is going to 1 that is not in the part of this cycle ; that means, 1 is going to 1 then 2 is going to 3 3 is going to 4 again 4 is coming back to 2.

So, this is the meaning of a cycle. So, this is a cycle of length 3 length 3. So, every permutation can be written as a product of cycle. Now, product of cycle not only product of ah cycle of length 2 basically will come to that and cycle of length 2 is called transposition ok; cycle of length 2 is called transportation.

Now, will show a theorem where it is telling that every permutation can be written as the product of cycle of length 2.

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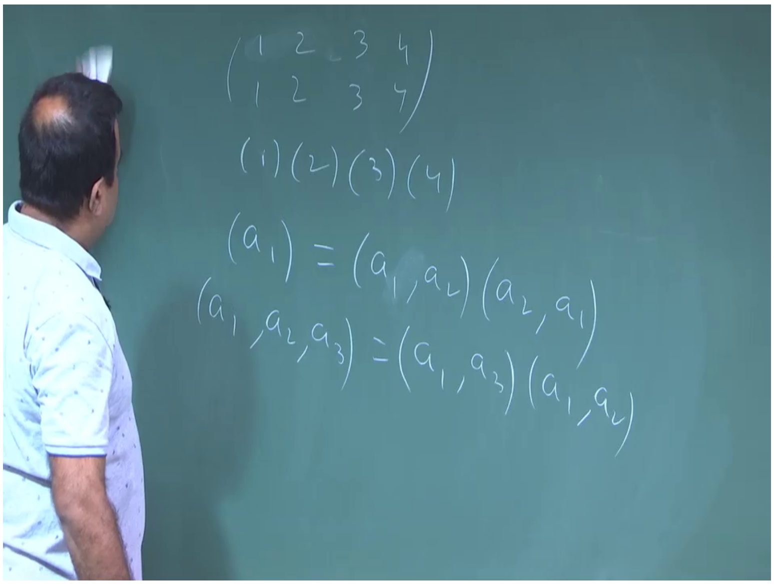

Now, suppose you have identity permutation identity permutation means a 1 a 2 say 4 we have 1 2 3 4. So, this cycle means 1 2 3 4. So, everybody is going to itself so, that is why it is a oneth length cycle and oneth length cycle how we represent a 1 length cycle. Suppose a 1 a 1 can be written as a 1 2 so, sorry a 1 a 2 and then a 2 a 1. So, this is the composition mapping.

So, what is the meaning of this. So, a 1 is yeah so, we first apply this so; that means, a 1 is going to a 2 and again a 2 is going to a 1. Now, if we have a cycle of length 3 a 1 a 2 a 3; now how we will write this in the 2 length product of 2 length so, this is a 1 a 3 and then a 1 a 2. This is the composition mapping. So, how to read so, a 1 is going to a 2 then again a 2 is going to 1 and a 1 is going to a 3 so, a 1 is going to a 2 a 2 is going to a 3. So, this is the way we write.

So, every cycle of different length can be written as a product of this one. So, let us take a another example.

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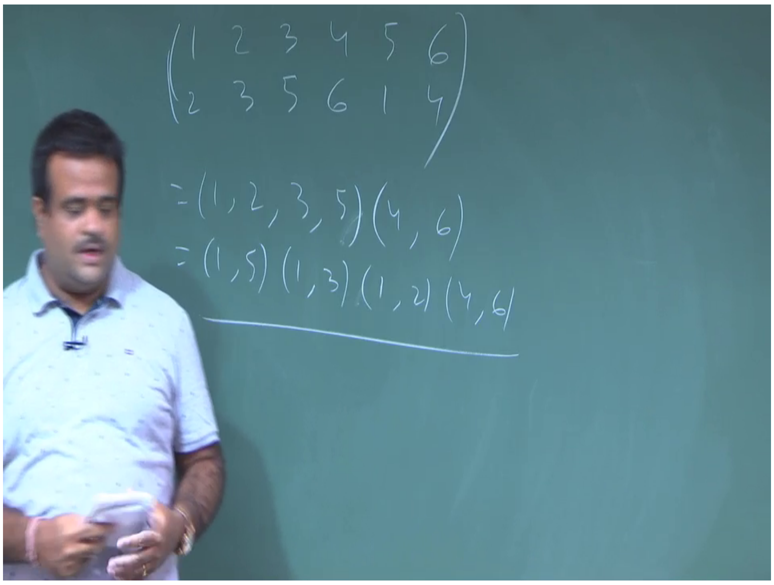

So, if we have say 1 2 3 4 5 6 these are permutation; this is 2 3 5 6 1 4. So, first of all how we write this in terms of the cycle so, 1 is going to 2 2 is going to 3 3 is going to 5 and 5 is going to 1. So, this is one cycle is closing and then we have anything missing yeah we have 4 missing so, 4 is going to 6 and 6 is going to 4.

So, this is the permutation we can write in this cycle of high cycle form and then again we want to write this is a 2 2 length cycle product of 2 length cycle. How we can write this? This is 1, 5 1, 3 this is composition 1, 2 and then 4, 6 we have; this is the product of transportation.

So, any permutation can be written as composition of product of transportation 2 length cycle. So, if the number of transportation is even then we call this an even permutation, if this is what we call this is not permutation. So, this is more or less some concept and permutation.

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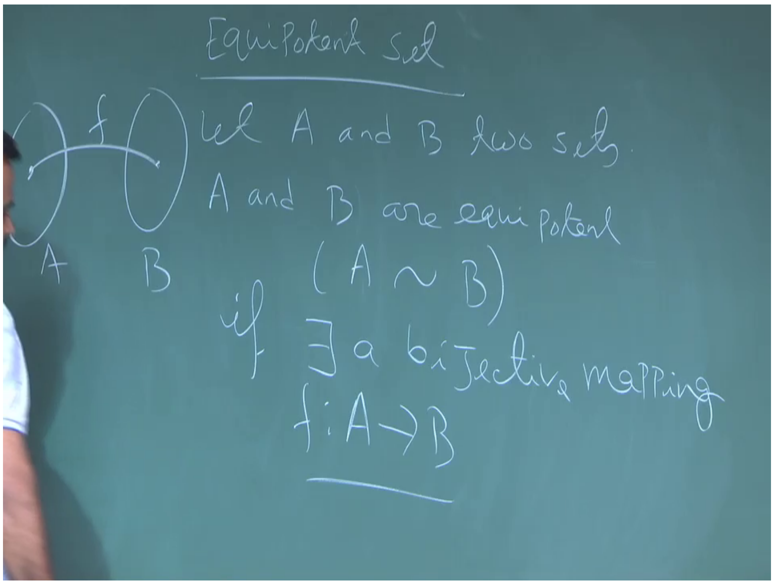

Now, we will move to the next topic which is called equipotent set. Suppose we have two sets let A and B two sets. Now, we defined a relation which is called equipotent relation. Now, A is A and B are equipotent are equipotent, we defined these as this symbol equipotent if and only if there exists a bijective mapping.

There exists a bijective mapping from A to B, if there is it is a bijective mapping from A to B then we say this sets are equipotent; that means, if we have a set A, if we have a set B. If we can defined a bijective mapping from A to B, if there exist a bijective mapping from A to B then we say this A to B sets are equipotent.

So, now this now we now we consider set of all sets and then we will see this relation is a equivalence relation, this equipotent relation between two sets.

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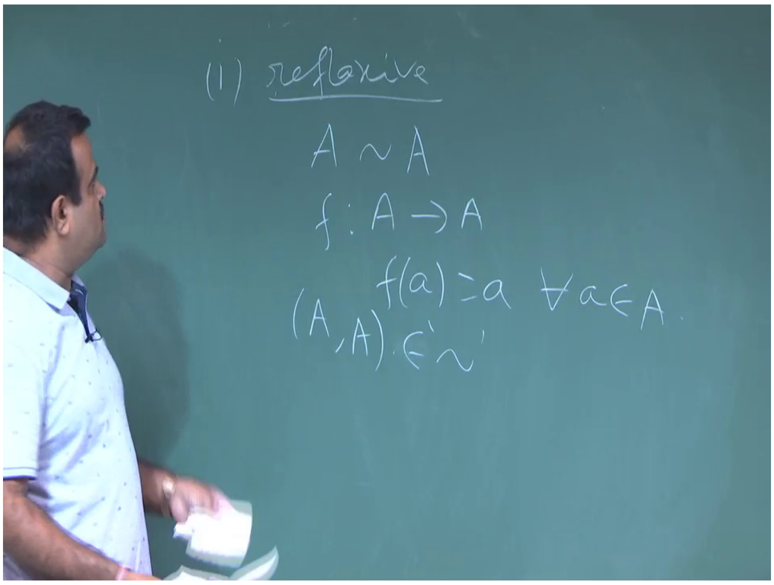

So, for that first of all you need to show this is reflexive. So, so, for that you need to show A is equipotent to A, this is obvious because if we have identity function like if we define a function A to A such that a of a is equal to a for all A. So that means, A, A belongs to this relation. So, this is symmetric.

Now, how to show I sorry reflexive how to show this is symmetric.

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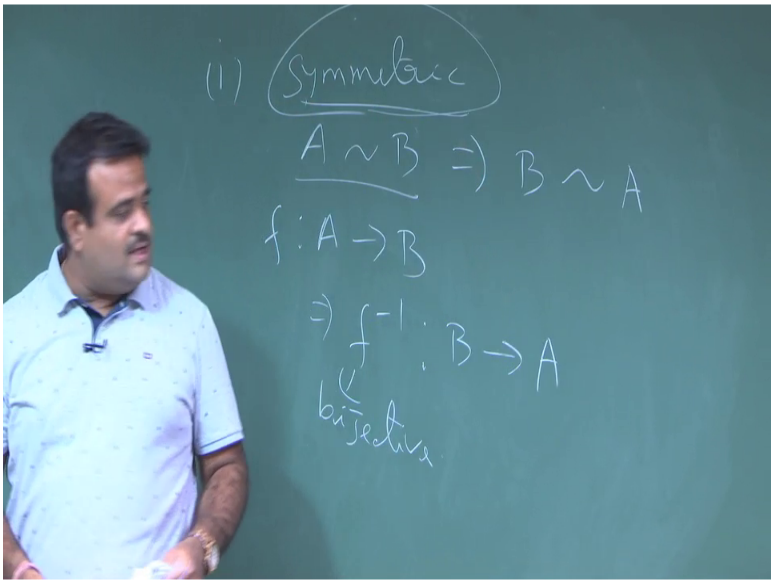

So, symmetricity means symmetric means if A is equipotent B this must imply B is equipotent with A. Now, what is the meaning of this with this meaning of this there exists a bijective from A to B then only they are equipotent. So, then we know if A is a bijective mapping then A inverse I sorry, f is a bijective mapping then f inverse exists f inverse is also found B to A and this is also bijective so; that means, this is symmetric.

Now, if this is symmetric then we can show whether this is a transitive property satisfying or not.

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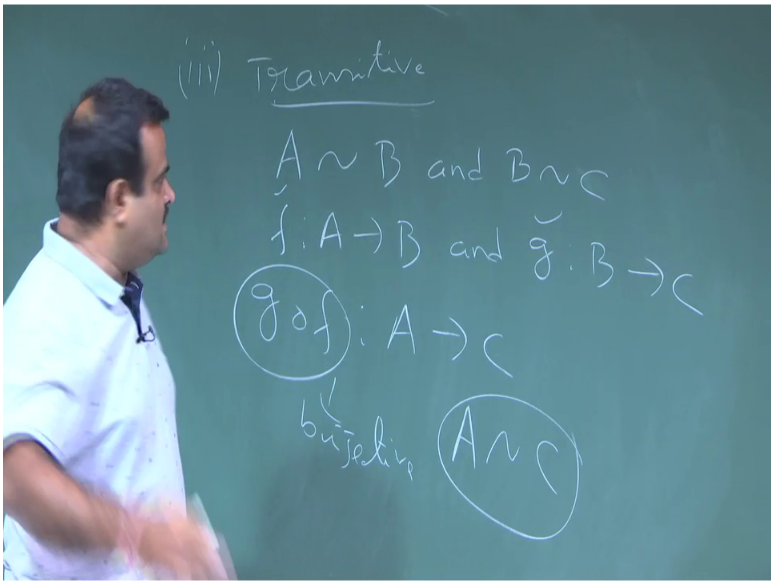

So, for that transitive so, we take two function so, we assume A is equipotent with B and B is equipotent with C. So that means, we have a function f bijective function f B to C and we have another bijective function g from B to this is A to B and that is B to C.

Now, if you take the composition function this is basically A to C and this is also a bijective. If this is bijective, this is bijective then this is also a bijective function. So that means, A is equipotent with C so; that means, this equipotent relation is a transitive relation. So, it is satisfying reflexive symmetric transitive so; that means, equipotent is a equivalence relation.

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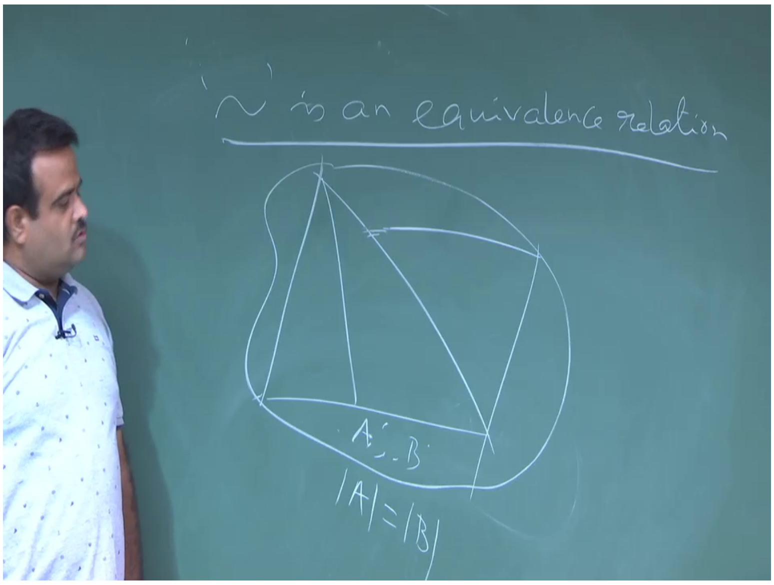

So, this is telling us this is an equivalence relation so; that means, it will partition. So, this is the set of all sets it will partition this in the equivalence classes, in the equivalence classes and is same classes; So, these classes the elements who are having same number of points; this we defined as a cardinality.

So, if A B are in the same classes their cardinality must be same that is the way ok. So, this will give us the equivalence classes. Now, we define enumerable set just yeah just another 2 minutes.

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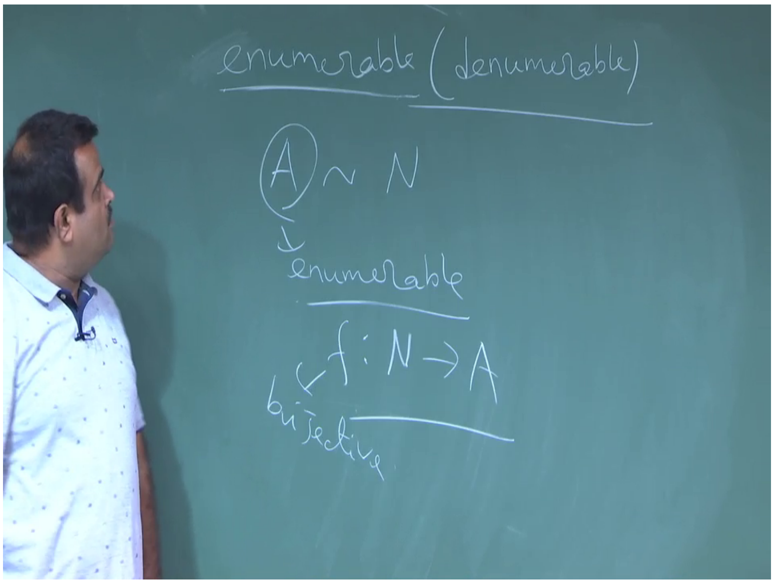

So, so, we just define enumerable or this is also called denumerable.

So, a set will be called enumerable if it is equipotent with the set of natural number; then A is called enumerable set ok. A set is said to be enumerable if it is equipotent with the natural number; that means, if there is it stay bijective mapping from N to A. If there exist to bijective mapping from N to A then we call this is as this is a innumerable number.

Now, then what do you mean by countable; countable set means if it is either finite or it is enumerable.

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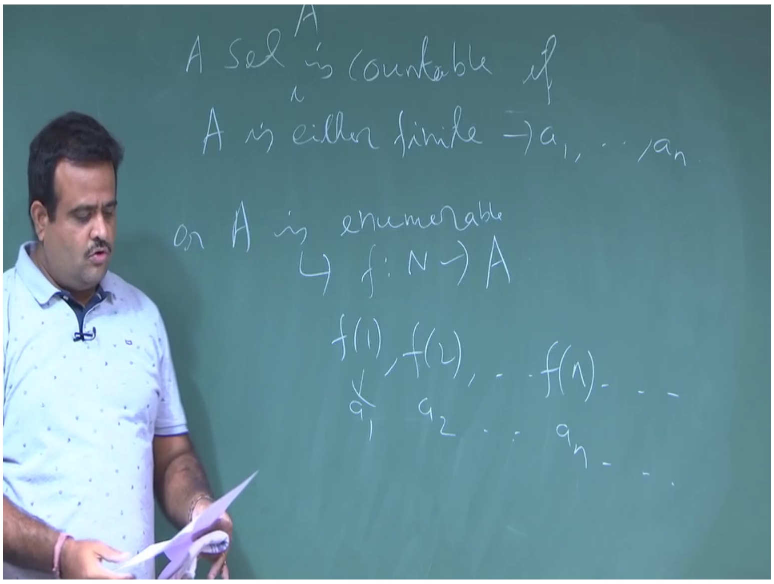

A set is countable if a set A if A is either finite or A is enumerable. So that means, if it is finite it can be written as a 1 a 2 a n or if it is enumerable then if there is a function from f N to A.

So that means, the A will be written as some f 1 f 2 bijective mapping f n like this so; that means, a 1 a 2 like this a n. So, A can be written as the sequence from a 1 a 2 2 a n. So, this is the definition of countable for countable set. A set is countable if it is finite or if it is enumerable.

Thank you.