Introduction to Abstract and Linear Algebra
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Lecture – 09
Binary Composition

So, we start the I will just some of the algebraic structure like group we start with the group and then we move to the algebraic structure group. So, before that let us just defined what do you mean by Binary composition or Binary operation ok.

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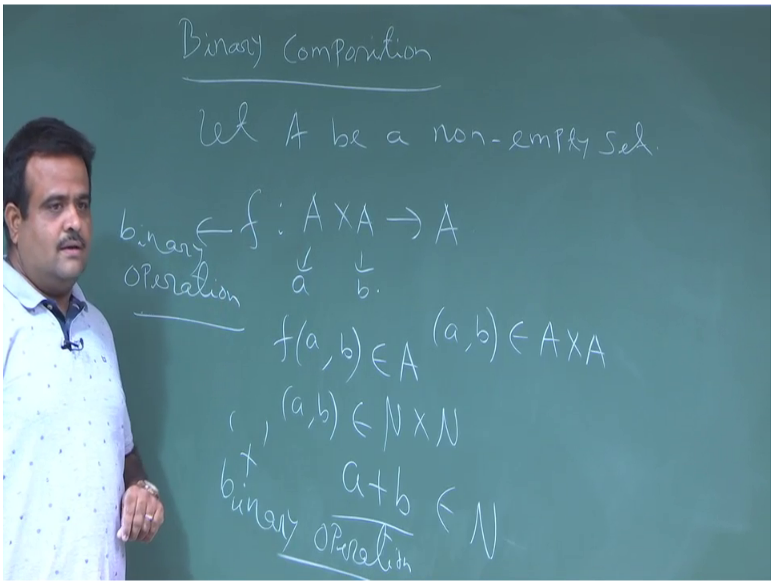

So, binary let a let A be an non-empty set of element then we defined a function f from A cross A to A ok, it could be A cross A to A.

So, basically we take an element from A and element another element from A and we apply this. So, we take an element from the Cartesian product A cross A and then we apply the, this function. This function is a basically function from the Cartesian product to this then this is if this is belongs to A then it is called closure property. So, basically, but binary composition means this type of function is called binary operator or binary composition; this is for binary operation ok.

So, like plus if we take say a to b natural number, natural number set. So, if you take a b from this then if we defined plus then this is this is basically this will also give us a natural number; if you add 2 natural number. So, plus is a binary composition or binary operation ok. Now, this is on a set A so, basically we take 2 elements from A and then we compose this with the operators this f or sometimes we denote this by in general star.

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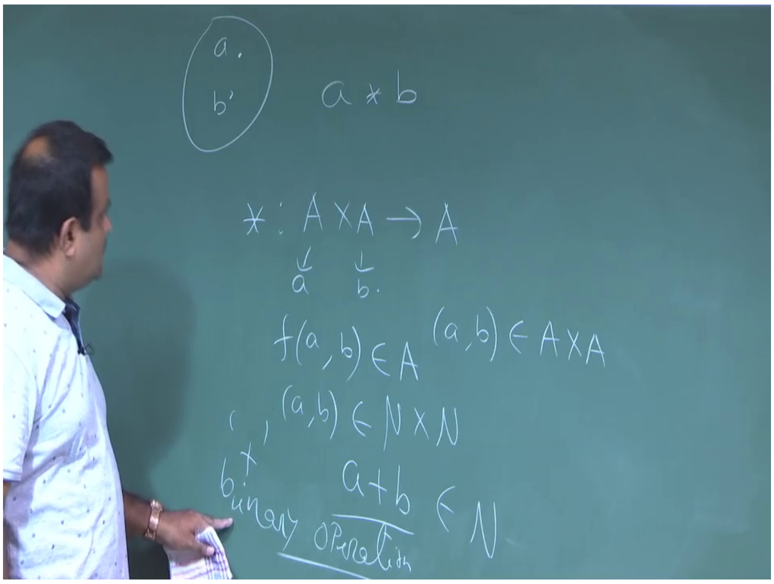

So, we take this is our A set we take 2 element from this a b. Now, we compose this by a operator star which is basically function form take 2 element from A and then it may give a element in A or may not, if it is giving an element in A then we call this is a closure operator. So, like addition addition on natural numbers. So, suppose A is N then addition is a closure.

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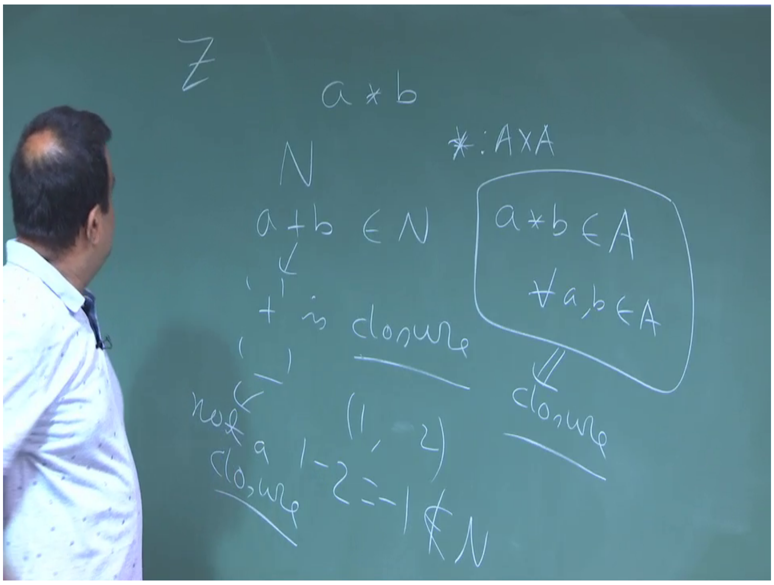

So, like if we have a natural number set and if you take a plus b then this is also belongs to N. So, addition so, this addition is closure, this is called closure property. So, what is closure property? So, if we have a operator A cross on A on A by a binary operator on A. So, it will be called closure if a star b belongs to A for all a b. Then this is called then the operation operator is called then the operation is called this binary operation is called closure operation or it is satisfying the closure property.

Now, for example, this plus is closure on this set N, but subtraction is not closure because if you take 2 natural number say 1 and 2 sorry 1 and 2, 2 natural number. Now, 1 minus 2 is basically minus 1 which does not belongs to N, this is an integer. So, this minus is the subtracted this is not a not a closure operator on N, but it is closure on i Z if you take Z as our set then minus is also a closure on Z ok.

So, so, this is the definition of a closure operator. So, a operator is need not be always closure on a set ok.

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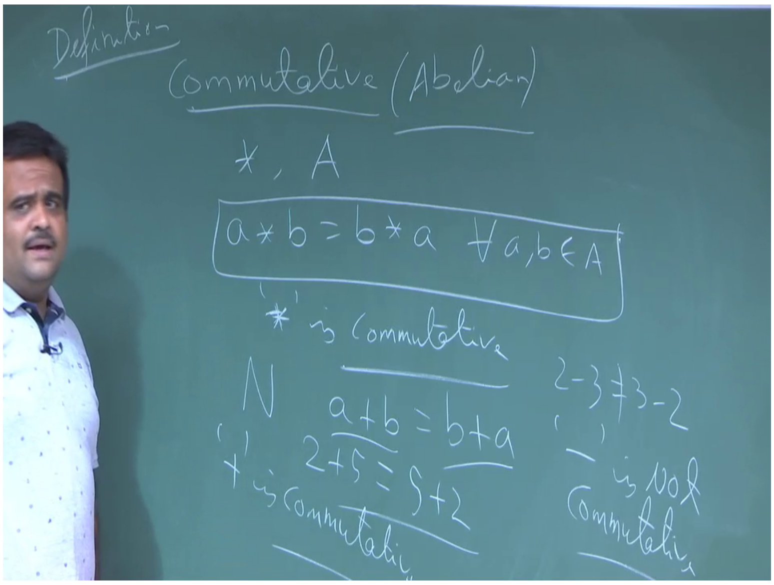

Now, we say a operator is commutative, this is the definition commutative this is so, suppose you have a operator star on the set A so; that means, it takes 2 element from A and it operates the star and it gives an element. If it is closure that element will belongs to A, otherwise it need not belongs to A, but for commutative we need this property a star b must be b star a and this is true for all a b ok.

So, a star b must be b star a and this is true for all a b. So, if this is satisfied then we call star is commutative. Similarly, we can define like for example, addition if we take a to be Z or N. Now, if we define a plus b, now this is the real number addition so, a plus b will know is same as b plus a. So, like 2 plus 5 is basically 5 plus 2 so, this plus is commutative, but minus is not commutative.

Subtraction is not commutative because 2 minus 3 is not same as 3 minus 2 2 minus 3 is minus 1 and 3 minus 2 is 1. So, this minus operates subtraction operator is not commutative. So, commutative also called Abelian; commutative property is also called Abelian property ok.

Now, we defined associativity when we call the operator is associativity property having associativity property. So, let us just defined that.

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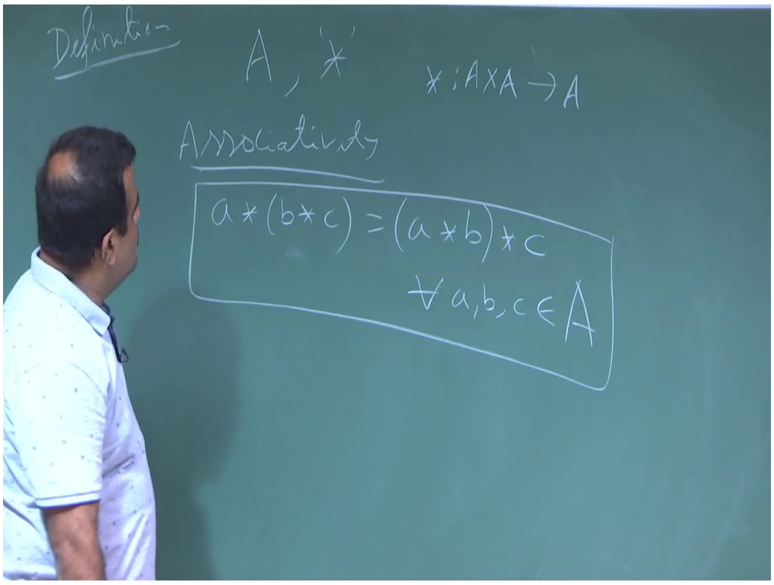

So, again suppose you have a set A and we have a operator on this set. So, this is basically a function from A cross A to maybe on A, if it is on a then it is a closure property. So, now, we call this is associativity if we take 3 element a b c; a star b star c is equal to a star b star c.

Now, for this we need to have this property to be closure because if it is not closure then a star because this operator is defined on A cross A I mean to A if it is closure. But if it is not closure then we are defining a a b star c now, b star c has to belongs to a because again.

So, b star c is an element of a then only we can operate so, this is some z so, a star z some sorry some d. Again similarly, this is some say e e star c so, that is why we for this associativity checking, I think we need to take this to be closure otherwise this has no meaning; because when we do the b star c this has to be a member in a, then although we can again operate with the other member of capital A.

So, that element will must be same as this one. So, and this must be true for all a b c belongs to this so, this is the property of associativity. Now, yeah so, now, we will take some example. So, plus is associative operation. So, if we define this plus say so, if we define this ah. So, a some of the example.

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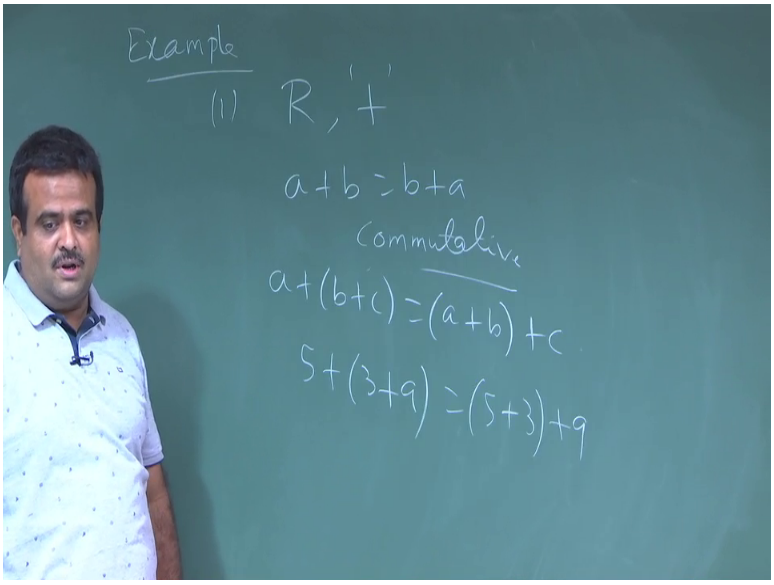

So, suppose we define R set of real number and will be we have a operator plus.

Now, we know, but the 2 plus is commutative because a plus b is equal to b plus c so, this is commutative. Now, also it is associativity because a plus b plus c is same as a plus b plus c we can take any 3 real number fraction also will ok, 5 plus 3 plus 9 is equal to 5 plus 3 plus 9. So, this plus is having associate this is coming from the real number property of real number.

So, this is this is this is true for all a b c.

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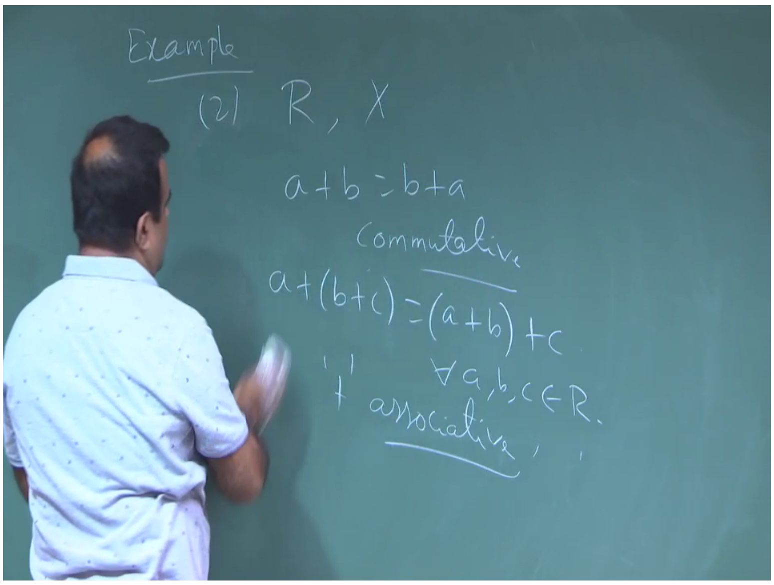

So, this is plus is both commutative and associativity associative. Similarly, if we have say dot multiplication real number multiplication.

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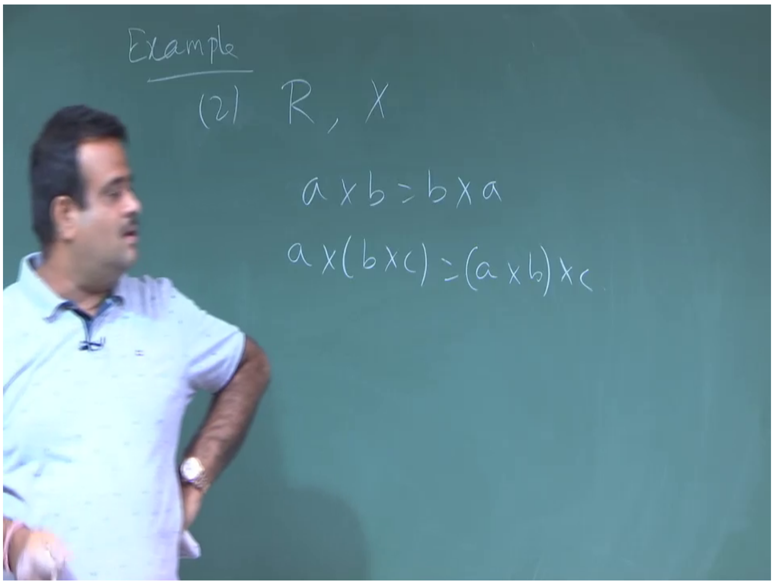

So, this is also because.

So, this is also because a dot b is equal to b dot a and it is also associativity is also there a dot b dot c is equal to a dot b dot c ok. So, this is also associativity.

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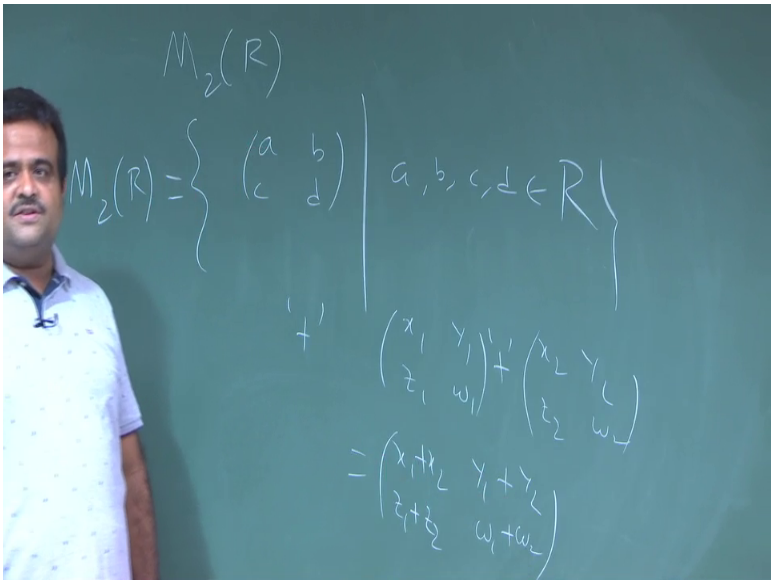

Now, if you take the matrices set of matrices say 2 by 2 matrices M 2 R where numbers are coming from real numbers. So, a b c d so, this set. This is a b c d belongs to R, this set is defined as all 2 by 2 matrices elements are coming from real number.

Now, on this matrix set if we define plus; we know how to define plus. We take x 1 y 1 z 1 w 1 plus this is the matrix addition x 2 y 2 z 2 w 2 then this is basically x 1 plus x 2 y 1 plus y 2 z 1 plus z 2 w 1 plus w 2 this is how we defined the matrix addition, this is plus.

Now, this these are all class of real number class because this entries are coming from the real number. So, this will give us a this we can show because of this these are real number addition. So, this is commutative also you can so, this is a associativity associative. Now, if we take the in into that means, matrix multiplication say how to define matrix multiplication say x 1 x 2 x 3 x 4 into y 1 y 2 y 3 y 4.

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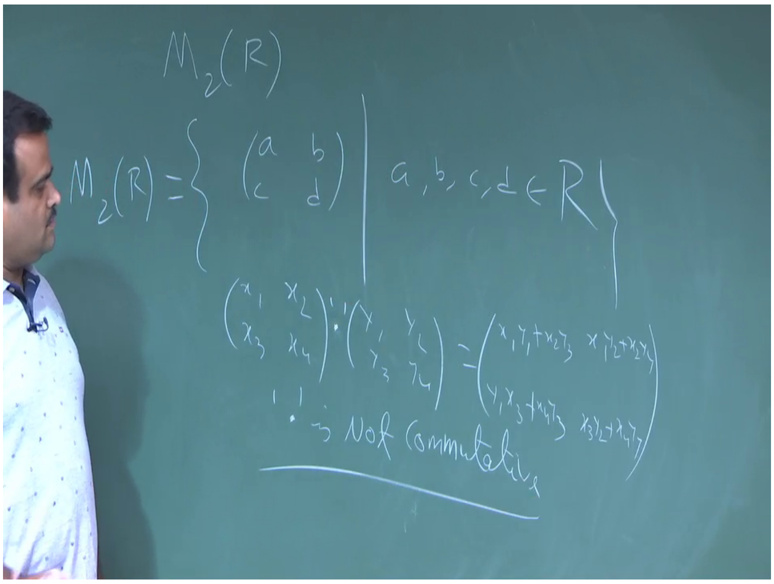

So, matrix multiplication is basically this will also give us the 2 by 2 matrix now, this. So, so, this is basically x 1 y 1 plus x 2 y 3 and then here x 1 y 2 plus x 2 y 4 and then we have y 1 x 3 plus this into this x 4 y 3, then this into this x 3 y 2 plus x 4 y 4. This is how we defined the product of 2 matrix this will also give us a 2.

Now, we know this, this is not a commutative operation. So, product is not commutative we can easily check that, this matrix multiplication this is not commutative ok.

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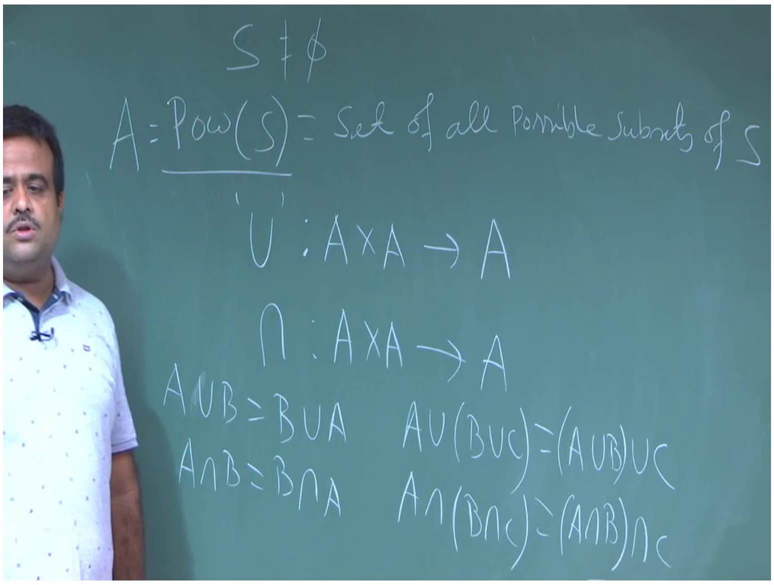

Now, we can take another example on say set suppose we consider suppose, you have a set S non-empty set we have a non-empty set S. Now, you consider the power set power pow of S which is basically set of all possible subset of S; set of all possible subsets of S which include the null set also and the set itself ok.

Now, now if we define the now this is our set A on which we are we would like to have a operation like so, we define the operation union and intersection; first of all union on the power set. So, so union this is also a closure operation. So, union is a if you take A cross A if you take 2 subsets, if you do the union this will give us a subset and intersection also if you take 2 subsets of S and if we do the intersection it will give us a subset of S ok.

Now, how to so, so the is this commutative like yeah. So, these are all this is commutative because we know the union operation like A A union B is basically B union A also intersection A intersection B is equal to B intersection A. So, these two are all these two are commutative operation and associativity yes, this is also associate associativity is there because A union B union C; we have proved writing in the loss of set theory A then B then whole union C.

Similarly, for intersection A intersection B intersection C is basically A intersection B intersection C. So, these two operation are both commutative and the associativity ok.

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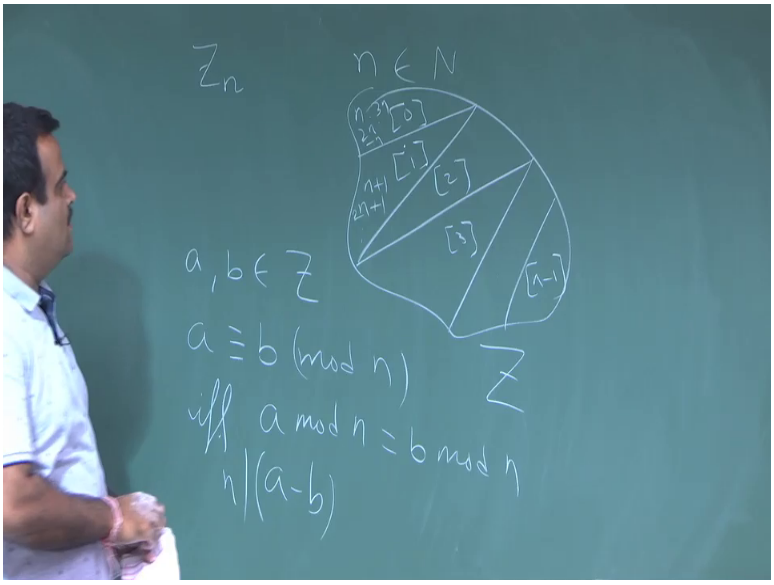

Now we will take another operation which is basically finite on the finite set. This is basically Z n Z n means, let n be a natural number. What we are doing, we are just taking this integer set and we are considering each integer we are taking mod n ok; that means, we take 2 a b 2 integer and we are defining a relation the modulo n relation if and only if. So, a is concurrent to b mod n if and only if a mod n is same as b mod n so; that means, we take 2 integer and we try to divide them both try to divided them by n.

Now, if they are giving us the same residue then they are in the same class so; that means, a minus b is a minus b is divisible by n. So, n divides a minus b so, that is the idea. So, so then they are same class and that class will be our this Z n basically. So, this is 0 class 1 class because if we divide any integer by n so, remainder will be either 0, 1, 2 up to n minus 1 2 3 dot dot dot up to n minus 1. So, these are the all points like n 2 n 3 n minus n because these are the points which are if we divide by n they will be just give the remainder 0 and these are the point we which is n plus 1 2 n plus 1 like this ok.

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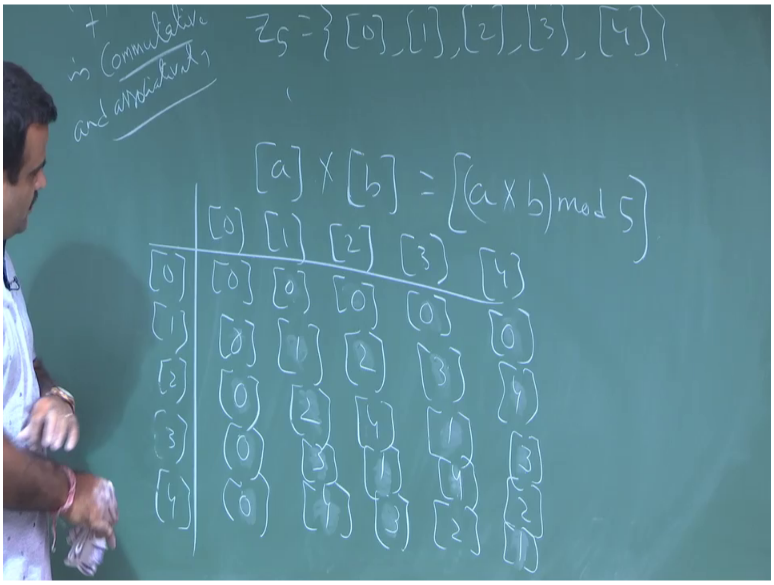

Now, this set Z n on this set we can define the operation like plus and multiplication like if you take n is equal to say 5 for simplicity. So, we are in Z 5 the Z 5 means 0, 1, 2, 3, 4 this is our Z 5 ok. Now, how we defined a operation like plus; plus means if you take any 2 element a b a plus b is basically we are defining as a plus b mod n. So, here n is 5 mod 5 and this will give us a another class ok.

So, this we can just since there are finite number of element we can just have a table on this; say 0 1 2 3 4 then 0 we want to see whether this plus is closure or not and also you want to see this is a commutative or not 2 3 4 ok. So, now with 0 if we add is up it will give us all this because 0 plus 2 is 2.

Now, with 1 0 plus 1 is 1 0 1 plus 1 is 2 this is 3, this is 4 then we have 1 plus 4 is 5. So, 5 mod 5 is 0 then this is 2, this is 3, this is 4 then again this is 0 then this is 1.

So, then this is 3, this is 4, this is 0, this is 1, this is 2 like this. So, the last one is this is 4, this is 0, 1 2 3 ok. So, this we can verify easily that this is a closure property is satisfying because all the elements over here are coming from Z 5. So, for simplicity we can just remove these brackets and you can just have this 1 0 1 2 3 4, but anyway this is ok.

So, this is all the elements now is this commutative this is we can check this is easily you can check commutative, we can take any 2 number say 1 1 2. So, this is giving us 3 which is same as 2 1 which is also this class. So, this is a commutative and also we can verify this is the associativity also so, this plus commutative and associativity ok.

So, again insert a plus we can just defined a into like so, this will be this will be a into b mod 5. So, if you do that then it will change like this so, if you multiply this all will be 0 0. Now, with 1 this will be 0 this will be 1 this will be 2 this will be 3 4.

With 2 so, these will be 0 for all so, with 2 this will be 2 and this will be 4 2 into 2 4 4 more this is ok. So, this is 4 and this will be so, 2 into 6 6 mod 5 is 1 and this is 8 8 mod 5 is 3 then come here. So, this is 0 this is 3, this is 6 6 mod 5 is 1 and then 3 3 into 6 9 9 mod 5 is basically 4 and then 12 12 mod 5 is 2. Then 4 this is 0, this is 4 and this 1 is 8 8 mod 5 is 3 and this is 12 12 mod 5 is 2 and this is again 16 16 mod 5 is 1 ok. So, this is the way how we define this.

Now, is this commutative we can check that if we take this 2 3, 2 3 is giving us 1 and this 2 3 is giving us 1. So, it is commutative. So, this is one example all right we have this now this is the over the finite set. So, if it is if the operator is over the finite set we can just have a table and then we can check whether their commutative or their associativity or not.

So, in the next class we will start some algebraic structure based on this operator and then we will slowly this is start the concept of group. Basically, we start with the group then we slowly go to the group.

Thank you.