Engineering Mathematics - II
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Lecture 2
Vector and Scalar Fields

So, welcome to lectures on Engineering Mathematics 2 and this is module number one lecture number two on vector and scalar fields.

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So, today we will cover that what are the vector fields and what are these scalar fields and most importantly we will also cover the concept of this directional derivatives which is very important and it is the generalization of the derivative what we have learned in calculus.

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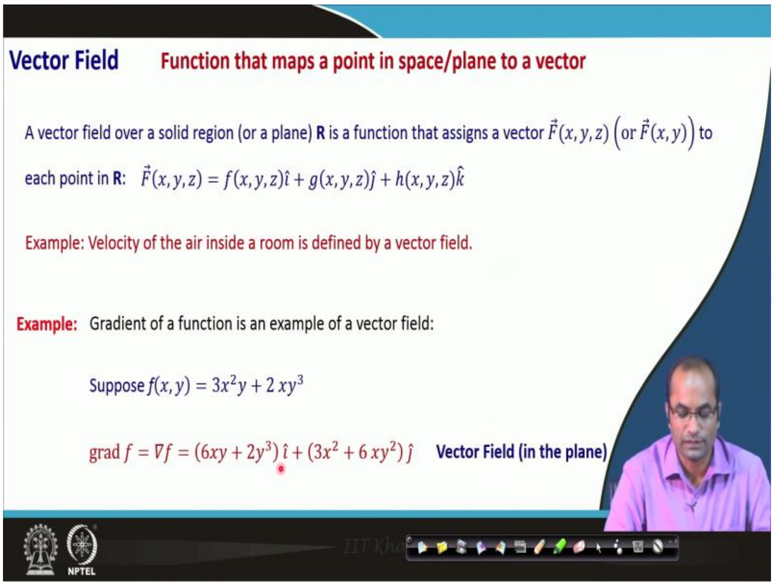

So, these vector field are these functions basically that map a point in a space to a vector. So, now instead of a scalar value we will have the vector value. Like in the previous lecture, we have learned the vector functions of a single variable. So, here instead of single variable we can have the several variable, but the output will be also a vector. So, we can define such a function by this vector here F, whose components are given by this fg and h.

So, what is important here that we are a supplying point here in this space and we are also getting as a output a vector So, this is what we call in vector setting as a vector field. So, these are the functions for instance from R3 to R3. The well-known example we can think of is the velocity of the air for instance inside a room is defined by a vector field. So because at each point, in the room, we have a function which can be velocity function, which is what we call the vector field.

So another example is the gradient which we have learned in the last lecture. So, the gradient of a function is also an example of the vector field. So, for instance, we have a function here of 2 variable 3 x square y plus 2 x y cube, and if you find the gradient, then we have a vector function here, which is given by this expression and then so this is the function which takes a point in R2 plane and gives a vector in R2. So, again we can call this as a vector field in the plane.

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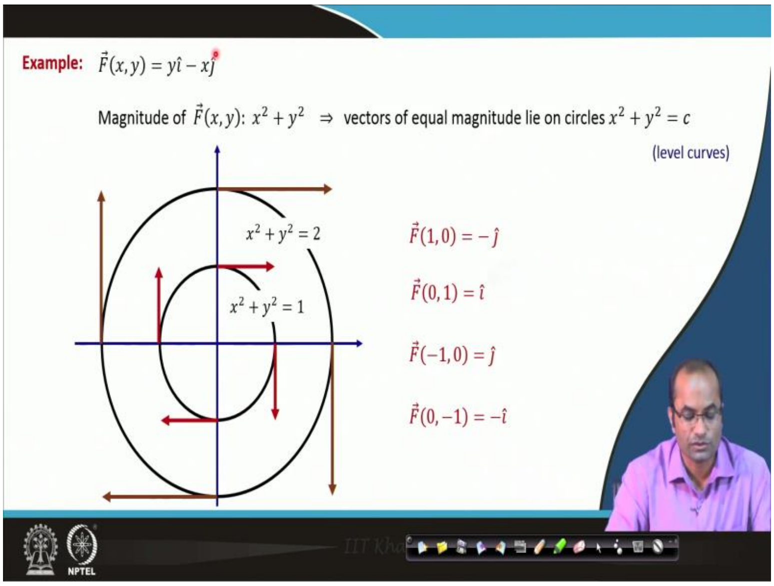

Now, the question is that how to visualize these vector fields that is another important component here which we will discuss through this example. So let us consider, that the vector field is given by this yi minus xj by this function. What is in important that the magnitude of this function here f at x, y is given by the z square plus y square. So what we can note down here that the vectors of equal magnitude because the magnitude is x squar plus y square.

So the vectors of the equal magnitude lie on the circle x square plus y square is equal to c, because the magnitude is c now. So if we have one, for example, c is one, then all the vectors of magnitude one will lie exactly on the circle, or we change the c to 2 for instance, then all the vectors of magnitude 2 will lie on the circle x square plus y square is equal to 2.

So through these level curves, we can define the vector valued functions. So in this case, we have drawn 2 cases layer 2 circles only. So x square plus y square is equal to 1. The outer one here is x square plus y square is equal to 2, but we can draw many more just changing the value of c. So for instance, now we want to visualize here that how this f 1, 0 at this point 1, 0 so we have this point here, 1, 0 and the value of this vector function is given by the minus j.

That means in this direction here, opposite to the direction of j. So this is the visualization. So we have a vector here at 1, 0 which is given by this minus j, and so on at each of these points on the circle, we will have a vector of the same length 1 because this is the circle here x square plus y square is equal to 1 and the magnitude of this given vector on this circle will be 1. So at any point if we draw a vector, so that magnitude will be 1. So for instance, we take another point the 0, 1 so x is 0 and then why is one so, at this point here x is 0 and y is 1. So it is given by I.

So, that means, we have this I so which can be again described by this vector. Similarly, we can talk about the another point minus 1 and 0. So, here we have the j, the vector is j so in this direction, and again 0 minus 1 we have in this direction. So, that is the visualization on this particular curve we have seen for this vector.

So, these all vectors have the length 1, but for instance if you draw on the outer circle whose radius is 2. Now, the vector will be just double in length, then what we have in the circle one similarly. So, we can have several concentric circles and on each circle we can draw the direction of these vectors so we can visualize that how this vector field looks like in this case, for instance, we have this 2 dimensional setting.

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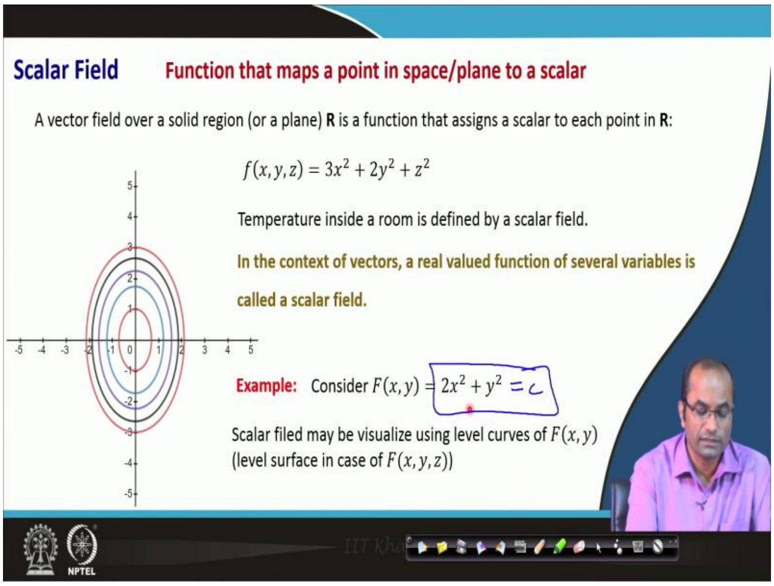

So, coming to the scalar field, so these are the funssssction that map a point in a space to a scalar value. So, scalar field means so, we have the value of the function is a scalar and not a vector. So, a vector field over plane or it can be a solid region is a function that assigned a scalar to each point in R so, as an input this will be a point in the… it can be in 3 dimensional space, but it will assign a scalar value.

So, for instance this one f x, y, z given by this one, so, this is a scalar values. So therefore, we have the we call it a scalar field. For instance the temperature inside the room is defined by the scalar field because temperature is a scalar quantity. So, at each point of… in the room, we have some function that can tell us the temperature of the room. So, in the context of this vector simply this aerial valued function of several variable is called a scalar fields, there is nothing new here only the term the scalar field will be used in this context of vectors otherwise these are the functions of several variables.

So, for example, we have f x y is equal to 2 x square plus y square. So, again here the visualization either if it is 2 dimensional we can just directly plot or there is another way which we can visualize by this level curves in 2 dimensional or levels surfaces in the 3 dimensional. So, here we will take like this 2 x square plus y square, putting some constant and for varying these constant we can have different different curves and these curves will tell that the value of this function is same on this curve.

So, for instance if you plot this for taking different value of the constant, so, if you said to this constant here and wearing this value of the constant we can get these parabolic goes and on each curve the value is at any point is same so, that is what the visualization we can think of for such functions.

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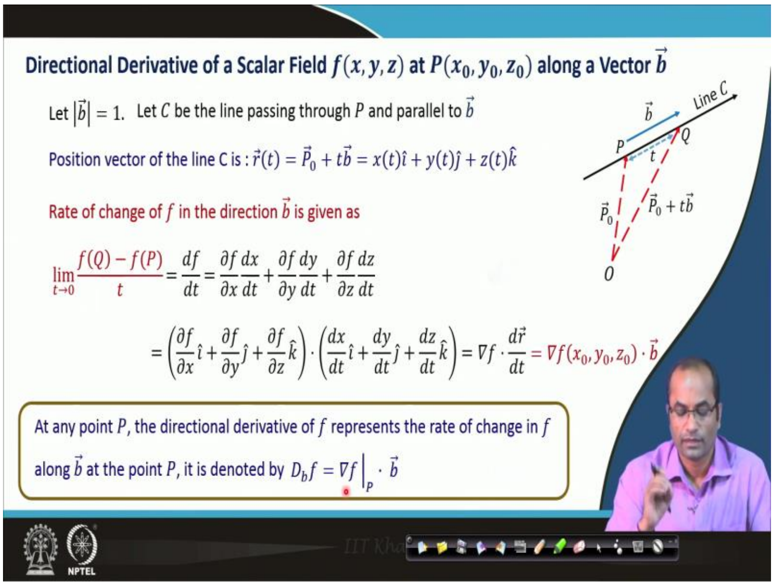

Coming back to this directional derivative, which I mentioned, it is a very important concept which we will study now. So we are talking about the directional derivative of scalar field. So, suppose we have f x, y, z scalar for it and that a point is x naught y naught z naught along a vector b.

So, what is the difference here what we are calling the directional derivative we have studied already the derivative mainly the partial derivative in the direction of x partial derivative with respect to y partial derivative with respect to z, but if you want, if you want to have the derivative of a function in a given direction not in the standard direction x, y, z then what how to compute that, because that is more general and more interesting, because our interest may be that how this function changes in a particular direction, not just with respect to x, y and z.

So, here the concept of this derivative will be generalized by this given direction b. So our interest is how this vector… how this scalar f this function f changes in the direction of b meaning. So this is our setting. So we have a point P our interest where we want to get the directional derivative in the direction of this vector b. Suppose, the b is a unit vector. So if you take another point q here on this line, which is passing through this p in the direction of b, then this can be given as p naught plus this t times b.

That is another point here the queue or the general point Q on the line. So if we consider this the position vector of this function here on this line c, that will be given by P 0 plus tb or we can define by general function as we do xt yt and zt but we can since the B is given and the P naught is given, so that Rt will be given by p naught plus tb. And the rate of change of this f in the direction of b, we can compute as that f evaluated at q minus because now we are in the direction of b only along this line we are moving in the direction of b from the point p.

So, what we are computing here that f at q minus f at p divided by this distance t here and then we are interested that what will happen to this course and st approaches to zero because we want to find this directional derivative at the point P. So that this is exactly the df over dt you want to compute and using the chain rule we can have because f is given by this one f x y, f is a function of x y z So, we can have the partial derivative with respect to x and dx over dt partial derivative with respect to y d by dt and so on. And now this we can rewrite in the vector setting as this which is the del of or the gradient of f into this dx dt dy dt dz dt and that is exactly the R prime the derivative of R.

So, what we have here the del f into dr over dt, but the interesting is here that dr over dt is nothing but… if we have this rt is equal to P naught plus tb. So, dr over dt is just the vector b. So, this is the vector b because that is the tangent line at that point. So, we have this dr over dt is just the b. So, the gradient f evaluated at x naught y naught z naught multiplied by this b will give us so b was the unit vector.

So, this will give us the change here, the rate of change in the direction of this b. So, at any point, the directional derivative of f represents the rate of change in f along b at the point p and this is denoted by this dbf or, and it can be computed as the gradient of f evaluated at the point p, and it is dot product with the unit vector B.

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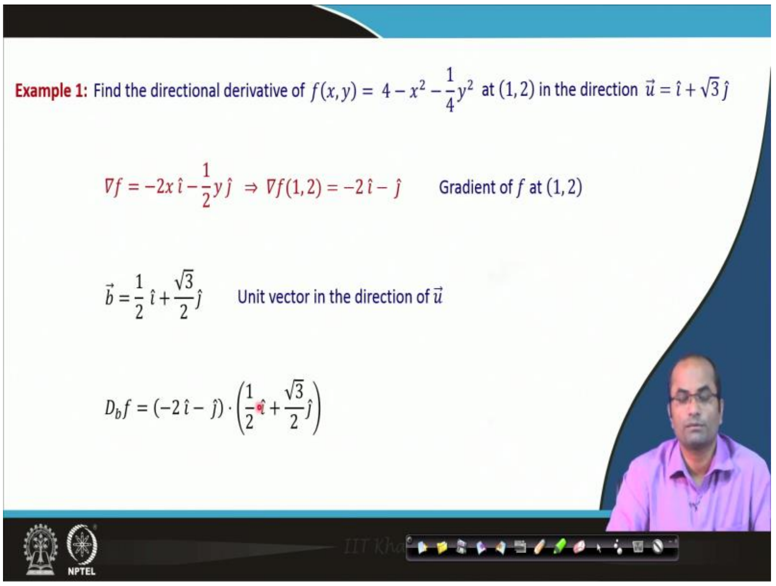

So we will go for some examples. So find the directional derivative of this fxy is equal to 4 minus x square minus 1 by 4 y square at the point 1 2 in the direction of this, so the direction is given i plus square root 3 j. So first, we need to compute the unit vector in this direction of this u to get this also we need to evaluate the gradient of f which is just the partial derivative of this with respect to it that is minus 2 x ith component and the yth the second component will be with respect to y that means you will get here 1 by 2 y so that is the gradient f. And we want to evaluate at 1, 2.

So this will be given by minus 2 y and minus j. So this is the gradient of this function f at the point 1, 2. The B, which is the unit vector in the direction of this vector u given u i plus the square root 3. So the length of this vector is 2, so we have divided this by 2, 1 by 2 and a square root 3 by 2, that is the unit vector in the direction of u and now the gradient, so the directional derivative of this F in the direction of B will be given by this dot product. So we have the gradient of f at that point. And then we have the unit vector B and this dot product will give us that directional derivative which is given by minus 1 minus the square root 3 by 2.

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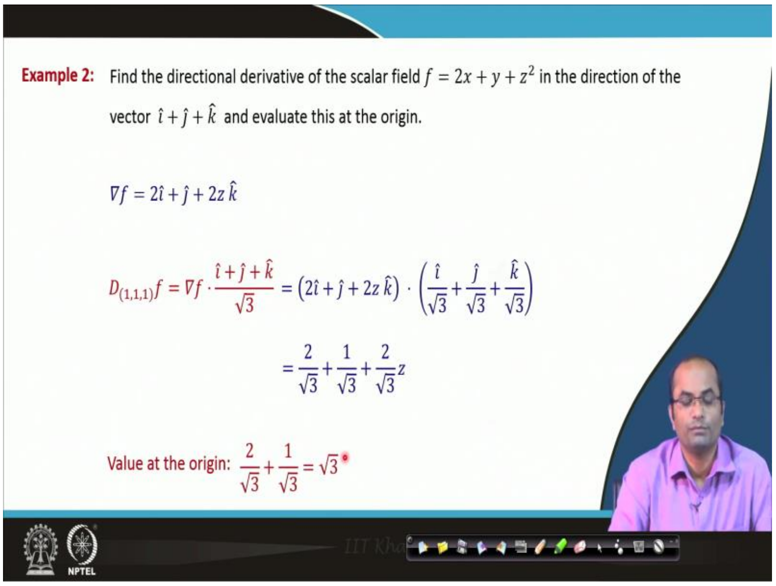

Another example, so we want to find the directional derivative of this scalar field 2 x plus y plus z square in the direction of i plus j plus k. And we want to evaluate that at origin. So, again, a similar example, we have to get the gradient of f that is 2 I plus j plus 2 zk. And then the directional derivative will be given if you multiply this by the unit vector, so the unit vector in this direction a plus j plus k will be given by this expression. And if you make this dot product, so we can find out the directional derivative at that point, sorry, at the point 111 in the direction of this i plus j plus k. And if you want to evaluate this at origin, so we can put this z to 0 and the value will be square root 3.

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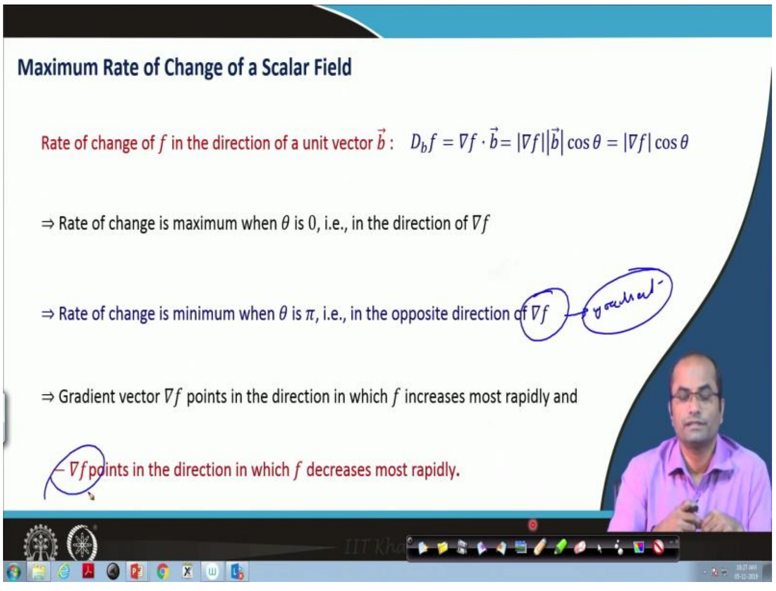

So, now, we want to get the maximum rate of change of a scalar function and where this directional derivative plays an important role. So, the rate of change of f in the direction of a unit vector B is given by this directional derivative that is precisely how we define this directional derivative. And if we see that this dot product here is the magnitude of this delta f, magnitude of b and cos theta.

So, this magnitude of b is 1 because we take this unit vector there. So this directional derivative or the rate of change of f in the direction of b is given by del f multiplied by cos theta. Now, it is important to note here that this rate of change in this direction of vector b will be maximum when the cos theta will be one that means the theta will be 0. So, the theta is the angle between the del f and the unit vector b.

So, if we take the unit vector v in the direction of del f. So if we take, if we compute the directional derivative in the direction of this del f, then the magnitude is going to be the largest one the maximum one. So, this is the point here that the rate of change is maximum when the theta is 0, that means, in the direction of the gradient f, the rate of change is minimum.

So, when this is going to be minus 1 for instance. So, the theta 1 theta is pi that means in the opposite direction of delta f, so, the rate of change is minimum when theta is pi and this is the direction that is opposite to the del f so, we got two directions in one direction the rate of change is maximum that is the direction of the gradient f. And there is another direction, where the rate of change is minimum, which is just opposite to del f. So, again to summarize that the gradient vector delta f point in the direction in which f increases most rapidly and the minus del f points in the direction in which f decreases most rapidly.

So, this is a very important concept which is very much used in many applications, because, by simply computing this delta f the gradient f, we are getting now the direction where this function this particular function f changes most rapidly and on the other hand we can also get by this minus delta f a direction where the delta f changes they decreases most rapidly.

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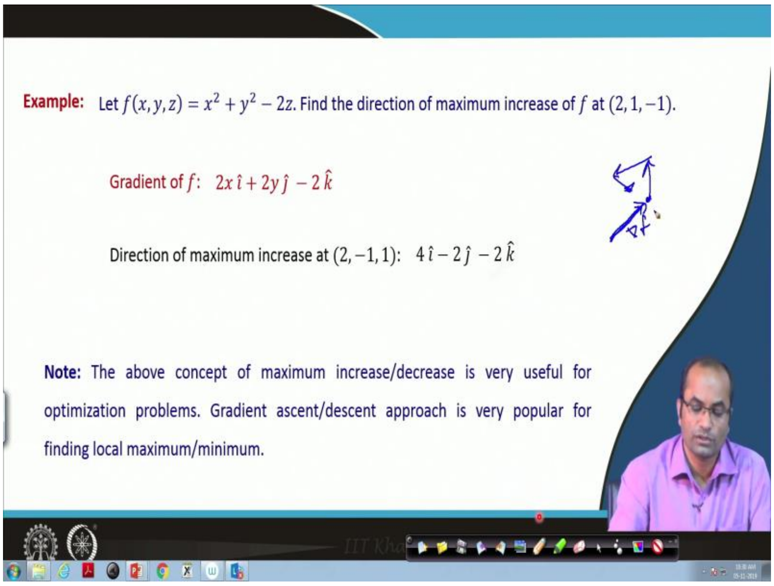

So here let us take an example here f x, y, z is equal to x square plus y square minus 2 z and we want to find the direction of maximum increase of f at this point 2, 1 and minus 1. So, we compute the gradient of f. So, gradient of f will be computed by taking the partial derivative with respect to x. So, that is going to be 2 x and the second component will be 2 y the third component will be minus 2. So the gradient of f will be given by 2 x 2 y and minus 2.

What do we need we need the direction 4of maximum increase at this point 2 minus 1 and 1. So the direction for the maximum increase will be given by exactly the gradient of f the direction of the gradient of f. So we will evaluate the gradient of f at this point, that means we have 2 into 2. So, that is 4 i, then we have minus 2 j and then we have here the minus 2 k. So, this is the direction of maximum increase of this function. So, if our interest is that, in which direction this function increases most rapidly, so, that will be the direction given by this gradient of f.

So, just a note this concept of maximum increase or decrease is very useful for optimization problems, which is well understood from this discussion. Because we are by finding this gradient we are actually getting the direction of maximum increase or by finding this minus grade f direction where the function decreases most rapidly. So, in this way if we move from a point in the direction of this gradient, we can either go in the direction, where the function is increasing most rapidly or decreasing most rapidly if we go in the direction of minus del f. So, in either way, this is useful to find the maximum value of the function or minimum value of the function.

So, there is a well known algorithm so called this gradient ascent or descent approach, which is very popular for finding local maxima and minima. So, we start from a point, compute its gradient and go so, we start from a point and then let us say there we find out the gradient of f. That is, we are finding the maximum for instance, so, we will go in this direction with some step. Then again we will find the direction then again we will find the direction and finally, we will… this process will take us where the function has its maximum or the local maximum. And similarly, if you go with the other direction minus del f, then we are able to search for the local minimum of the function.

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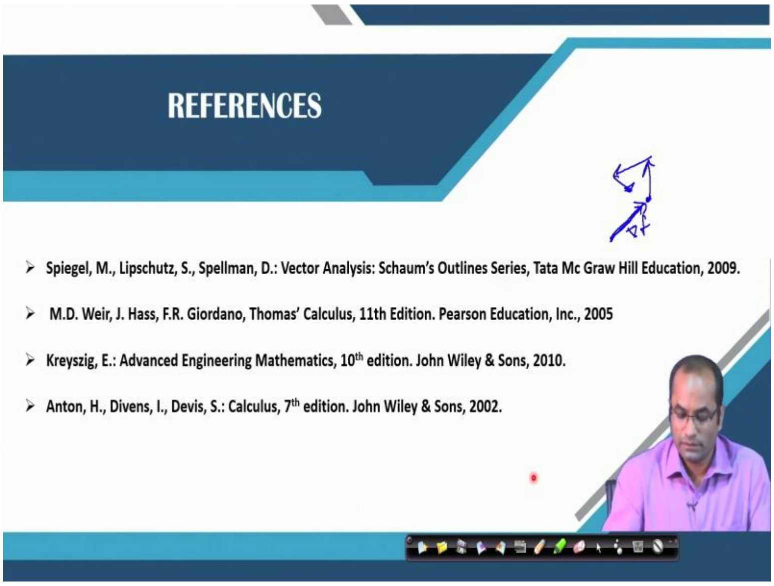

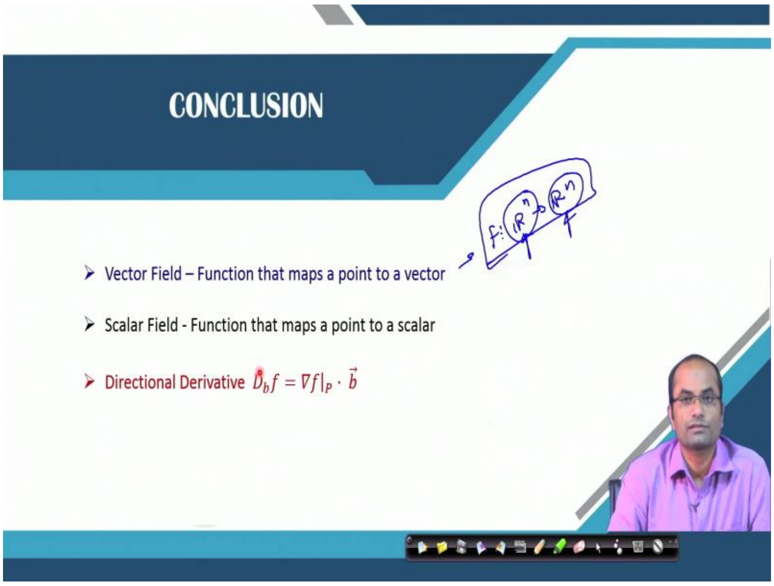

So, these are the references we have used to prepare this lecture and just to conclude, so, we have defined that what are the vector fields so, these are the function that maps a point to a vector so, in general we have function here from Rn to the Rn. So, the input will be also from the Rn and the output will be also in Rn that is what we call the vector field and the scalar field these are the functions of several variables which we studied in the calculus.

So, these are the functions that map a point to a scalar and the important concept which we have covered here, that is the directional derivative, which we can compute just by the gradient of f at a point P. And then if you multiply this by the direction where we want the rate of change of this function and this is the unit vector, so you will get this directional derivative of the function which has several applications in optimization, for instance, and other numerical process. So I thank you for your attention.