Engineering Mathematics II
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Lecture 24
Roots of Algebraic and Transcendental Equations

So, welcome back to lectures on Engineering Mathematics II and this is lecture number 24 on Determination of Roots of Algebraic and Transcendental Equations.

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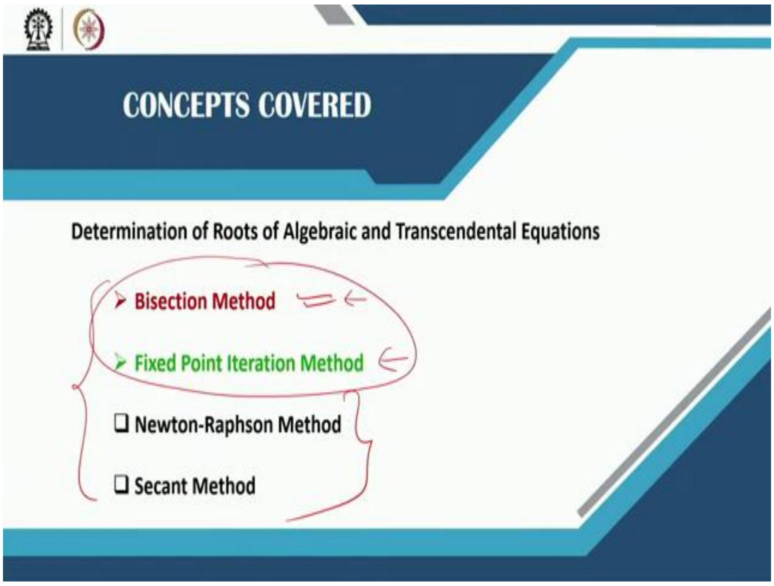

So, today will be talking about the several methods which I mean numerical methods that can be used for determining the roots of Algebraic and Transcendental Equations, one of them is the bisection method one of the simplest approach to get roots of Algebraic and Transcendental Equations.

The second approach we will be talking about the fixed point iteration method, and then there are two more approaches which we will cover in the next lecture. So, Newton Raphson method and Secant method. So, basically these four approximations algorithm will be talking about in this in two lectures. So, the first two we will be covering in this present lecture and the next two will be the topic of next lecture.

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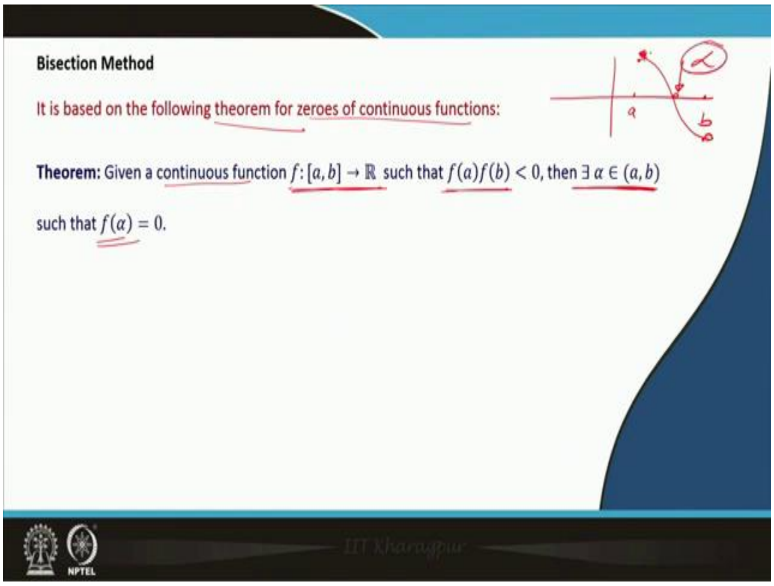

So, coming to the Bisection method. It is based on the following theorem of for zeros of continuous function. What is this theorem? That given a continuous function f from this a, b to R such that fa fb is less than 0. So, the product of the value of the function at a and f at b is negative and that indicates that.

So, for instance this is a and this is b so, one for example is positive and another way it is negative or it can be negative at a and then it can be positive at b. So, that the product is always negative. So, one of them is positive and the other one is negative then only this is possible. So, naturally the if the function is continuous it will cross the x axes so, it must have a root then so, that is the principle behind this bisection method which we will be exploring a bit more in this lecture. So, there exist alpha in this interval a, b somewhere in between a and b so, that f alpha is 0 so, this is exactly the point alpha where this graph of f meets x axes.

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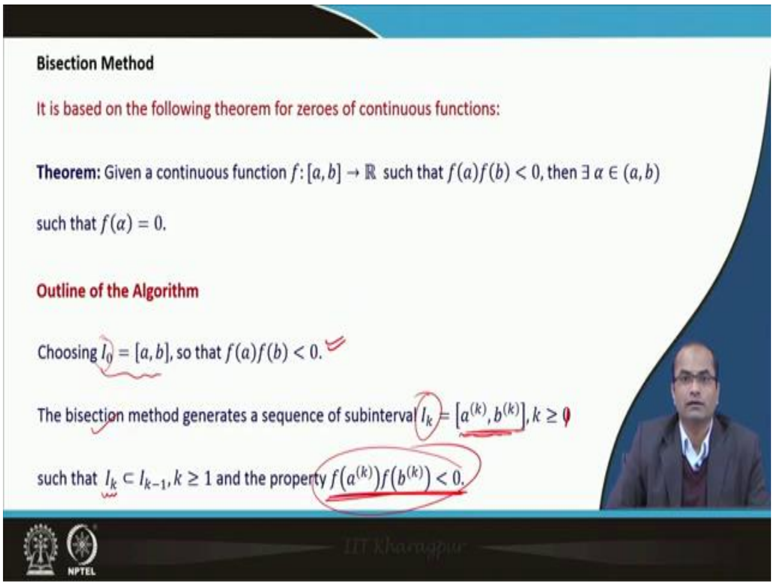

So, based on this principal we will develop an algorithm to find zeros of a continuous function and the outline of the algorithm is as follows that we choose the interval I naught that is the whole interval ab which makes sure that there is a root between these between the points a and b or in the interval a, b.

So, we are calling this let say I naught this interval a, b which has this property or which ensures that there is a root in between a and b. The bisection method generates a sequence of this subintervals that means we are calling it Ik. So, k greater than equal to 0 or greater than equal to 1.

So, I naught is already defined so, we can go with let say I1 also there so, a naught, a1, b1 and a2, b2, so we will go for the sequence of subinterval and what will be the property of these subintervals that the subinterval here will be the subset of the earlier one the for example I1 will be the subset of this I0 and I2 will be the subset of I1 and so on.

So, basically we are narrowing down the interval subsequent interval and what is the property of these intervals so these end points again every time this property is fulfilled that f ak bk f bk is less than 0. That means while narrowing down the interval we are making sure that the root lies in this new interval.

So, this process of narrowing down the interval will lead to the root at the end or in the case when this k approaches to infinity that we will also show that it this process will lead to the determination of the root. Or every time we are narrowing down the interval that means we are getting better approximation for the root because this condition makes sure that the root lies between this end points of the new interval.

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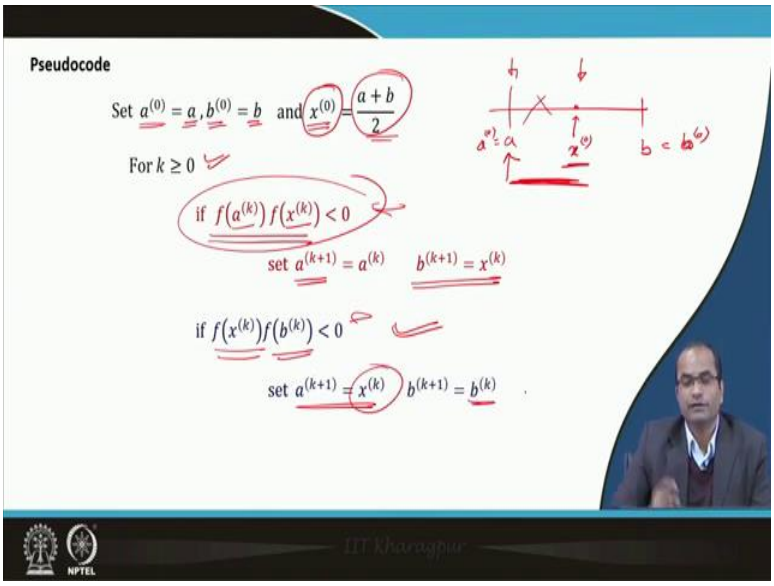

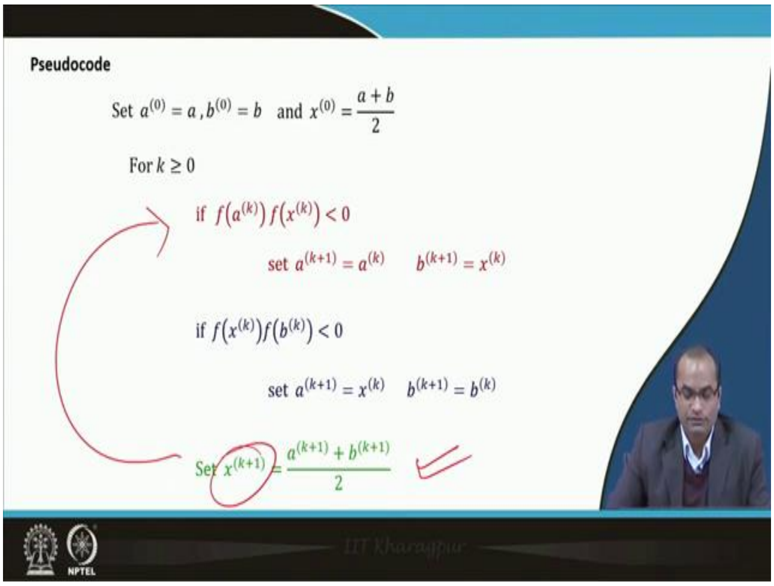

Well, so just a Pseudocode which can help immediately to code or the write a program in the computer. So, what we will do, we will set a naught, b naught as the given interval a, b where, which has the property that fa fb is less than a 0, and then we will take the middle point this is what the bisection the name is also coming.

So, we are bisecting the given interval into two equal parts by taking this as x naught which is the middle point of the two end points and then the algorithm will go as follows. So, now we will check for k greater than equal to 0 so, let say k is 0. So, we will check whether a0 or x0 is negative if this is the case that means.

We have for example, this was a and b which we are denoting as a naught and b naught also and then middle point we are calling it as x naught and then we are checking whether this f, so the function value here and function value their they have this property that the product is negative that means the root is somewhere in between this interval and if this is the case we will set now the new interval as the left point of the new interval again a because our interval is this one now.

Because we know that the root lies in this interval and the other end of this (inter) new interval will be x naught, so this bk plus 1 will become xk, if this is not the case then fxk into f bk will be negative, one of them will be negative because there is a root in between so either it will be there or it will be between xk and bk.

So, if is, if it is between xk and bk we will set our interval accordingly. Now, the left point we will take xk and the right point we will take bk. So, our new interval is set now. And then we will go for the middle point again as xk plus 1 and then we will move for iterate for this k to get these sequence of these xk's which are actually the approximation of the root and from the algorithm it is clear that as we move further the xk will be a better approximation certainly.

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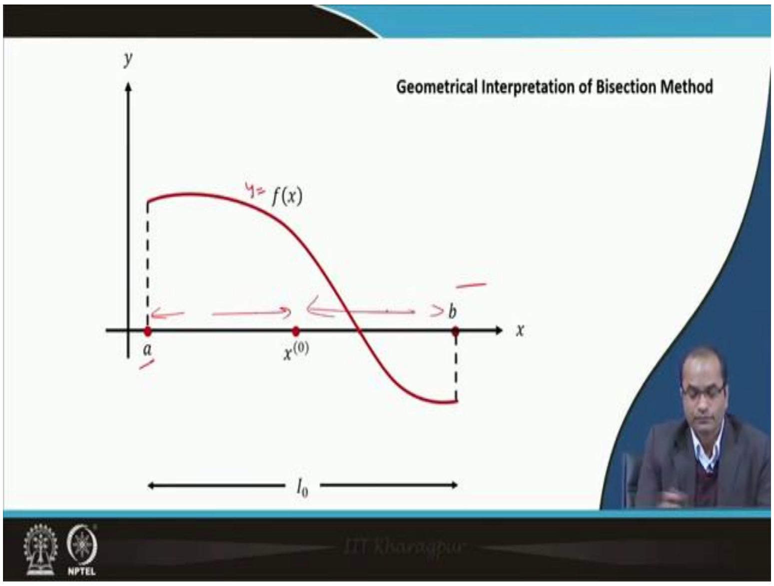

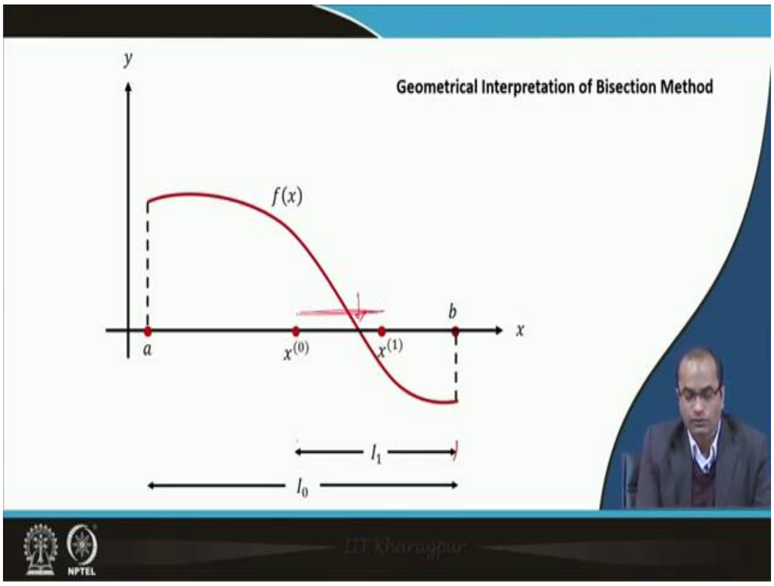

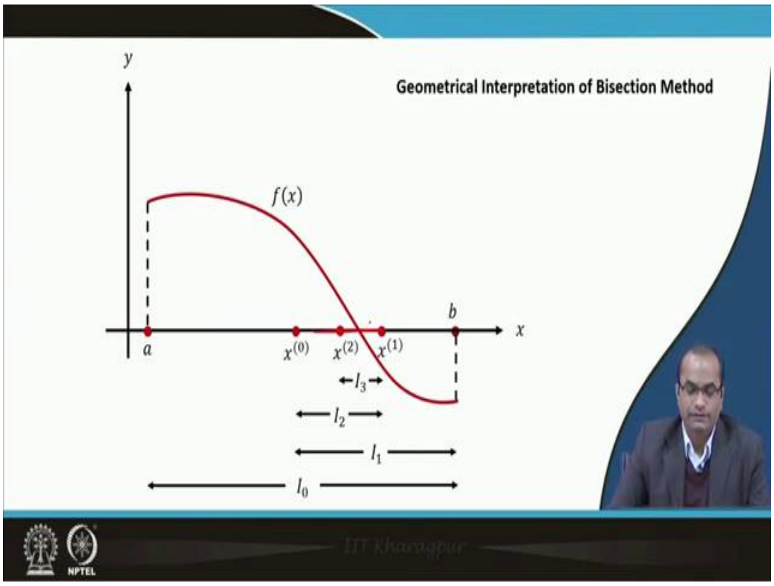

Coming to the geometric interpretation which is also clear which was clear from the algorithm but we can again look into it so, suppose this is the curve of this y is equal to fx and then these are the two points a and b, where we are sure that there is a root. So, fa and fb is negative so this graph will certainly cut this x axes somewhere.

This is the interval I naught in our notation and then we go with the middle point so that is x naught and what do we see that there are two intervals now one is this side here from a to x naught, other one is here from x naught to b, but that condition again we will check whether fa x naught the product of the function value at a and x naught or x naught and b which one is negative so certainly because root lies here so the other one will be negative so we will have this I1 our interval of interest is I1 which is the right hand side of this whole interval.

And now also, this is again we will take the middle point suppose this is x1 and then again with this criteria of checking where the root lies and it will lie obviously in this left interval now so the root will lie here and therefore, the I2 will become this interval from here to here again we will go for midpoint and then see that this is I3 and so on. So, we are narrowing down this interval and then the middle point of it will give the approximation of the root so as we proceed further we will have a very very small interval and then again midpoint of this interval will give the approximate root of this function fx.

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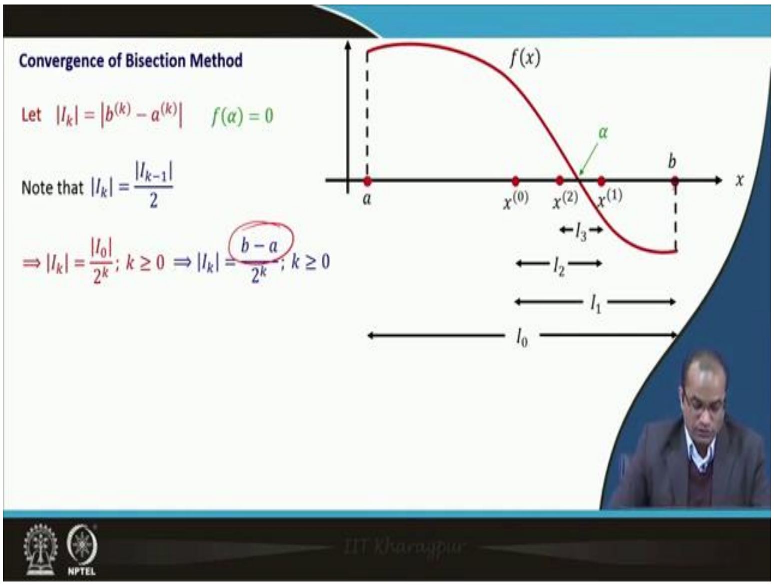

Well, so talking to the convergence of the bisection method which is the beauty of this method is that it always converges which we can see again mathematically though geometrically it is clear that it will converge because we are bisecting the interval always into two parts and narrowing down the intervals so certainly it will go.

It will go to the actual value as k approaches to infinity. So, we denote the length of this interval by Ik and the absolute value here so bk minus ak the absolute value we are denoting the length of the interval. And we know that f alpha is 0 so we are searching for this alpha and note that this Ik is half of the Ik minus 1 so the I1 is for example the half of the I naught the length of I naught or I2 is half of the length of I1 because we are bisecting the interval every time.

And by just induction we can, we can see because here we have like I1 is I naught by 2 and then I2 if we go then that is I1 by 2 and I1 is already I naught by 2. So, it is a 2 square so, I2 here is giving us this 2 square, I3 will give 3 square and so on. So, Ik is giving here 2 power k with this I naught and I naught is nothing but the b minus a that is the beginning interval which is b minus a.

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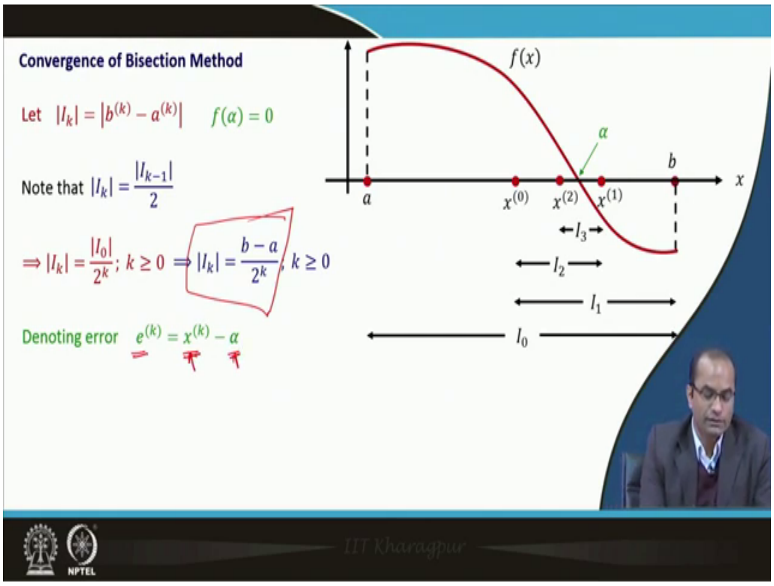

So, we have this relation that the length of this Ik is equal to b minus a divided by 2 power k and now if we denote the error by this xk minus this actual value alpha so this is the approximate value, this is the actual value. So, this is the error ek which we are denoting by xk minus alpha.

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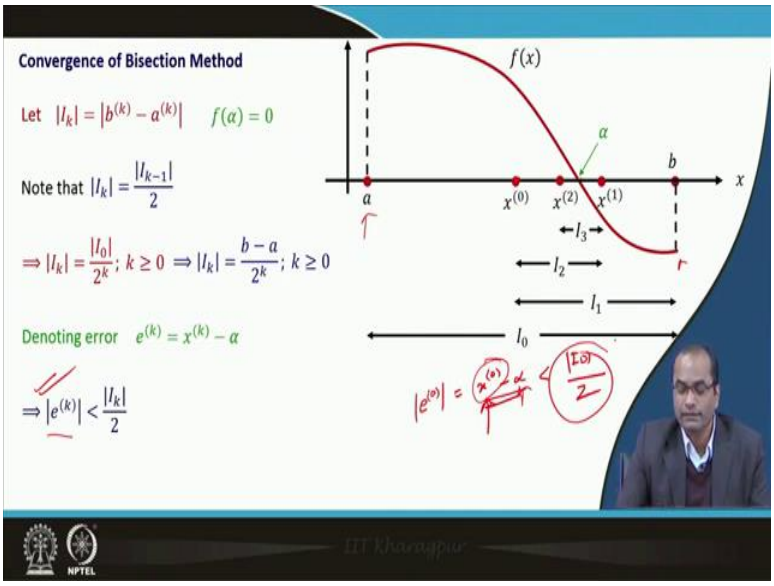

And what we know also that this error will be less than Ik by 2. Why is so? So, for instance consider I naught so, e naught so, the error at the beginning. So, what is the e naught? e naught is just the x naught and minus the alpha so x naught is somewhere there and minus alpha which is somewhere in this interval. So, this x naught minus this alpha if we look at it must be less than the half of the interval I naught because I naught was from there to there and then x naught was the middle point and we know that somewhere in this interval a to b the root lie.

So, definitely the difference between this x naught and alpha will be less than half of the interval length. So, this is what we have written here. So, this will be less than the interval length by 2 because this is somewhere in between, this is exactly in between and this is going to be either the right side of this or the left hand side of this. So, this distance between x naught and alpha will be certainly less than I naught by 2.

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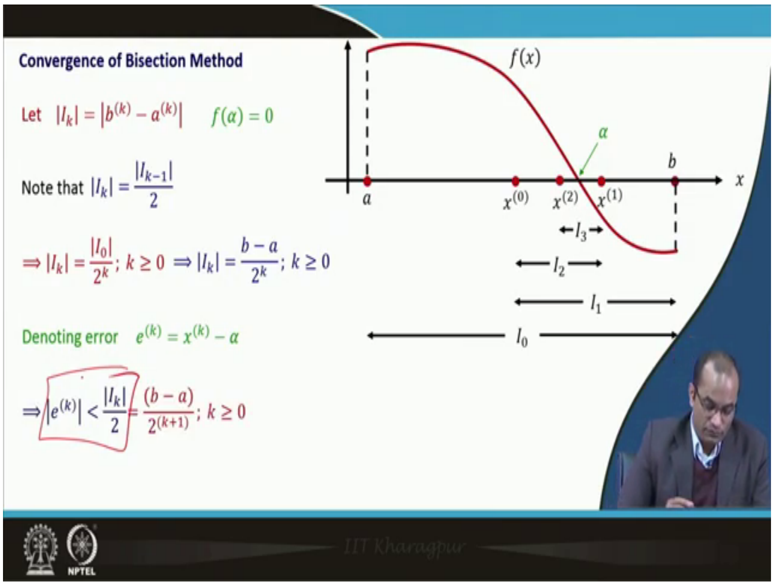

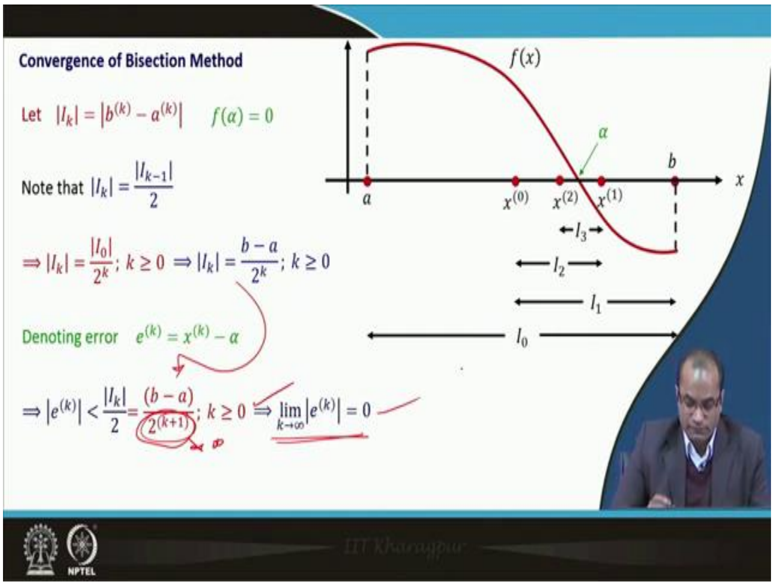

And the simple logic can give us in general that ek will be less than Ik by 2. So, having this now we know the relation of Ik which we can substitute from there b minus a2 power k plus 1 and this is true for all k and then if we take the limit that k approaches to infinity. So, this will go to infinity here and then b minus a is some number so, going to infinity means this will go to 0.

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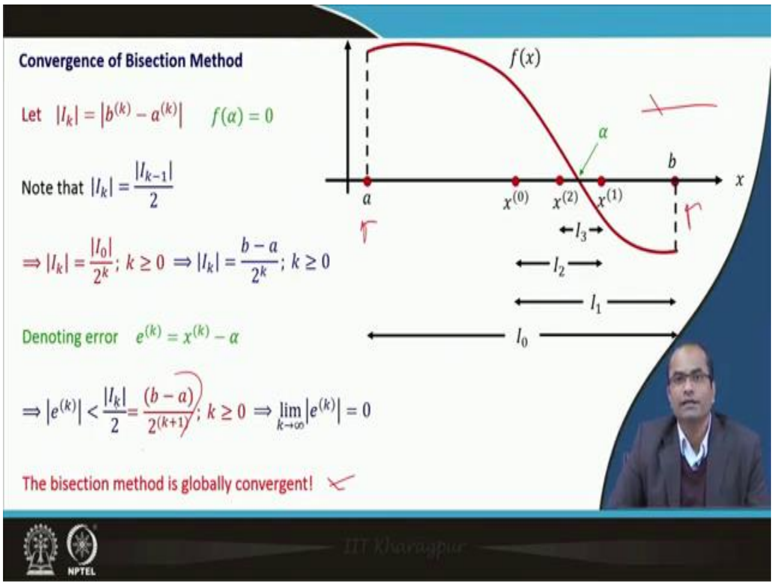

So, the error will definitely go to 0 which is clear from the geometrical interpretation as well or mathematically we can see here. So, what is the advantage of this bisection method that it is globally convergent, the convergence is global we do not have to worry, we do not need any other condition then to have this continuity of the function and which has some root in a given interval so we have to find that interval.

So, once we have that the convergences guaranteed, we do not need any other condition on the function or its derivative, etc. So, that is the main advantage of this method but the disadvantage if we talk about the convergence is not very fast because we are just dividing this by half and half so it is kind of linear convergence which is the only drawback of this bisection method.

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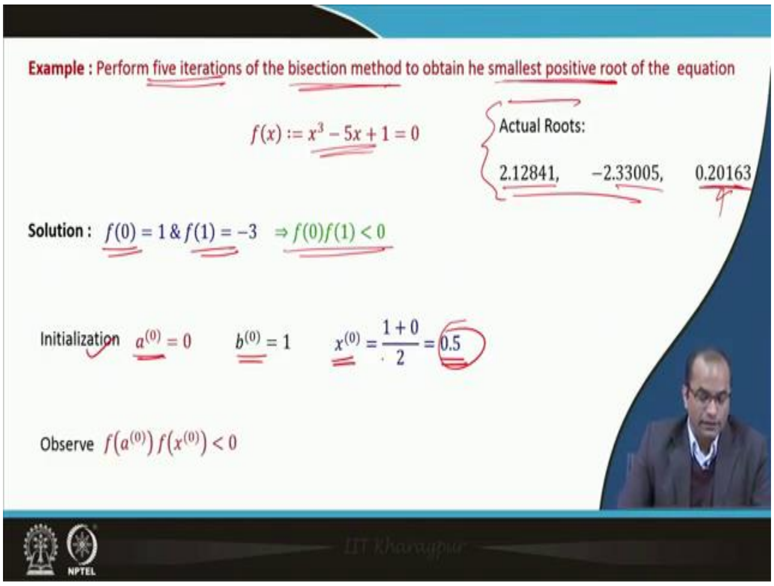

So, we can perform here five iterations of the bisection method to obtain the smallest positive root of the equation. So, first we have to realize the interval where this root lies and that is another difficulty which always we have in all kind of iterative method that we have to choose some initial gas.

So, here the initial gas is the interval we have to find and if we take a actually the actual rule because in this case we can find out so there are three roots here this smallest is actually 0.201 close to this 0, other two are 2 and minus 2. So, if we look at this equation also so we have f0 as 1 and f1 is minus 3. So, this there is a root between 0 and 1 and certainly that is going to be here the smallest root we have. So, f0 and f1 is less than 0 so there is a root in this interval 0 and 1.

So, we will do the initialization in our algorithm that a naught we will take 0 and b naught we will take 1 and then we will go with the x naught which is the middle point as 0.5. So, 0 plus 1 by 2 it is a 0.5 so this is our first approximation 0th approximation or initialization which says that x naught is 0.5.

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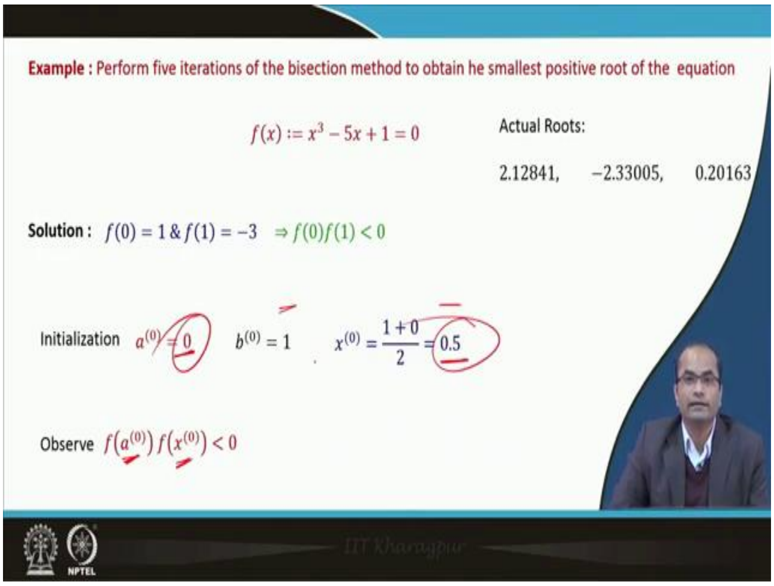

And now, we have to check whether the root lies, now we have bisect the interval so, we will see whether the root lies between 0 and 0.5 or it lies between 0.5 and 1 this is what we have to check. So, we have to check the condition so, in this case we observe that fa naught and fx naught, this is less than 1 that means the root lies between 0 and 0.5 this is the way we will find the next interval.

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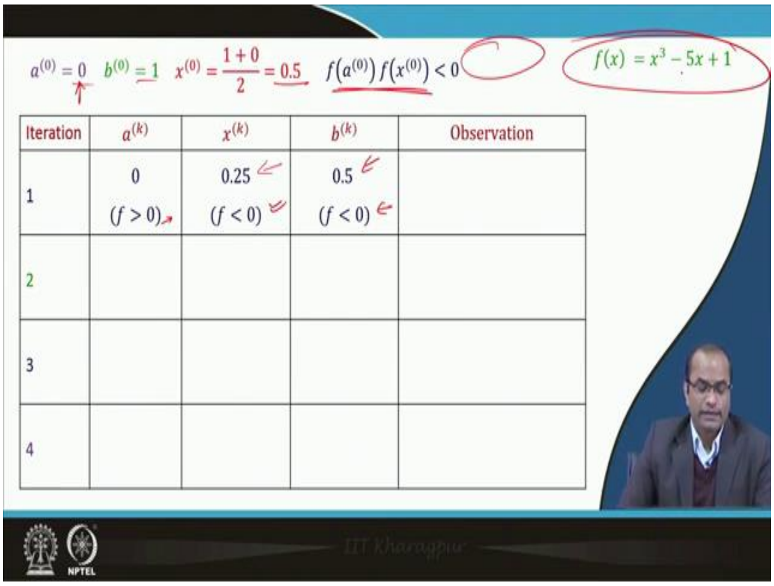

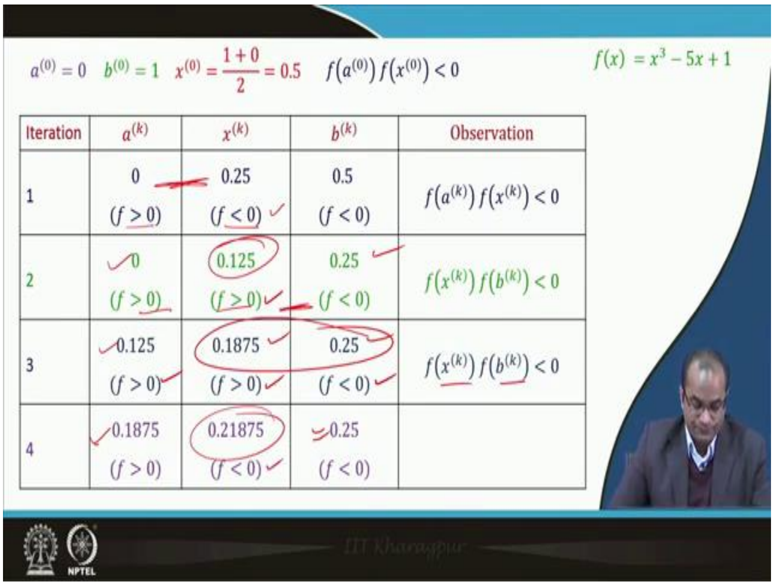

So, let us go with the iterations. So, we have already seen that 0, 1, then 0.5 and we have observed that this root lies between a naught and x naught. So, as a iteration first after this initialization here what we will do. So, a1 now we will check a1, so a1 is 0, so a1 we have taken the 0 the next interval the left boundary is 0 and the right boundary we will take 0.5 where the function is negative, here the function is positive one can always check, what has to check basically whether this sign of this function.

Then we will go with the middle point of the 2 here ak, bk this is a middle point xk here again we have to check the function value whether it is positive and negative so it is coming as negative in this case. So, what we observe here it is positive, here it is negative so, the root lies between exactly this interval 0 and 0.25.

So, we will set then this is our interval and 0.25 will be our interval and then we will take the middle point here 0.125 and check again the sign of the f is positive there so here is positive here is positive. So, there is no root between this we will have a root now in this interval f positive, f negative.

So, our interval will become now 0.125 and 0.25. So, 0.125 and 0.5 is the new interval and then again the middle of this will give 0.1875 and here f is positive, here f is positive but here f is negative. So, now this will give the new interval because xk and bk this product is less than 0 and so a new interval it is 0.1875 and this is 0.25 and again we will take the middle point here so we have the new value 0.21875 and f is negative there.

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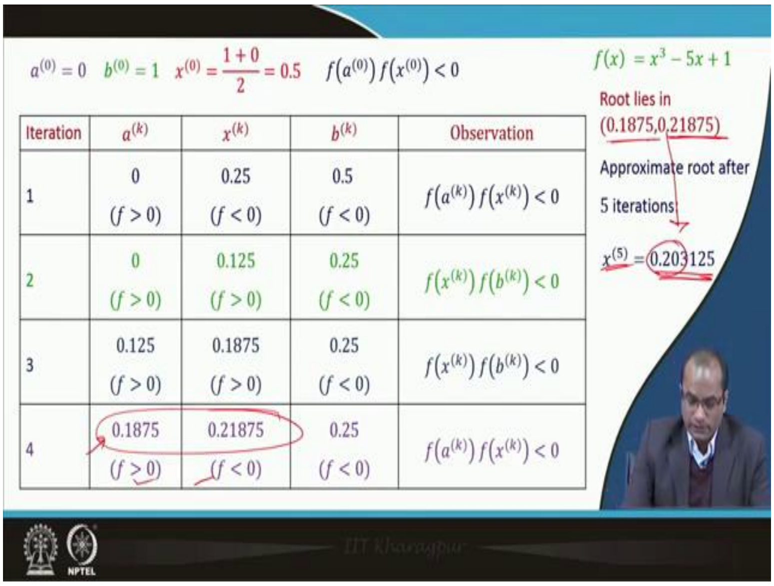

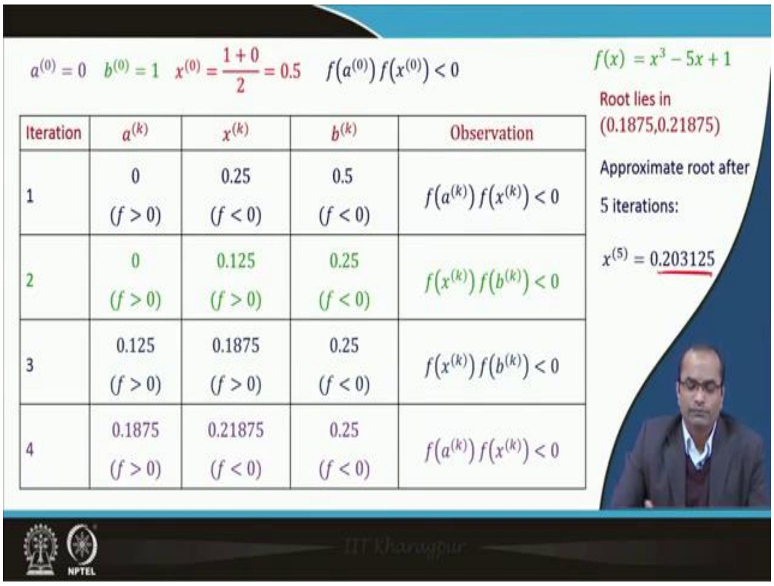

Having this now. So, it is positive negative that means we know that the interval is now after this fourth iteration our new interval is 0.1875 and 21875. So, the root lies in this 0.1875 and this 0.1 0.21875 after fourth iteration. So, the x5 we can get now just the middle point of this so we will take the average here and the average will give this 0.203125 that is the x5 after the fifth iteration.

So, we can see it is the actual root was also 0.20 something so it is matching it is getting closer to this after even five iterations, if you go further naturally this root will improve. So, this a simple algorithm which we can, which we have demonstrated with this simple example and it is always just checking the interval where the root lies and then narrowing down the interval and that is all. So, it is a very simple algorithm to compute the root of a function.

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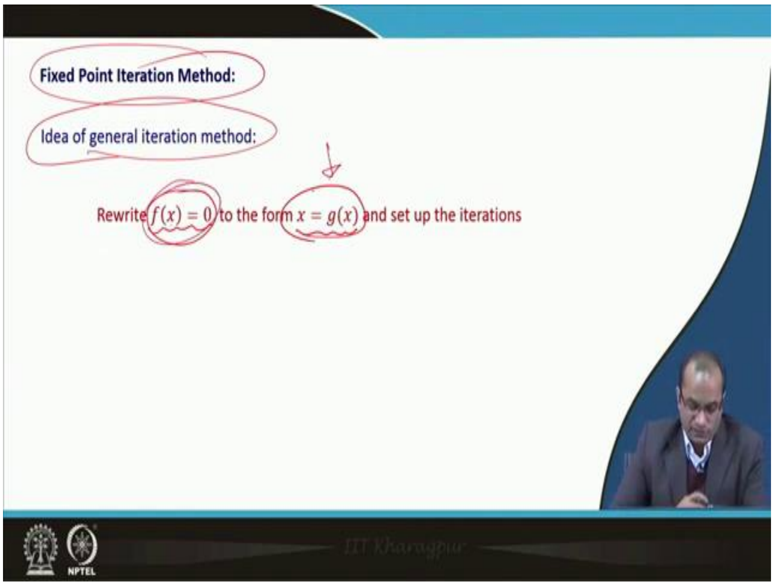

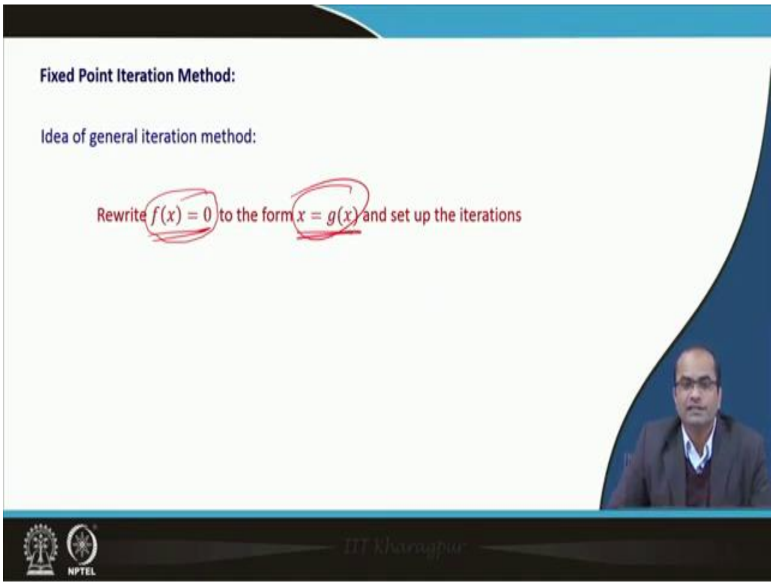

The another algorithm which we will be talking about, another method that is fixed point iteration method so it is a proper iteration method which we will now discuss. So, the idea of general, indeed a general iteration method lies on this principle that if we rewrite the given function fx equal to 0 in this form that x is equal to gx.

So, we rewrite our fx equal to 0 equation into this form x is equal to gx, we will bring x one side the everything else we can take to the right hand side so that we have this form x is equal to gx and once we have this form x is equal to gx we can set up the iteration because finding x from here x from this fx equal to 0 that is the root. We have already written as x is equal to some other function gx.

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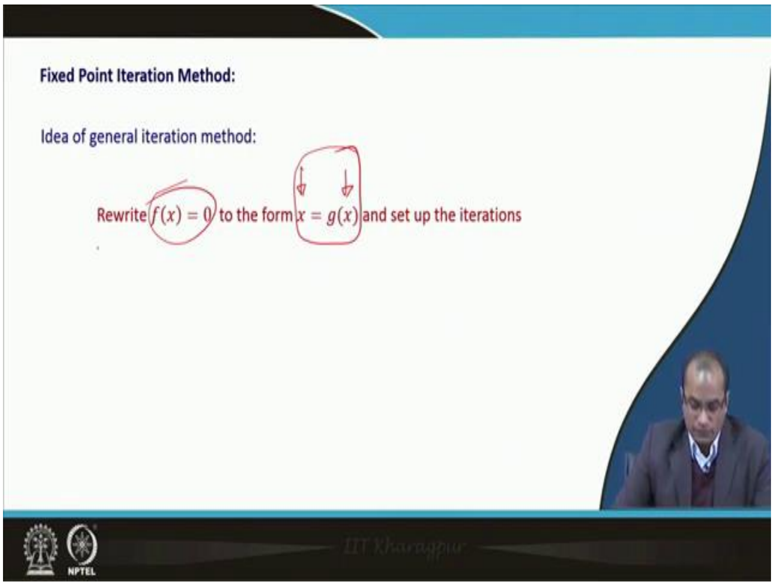

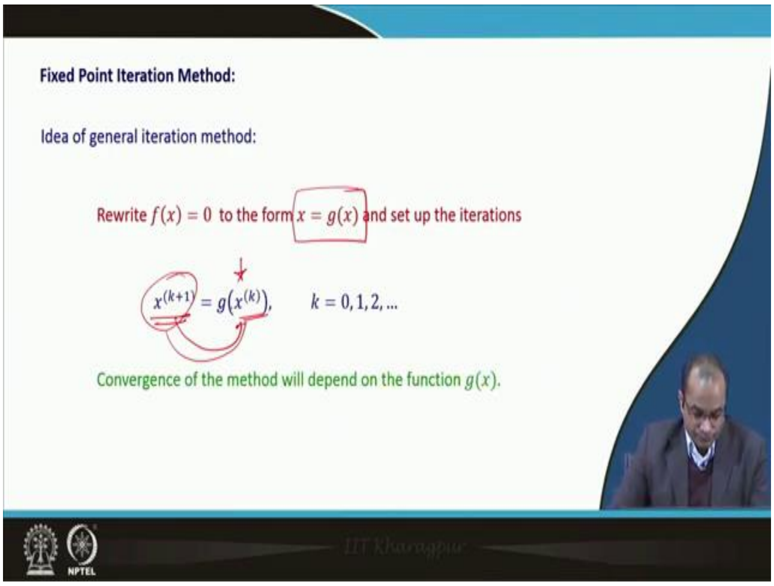

So, here also we are looking for x which satisfy this equation that means that is the root of this equation. So, to search which x satisfies this equation we set up the iteration at this point by saying that this is xk plus 1 and gx k. So, we will choose the starting value, approximate value here let say x0 compute gx0, we will call it as x1 then x1 will go here, then we will call x2, x3 and so on. This will be a sequence of the approximations and we will show that such a sequence under some condition converges to the actual root not always as in the bisection method.

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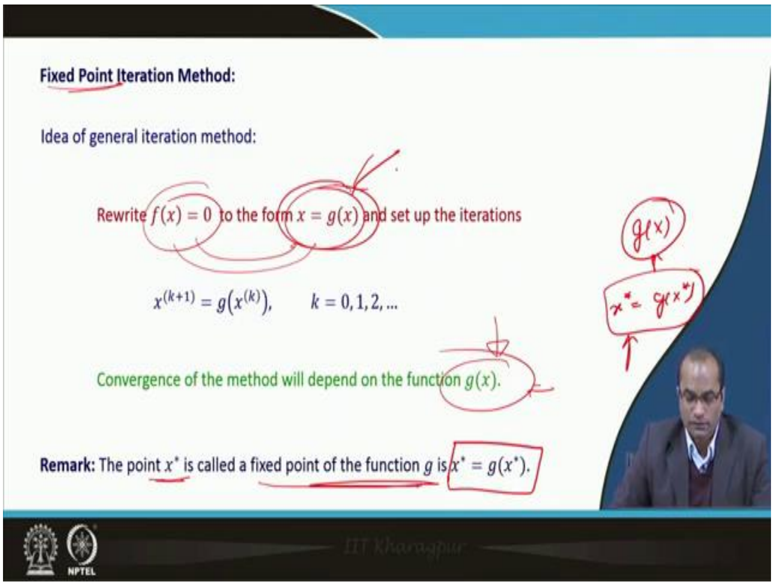

So, the convergence will naturally depend on that how we rewrite this fx equal to 0 to x is equal to gx that is what the convergence actually depends so because there are various ways we can rewrite this fx equal to 0 in this form x equals to gx, I mean this is not a unique way that we have to write this fx equal to 0 in this form.

So, there are several ways and the convergence will depend naturally and we will demonstrate this that how do we define this gx. Why this fix point name is, name has come for this method? Because the point this x star is called fixed point of a function g if we have x star is equal to gx.

So, a function is given, for example, gx is given and the point x star is called the fixed point of this, if we have this property that x star is nothing but gx star. So, based on this because we are looking here for this fixed point of this gx we are calling this a fixed point iteration method.

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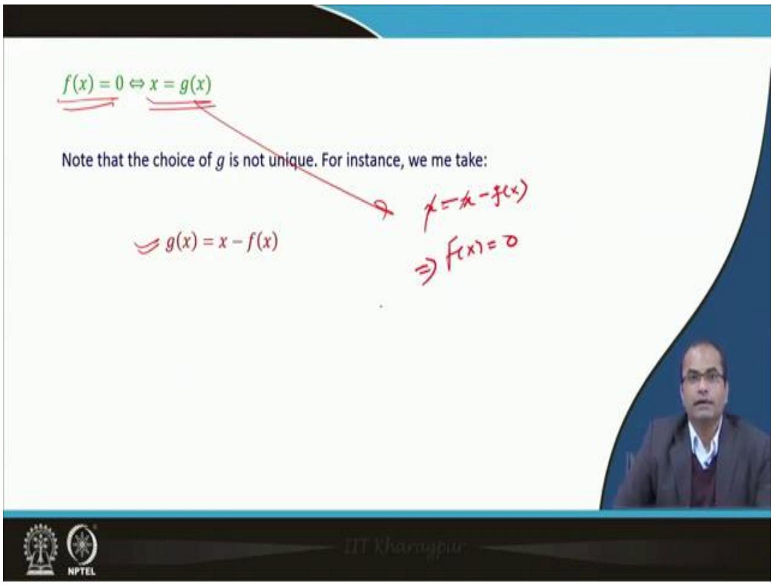

So, having this fx equal to 0 we rewrite into x is equal to gx and there could be several forms it is unique for instance we can take gx as x minus fx so what we have now here for instance x equal to x minus fx if we have taken this x x gets cancel and again we will get fx equal to 0.

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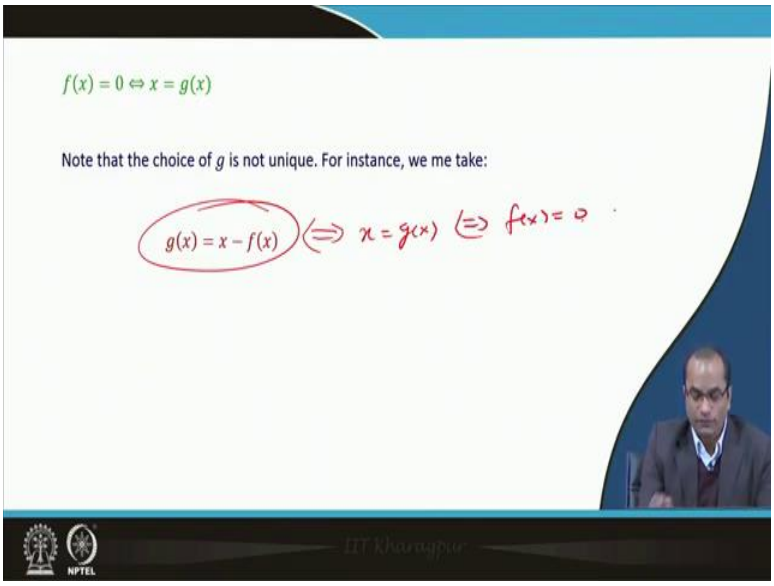

So, this is the function, one of the function gx we can always choose as x minus fx and this is nothing but this is equivalent to saying that fx equal to so x is equal to gx here we will give exactly fx equal to 0. So, this is one of the function which we can work with.

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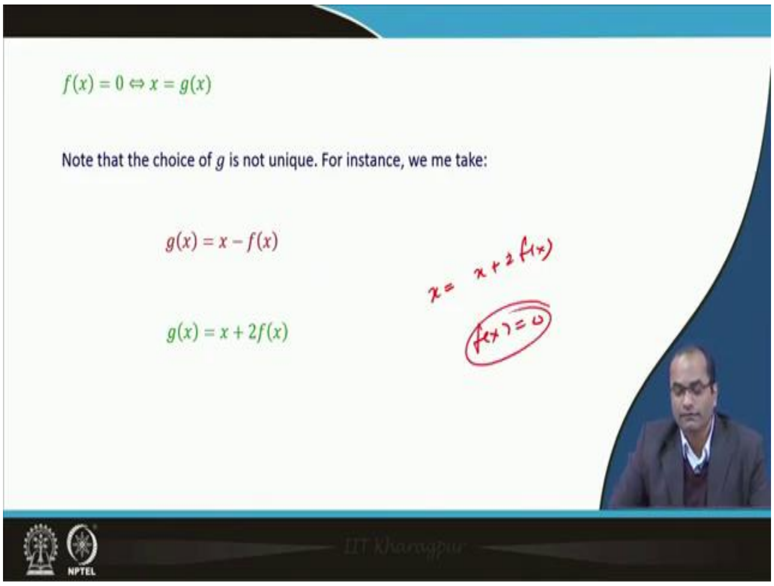

Another one we can say x plus 2 fx. So, there are various ways to define this here we have like x is equal to gx so x plus 2 fx, again x x gets cancel and then you will get again fx is equal to 0 so several ways not only 2 we can have 3, 4, 5 whatever.

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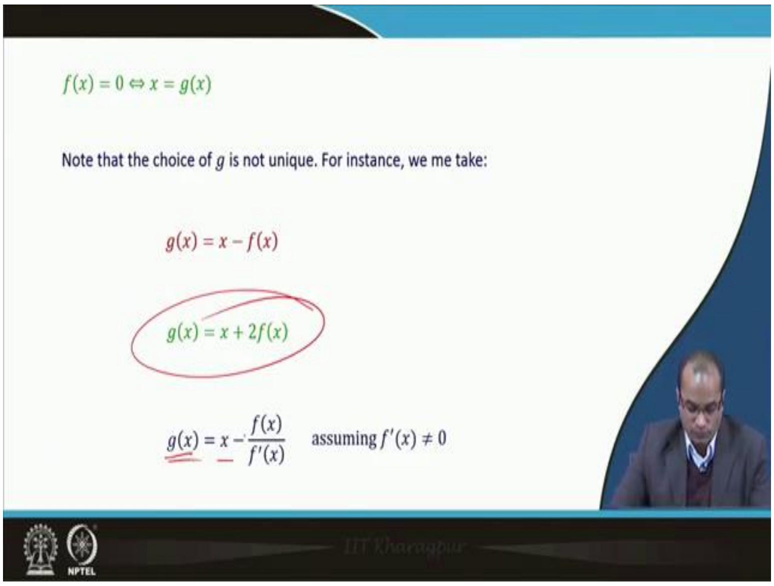

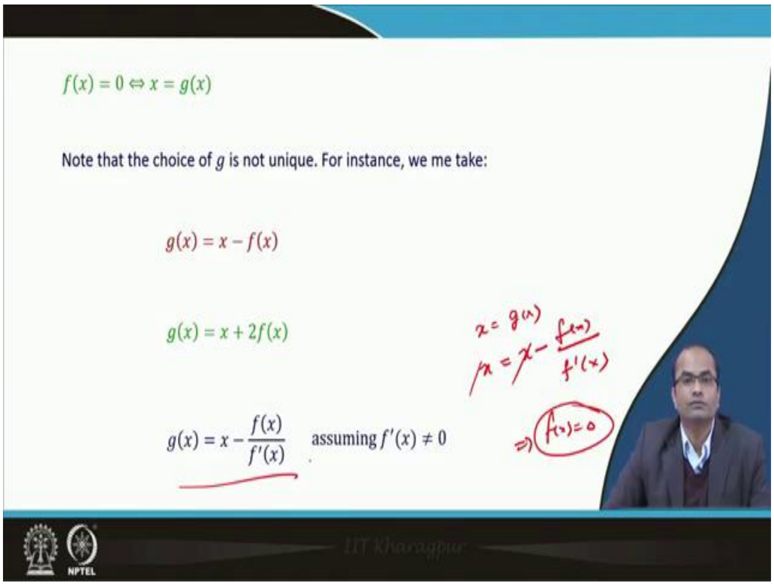

So, this is one way so, there are infinitely many ways to define this gx, one more possibility which is little bit complicated we can have x minus fx over f prime x for instance so here also x is equal to gx if I put this gx there fx over f prime x and this gets cancel and then again you will get fx equal to 0. So, this is also equivalent to fx equal to 0.

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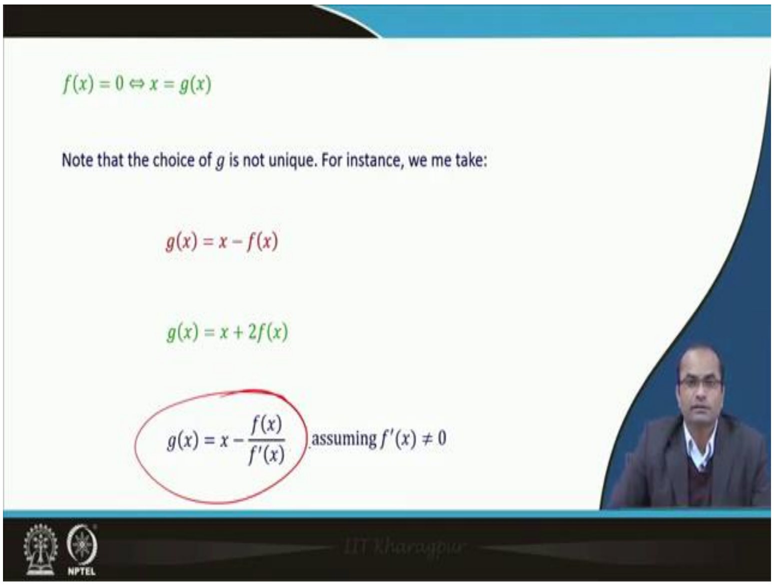

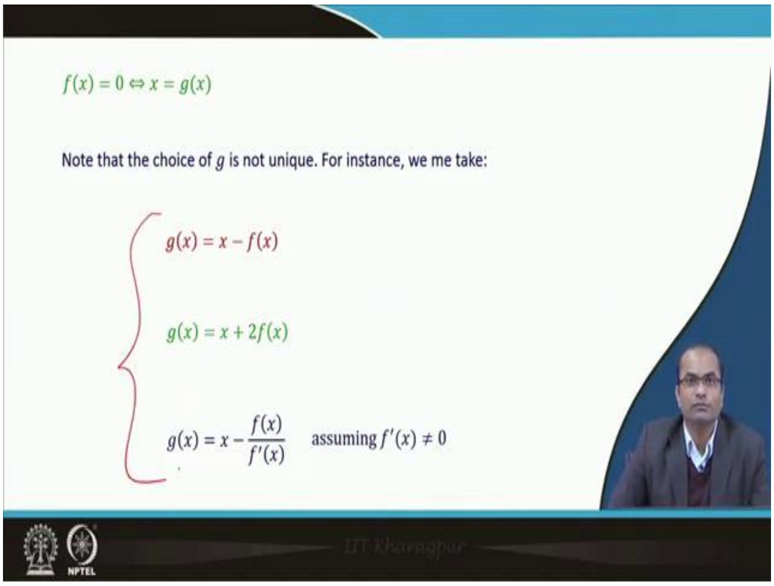

So, here indeed this particular g which leads to a another method Newton Raphson method which we will discuss in the next lecture. So, what we have seen here that there could be infinitely many possibilities for defining gx.

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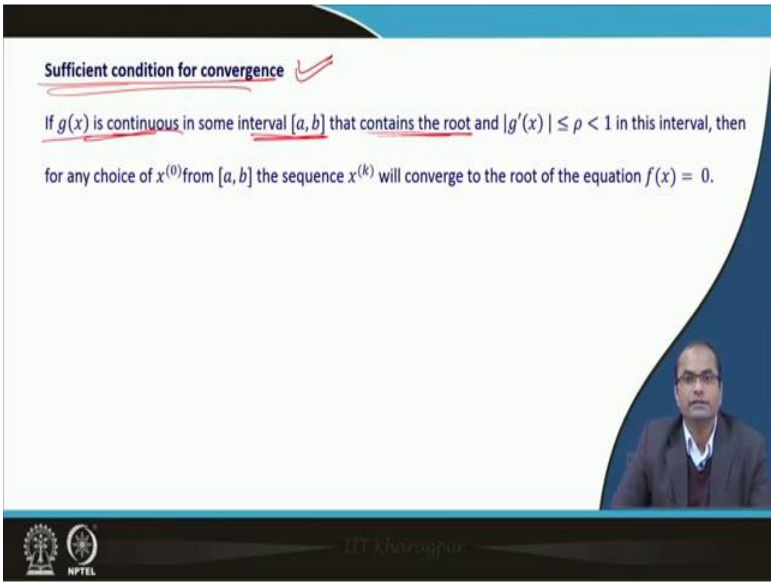

Now, will be talking about that which g we should choose so that the convergences guaranteed and this is what we have here the sufficient condition for convergence. So, if this gx is continuous that is very natural condition we have here in some interval this a to b and that contains the root. So, we have a interval where the function is continuous and we are sure that this interval contains the root.

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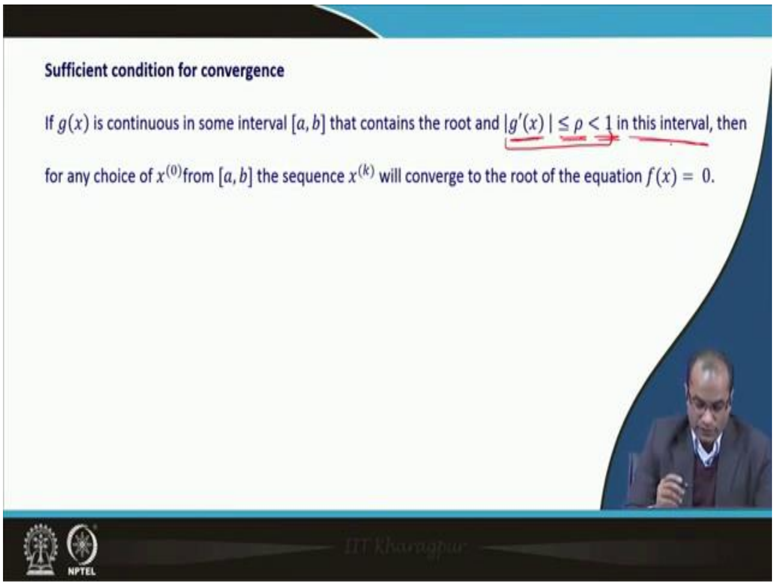

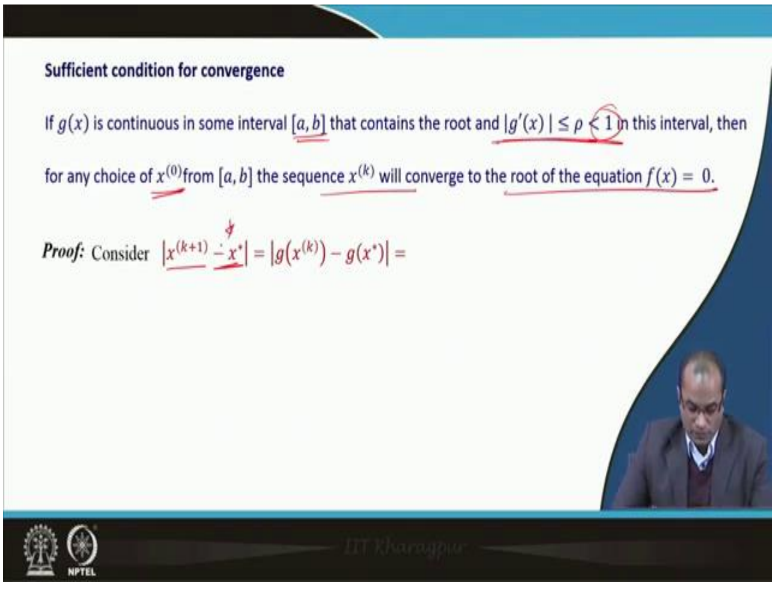

And, then we have, if we have this g prime the derivative of g, the absolute value of the derivative of g less than equal to rho and it is less than 1 in this interval. So, in the interval ab if we have this condition that the derivative is strictly less than 1 then for any choice we take, any initial choice we take in this interval ab the sequence xk.

So, we are getting x1, x2, x3 and so on, and this sequence will converge to the root this is what we will observe now. So, going to the sketch of the proof so we consider this error here xk plus 1 minus this xk x star, x star is the actual fixed point. So, the root of this equation fx equal to 0.

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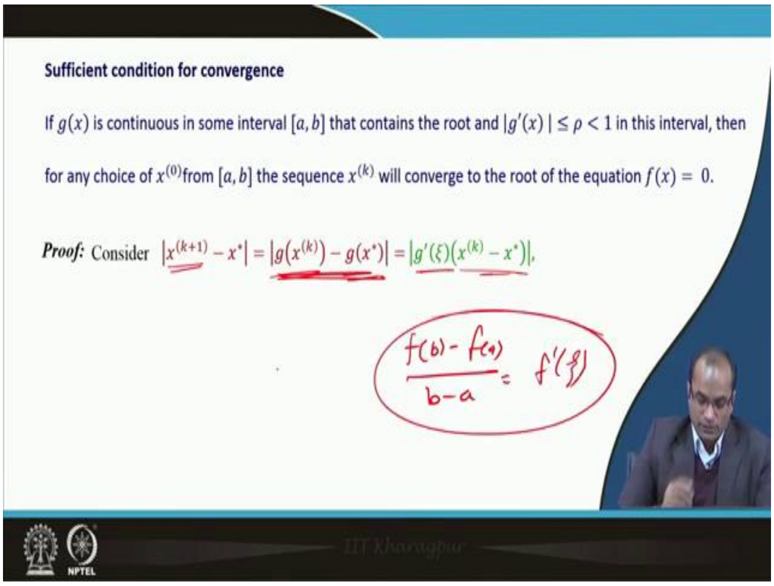

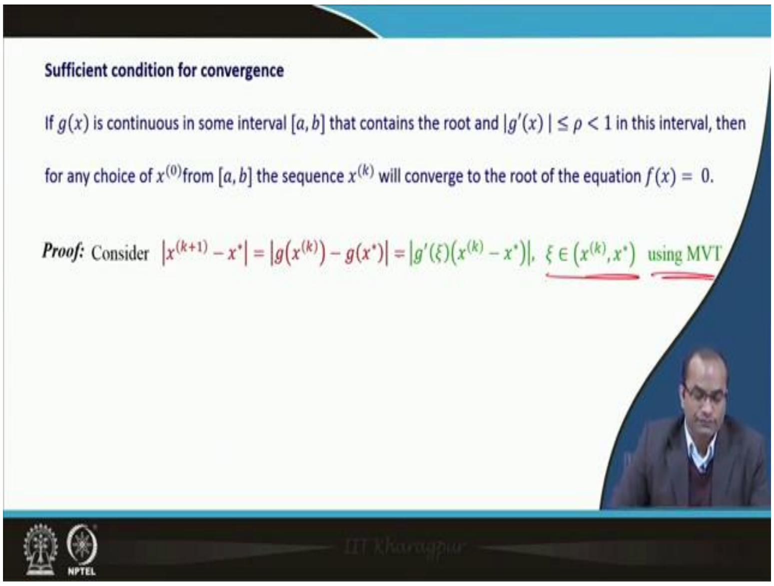

So, xk plus 1 minus x star xk plus 1 from the algorithm we have gx k and minus this x star is equal to basically gx star. This is because this is the fixed point. So, having this what we, we apply the mean value theorem here gxk minus gx star by mean value theorem we can write g prime so there is a point between this xk and x star.

So, that we have g prime at xi and xk minus x star so, that is the mean value theorem which we usually write like fb minus fa over this b minus a and there is a point here xi. So, this fixed point this mean value theorem we have applied there so this xi lies between somewhere in this point x star and xk and this is the mean value theorem.

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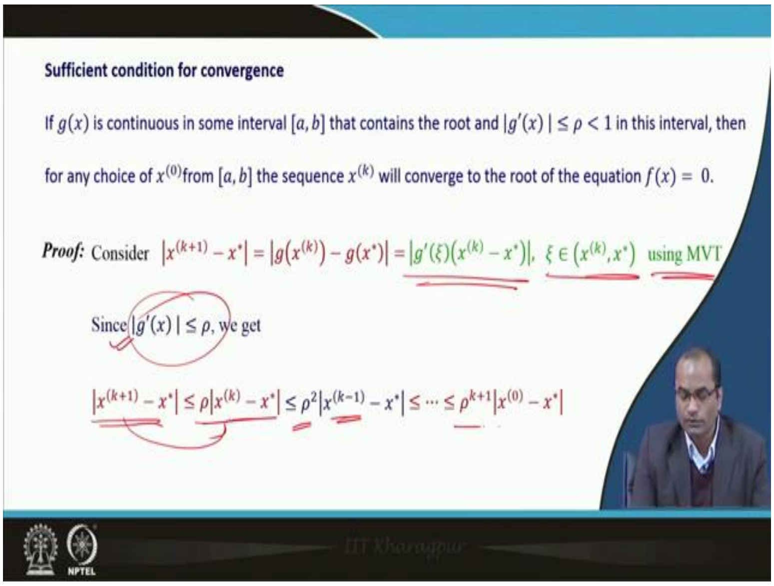

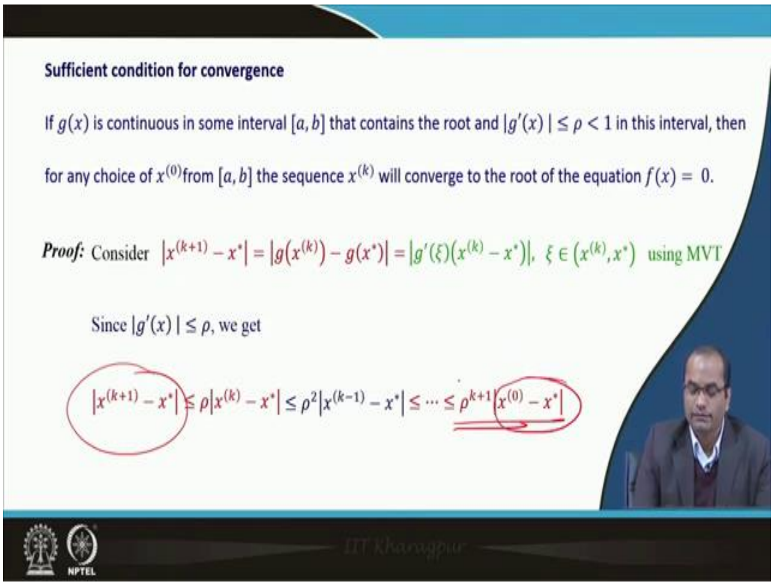

So, now, we know that g prime the derivative is less than rho in the whole interval a b so, weather wherever we are here so this g prime is less than equal to rho and using this there so xk this error is less than rho into xk minus x star and again we can do this by induction that it is a rho 2 and then here k minus 1 will come then rho 3 and so on. So, we have this relation that this error here at k plus 1th step is less than the rho power k plus 1 and the error at the beginning, x naught minus x star.

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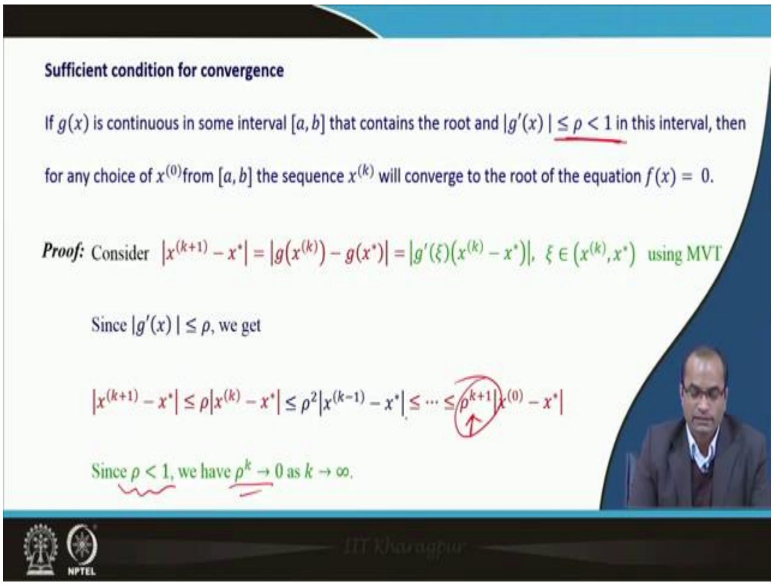

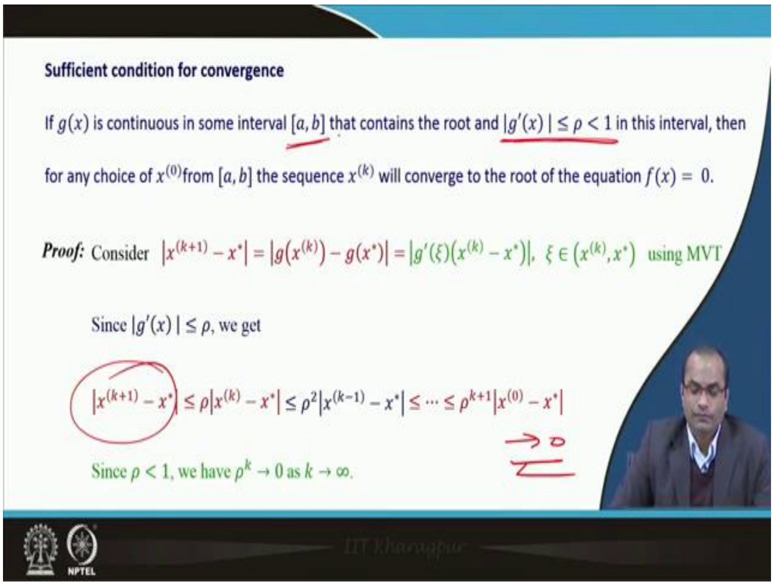

Having this it is clear since rho is less than 1 that is the condition we should have that the derivative is less than 1, though this rho k with this will go to 0 because rho is less than 1 and this error here as k approaches to infinity will go to 0 so that is the proof here we have. So, what is the condition that this g prime should be less than 1 in the interval which contains the root and the initial guess.

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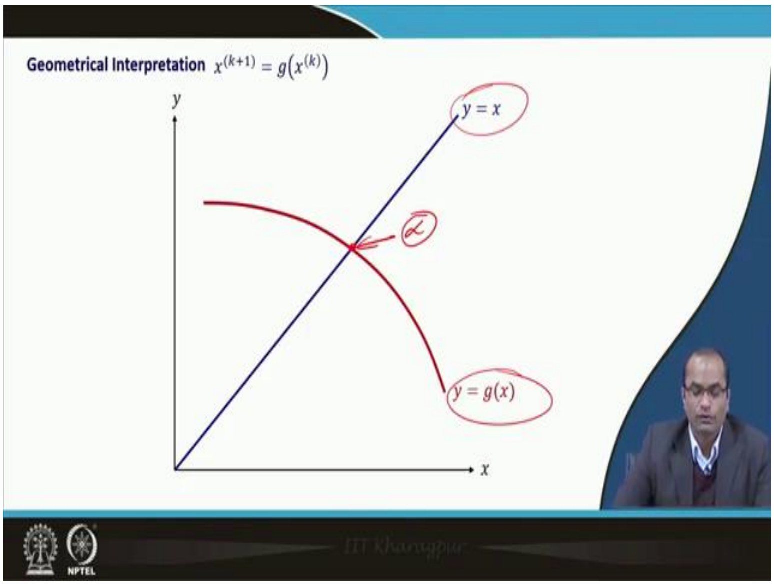

Coming to the geometrical interpretation of this. So, we have y is equal to x this line and we have y is equal to gx the graph of this curve and looking at where x is equal to gx so, this is the point here so, it is a root actually alpha which we call and this is what we are looking for.

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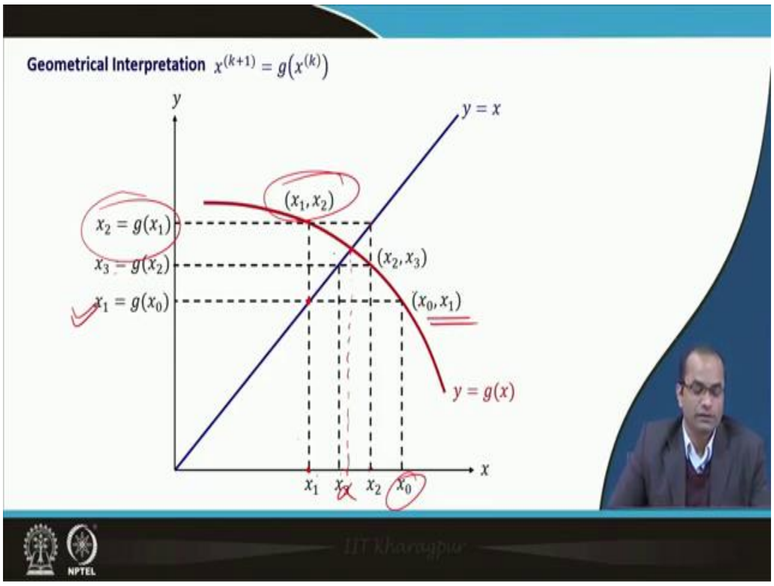

So, we start with some initial guess x naught for instance here then we compute x1 by this gx naught and so this is the point here x naught x1, x naught x1 and then so this is exactly x1 at this point because here y is equal to x line it crosses. So, this is the height was this x1 and here both are equal so this is also x1 now here and this cuts this graph there which will be x2 now. So, this is our x2 or this is x1 x2 and further we will go so, this is your x2 here. So, what we observe that we started with somewhere here x1 was there now and x2 is closer too.

So, this is the actual root we are looking for this is our alpha, so we are going close to this alpha and then we have further the x3 will come and then we have this x3 here which is further closer to this point and then we will continue this and will approach finally to this point, so this is the idea of this fixed point iteration method.

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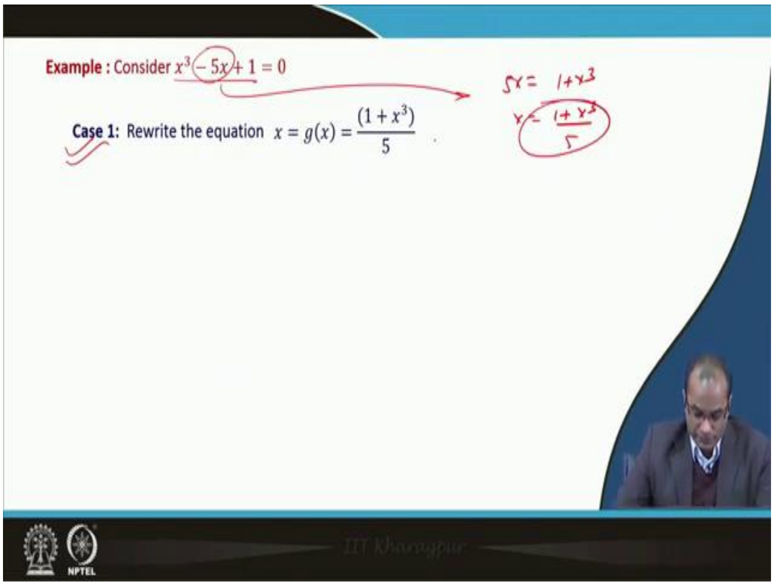

So will go with the example here x cube minus 5x plus 1 the same example which we have consider earlier and let us compute that root which was 0.20 and so on. So, we rewrite the equation now so, there are several ways of rewriting so the first approach we have we will consider other way also for rewriting this one. First we are taking here that we have taken the 5x is equal to 1 plus x cube and then x is equal to 1 plus x cube by 5.

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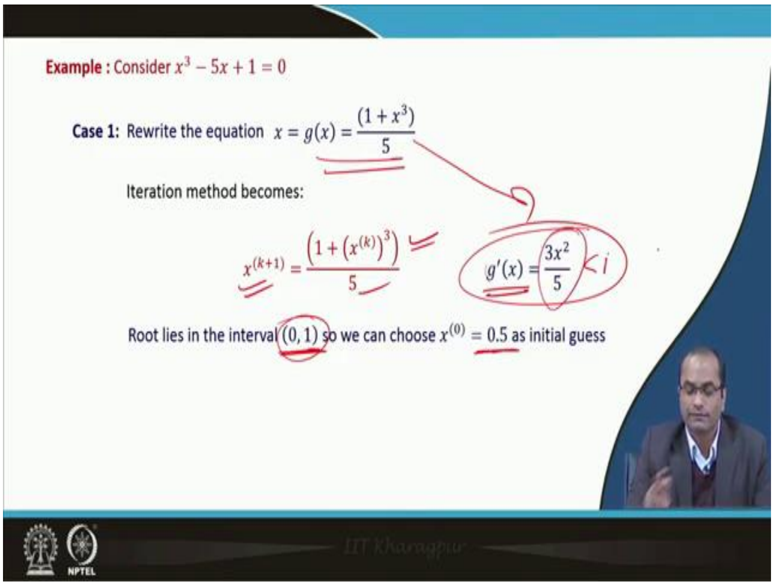

So, this g we take and then set up the iteration so, xk plus 1 is equal to this 1 plus xk cube by 5 and note that this g prime which we can compute from there it is a 3x square plus 5. So, if we know that the root lies between this 0 and 1 so, if we choose this 0.5 as the initial guess in this interval and we are sure also in this case that whatever values in this interval this g prime x is less than 1 because the 0 to 1 and then we have here the condition.

So, the sufficient condition for the convergence is satisfied that means if the sufficient condition is satisfied for sure this scheme will converge if it is not satisfied then we are not sure whether it will converge or it will not converge because we have here only the sufficient condition not the necessary insufficient condition.

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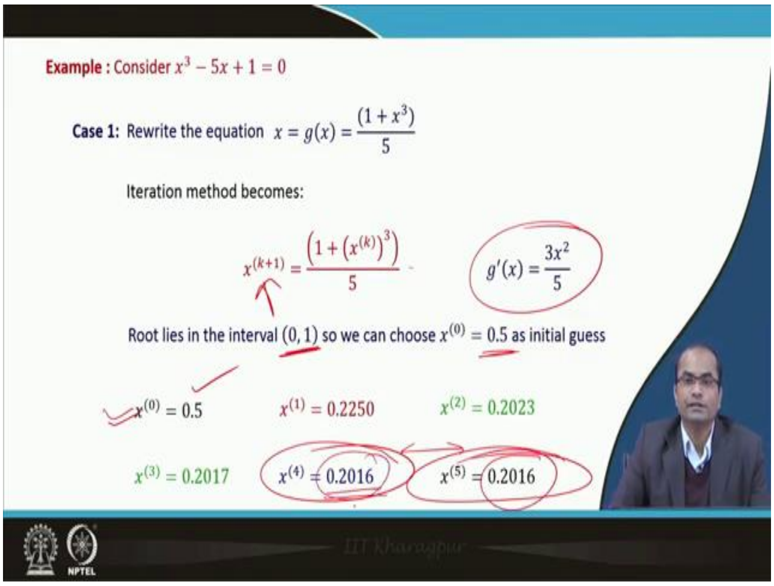

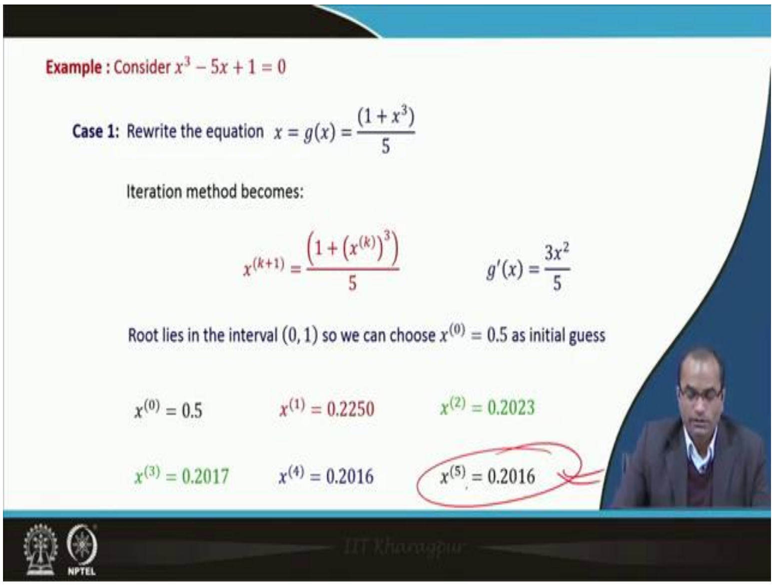

So, having this we start with the x naught for instance 0.5 we can choose actually any initial guess between 0 and 1 and this condition makes sure that it will converge not only for 0.5 but any other guess also we can take 0.1 we can take 0.8, 0.9, etc. It will converge. So, for instance if we take 0.5 then we compute x1 from this iterative scheme 0.22 then 0.20 and so on and we see this 0.2016 here also it was 0.2016 so after this fifth iteration itself the schemes seems to be convergent. You have 0.2016, here also 0.20 so there is no change happening up to this four digit. So, this is a very good approximation if four digits are considered 0.2016.

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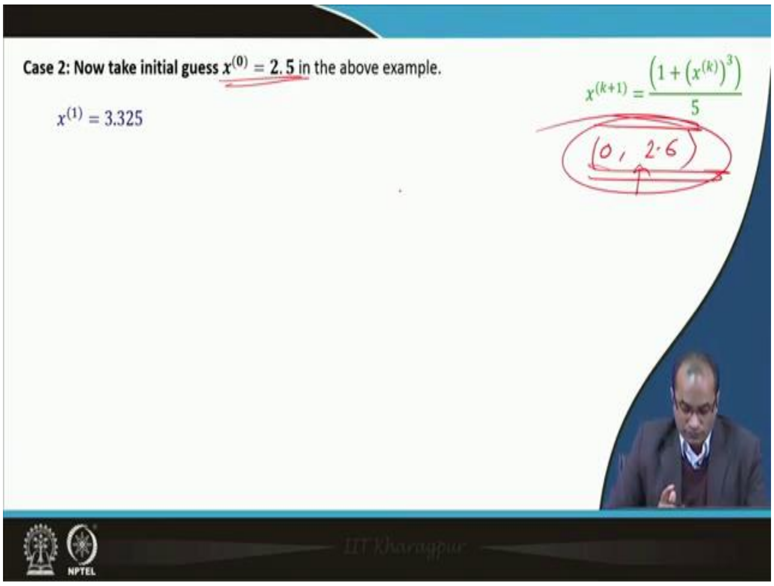

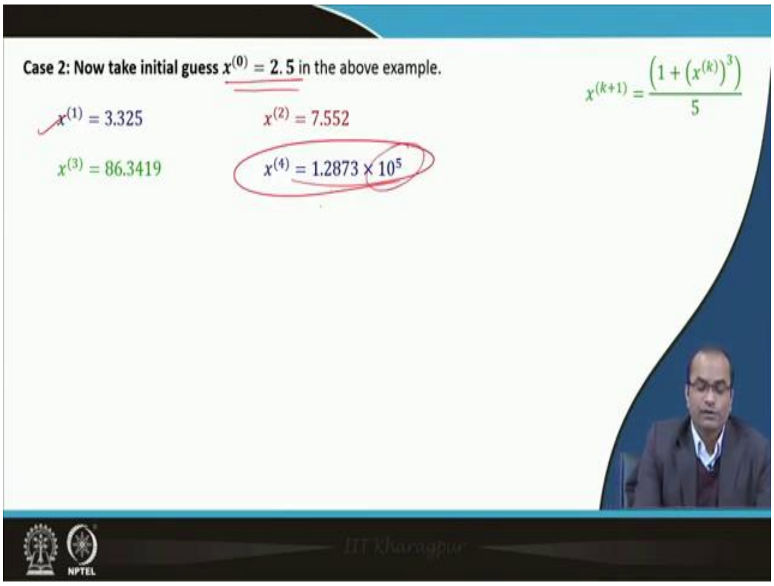

What we will also see in the same scheme for example if we take initial guess here 2.5, so, having this initial guess 2.5 that means our interval which contains the root which was 0.20 so up to this 0.25 so our interval let say from 0 to 2.6 or 2.5. So, this interval contains the root but the problem is now if we check that g prime for this value this will be greater than 1.

So, now this time the convergence is not guaranteed if we take this initial guess it may converge, it may not converge, say for example if we compute x1 here, x2 , x3 and x4 itself is telling that it is going somewhere else 10 raise to the power 5 a very large number.

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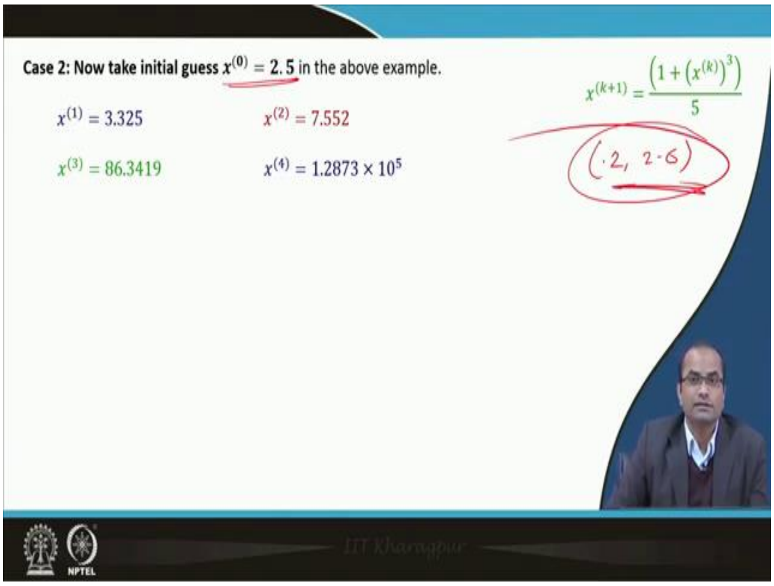

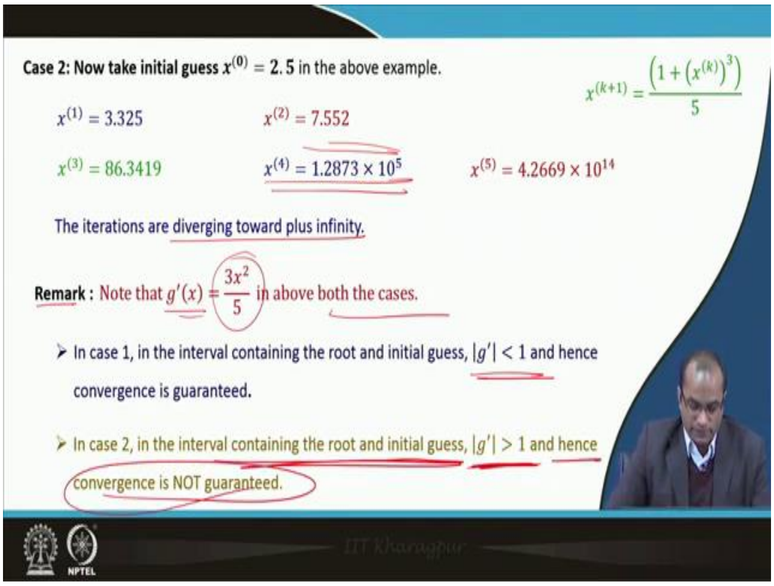

So, it is not converging in this case because our initial guess we have taken 0.25 and in if the interval we consider which contains the initial guess as well as the root that means it is 0.22 like 2.6 something if they will consider this interval there are points here where f g prime is not satisfying their sufficient condition. So, here we were not sure actually whether it will converge or it will not converge but actual computation shows that it actually does not converge.

The iterations are diverging toward plus infinity. So, what we have seen now just a remark that g prime was 3x square by 2 in both the cases. In the first case the g prime was less than 1 and hence the convergence was guaranteed, whereas in the second case the interval containing the root and the initial guess the g prime was greater than 1. At some points and hence the convergence is not guaranteed which is also reflected in this a numerical calculation.

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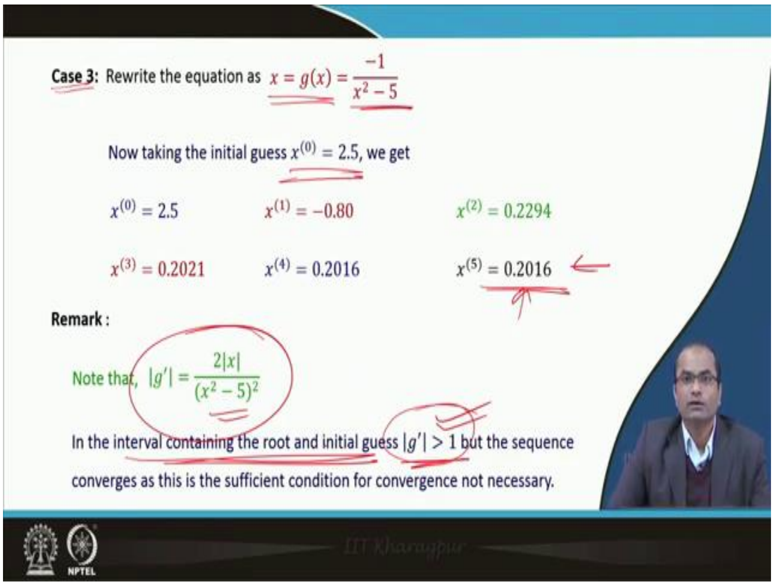

What we do now if we rewrite the equation for example in this way that x is equal to minus 1 over x square minus 5, so there are several ways of rewriting this, this is one of them and now if we take the same initial guess 0.25 and we check now that it is converging to this root. So, by just rewriting the equations the same initial condition may converge to the determination of the actual root.

But here what we note that g prime is this two times is absolute value x x square minus 5 whole square and in this case also the interval containing the roots and the initial guess g prime was greater than 1. So, what is the message because this is the sufficient condition that g prime absolute value should be less than 1 then we are sure that the iteration will converge. For example, in this case it is greater than 1 it may converge and it may not converge and the numerical calculation shows that actually it converge in this case.

Whereas, just in the before case 2 what we have seen that g prime was greater than 1 and the sequence does not converge. So, these are the sufficient conditions so if we have that condition fulfilled then we are sure that it will converge, if it is not then it may converge like in case 3, it may not converge like in case 2.

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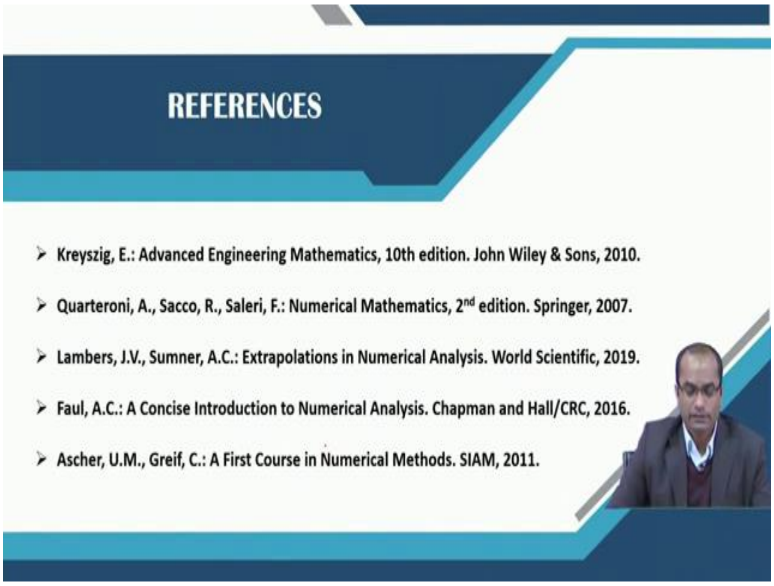

So, these are the references we have used for preparing this lecture.

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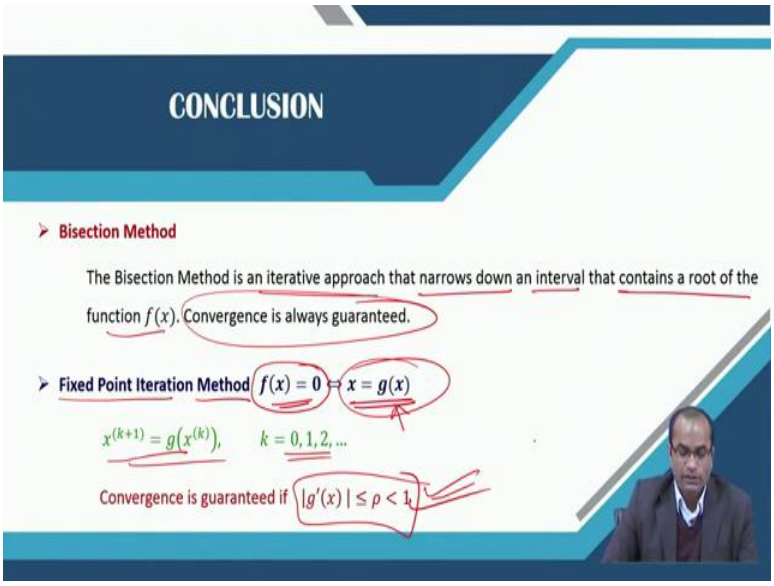

And just to conclude we have discussed the bisection method and this was an iterative approach that narrows down an interval that contains root of a function fx and most important in that approach the convergence is always guaranteed.

The second approach we have discuss the fixed point method where it was based on rewriting this fx equal to 0 in this form x is equal to gx and having this we can set up the iterations there and the convergence of this method was dependant on this g and if we have this condition that g prime is strictly less than 1 then the convergence is guaranteed and as a we have seen also several cases that it is actually sufficient condition. If this condition is not fulfilled in one case the convergence was achieved, in other one it was not achieved, so, that is all for this lecture and I thank you very much for you attention.