Engineering Mathematics II
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Lecture 26
Polynomial Interpolation

So, welcome back to lectures on Engineering mathematics 2 and this is lecture number 26 on polynomial interpolation. A very important topic in numerical analysis.

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And in this lecture we will be talking about what is actually the polynomial interpolation and then we will be also talking about the existence and the uniqueness of such polynomial interpolation and finally we will derive a formula which will provide us the error in such interpolating polynomial.

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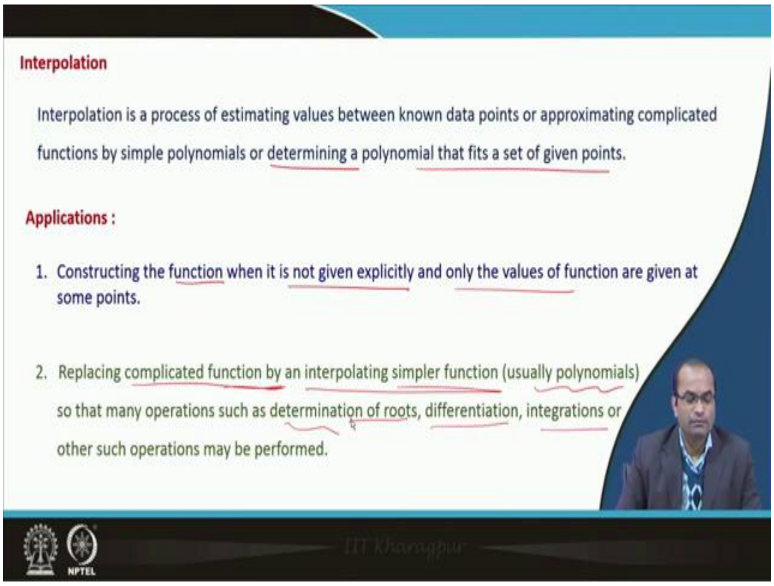

So, coming to the interpolation, it is a process of estimating values between known data points or approximating complicated functions by simple polynomials or determining a polynomial that fits a set of given data points.

So, it is a various ways one can define the interpolation, so the applications they are like constructing the function when it is not given explicitly only the values of functions are given, so we will construct a polynomial which passed through or fit the given data point, so that is the one of the major applications of these interpolation.

The second one that replacing the complicated functions were by an interpolation simple functions usually the simple functions we take polynomials because polynomials are really simple function because many operations on these polynomials such as the determination of roots, differentiation, integration, and many other operations can be performed easily.

So, given a complicated function we will approximate it by some polynomial of some degree and then we can approximate basically the roots or we can do differentiation, integration et cetera such operations can be performed.

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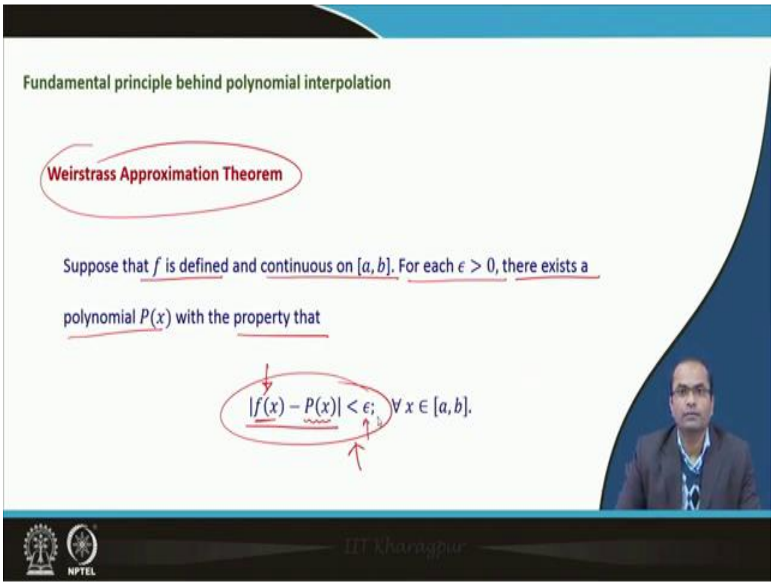

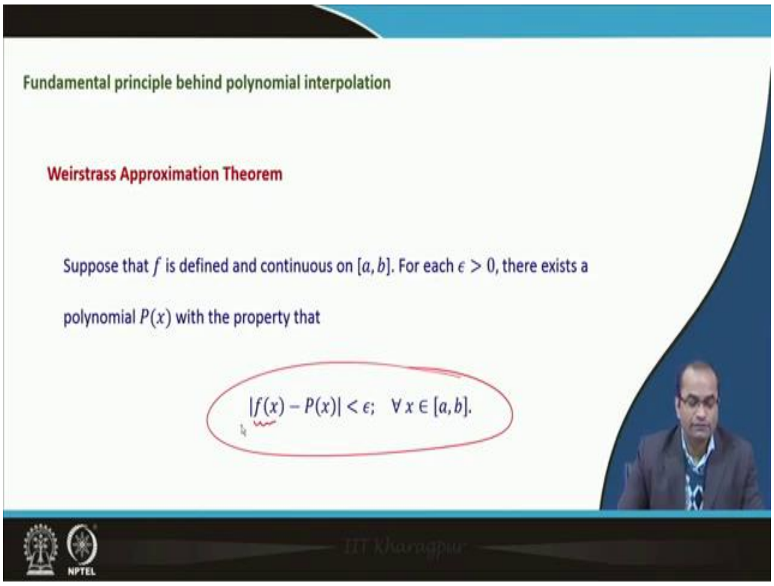

So, what is the fundamental principle behind this polynomial interpolation that will be clear from this Weirstrass approximation theorem, which says that suppose this f is defined and it is continuous on some interval then for each epsilon however small, for each epsilon there exist a polynomial P x with the property that f x minus P x is less then epsilon.

So, this relation tells that given f x here a continuous function on a, b we can have a polynomial which will approximate this function which as good accuracy as we want because epsilon we can fixed and it can be very very small and then we can find a polynomial Px corresponding to this given continues function fx.

So, that the base of this interpolation or having a polynomial approximation of a given function. Because this theorem says that for any given function which is continuous we can find a polynomial with as good accuracy as we want. And based on these Weirstrass approximation theorem then we will derive the polynomials corresponding to a given function.

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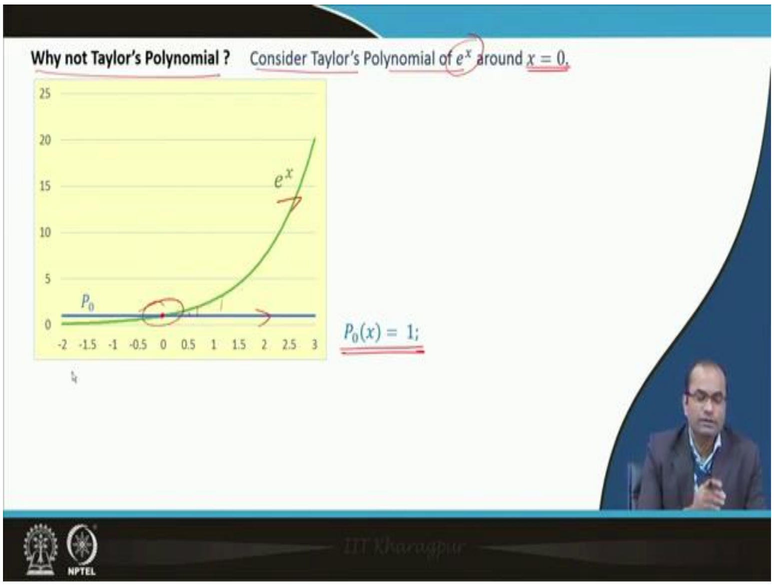

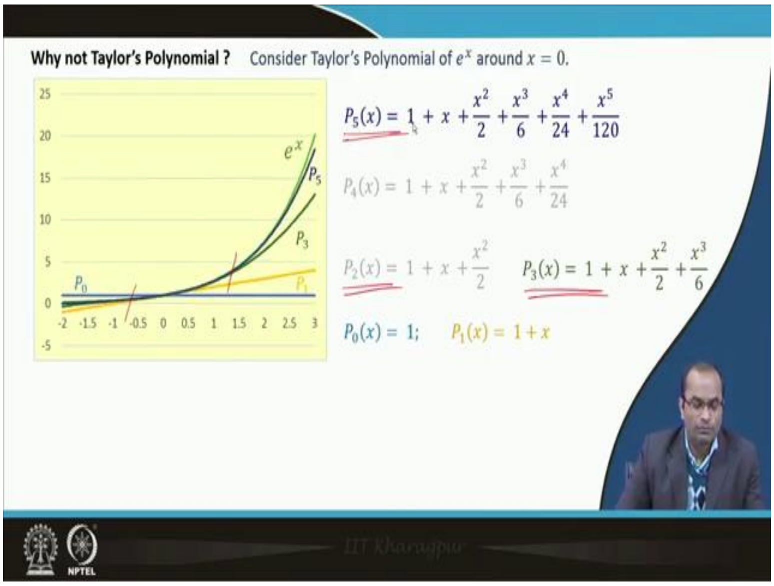

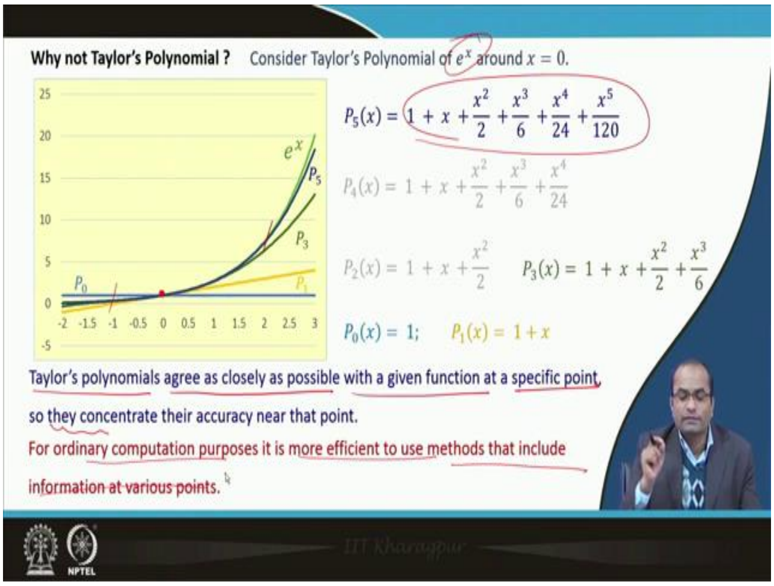

So, but the question is that but why not Taylor's polynomial, because we have already studied Taylor, Taylor's polynomial which was also used for approximating a given function. And now we are talking about some other polynomial with not exactly the Taylor's polynomial so the question is that, why not just the Taylor's approximation or the Taylor's polynomial because they are also doing the same job given a function we are approximating a function in terms of the polynomial which is in this case Taylor's polynomial.

So, let us just demonstrate through this example and then we will give the reason that why not Taylor's polynomial we are talking about some other kind of polynomials here. So, constructing the Taylor's polynomial for instance this e power x the exponential function around this x equal to 0. So, the first polynomial of this degree 0 that is a constant function that is 1.

Because that is just function value there so f 0 e power 0 is 1. And if we plot this so this was the exponential function here and this is this constant function so it is having very good accuracy at this point or naturally it is matching with the function itself at x is equal to 0. Other than that point the accuracy is not as good and it is actually getting worse when we are going far away from this point. So, the accuracy is more or less concentrated around this point of expansion that is the one point here we should not.

If we go ahead we construct this polynomial of degree 1 so that would be 1 plus x that is the Taylor's polynomial here. And if we plot this now it is linear function 1 plus x and that is given here. And now we have higher degree polynomial and then we are getting better approximation in the neighborhood again of that point but if we go far again this approximation is not at all good.

If we construct further, for example, second order polynomial or the third order polynomial so we plot here the third order polynomial for the instance now we are matching quite a bit in in a larger neighborhood around this point x equal to 0 and similarly, if continue and construct for instances fifth order polynomial then we do c here the accuracy is now in a much wider interval.

So, what is the point now here? I mean the question is that why not this Taylor's polynomial, we have that Taylor's polynomial here which is approximating the given function for instance the exponential function. So, the main issues here the Taylor's polynomial agrees as closely as possible with a given function at a specific point which was the point of expansion or basically in other word we say that they concentrate their accuracy near that point. So, as we are going far away from that point the accuracy is not at all good but in the close neighborhood of that point the accuracy is very good.

The second point here to construct the Taylor's polynomial we do need information of that function at x is equal to 0. So, we need the function value at that point, we need its derivative, second derivative, third derivative and so on. Depending on what degree polynomial we want to construct.

Whereas in the interpolation we will again, we will be constructing the polynomials but there we need the function value at different different points not at one point the values of higher order derivatives rather we need the function value at different different points, so we will naturally come up with some other polynomial not the Taylor's polynomial which also does the purpose of approximating a complicated function by a polynomial.

So, for ordinary computation purposes, it is more efficient to use methods that include information at various points not at just particular point because in many applications, many experimental results we have at different different points the given data and then we want to know that which polynomial fits the best, the given data points.

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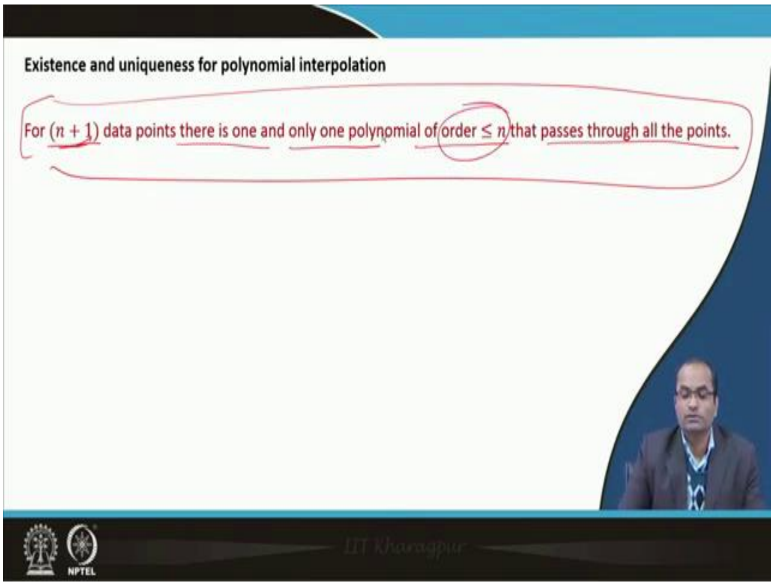

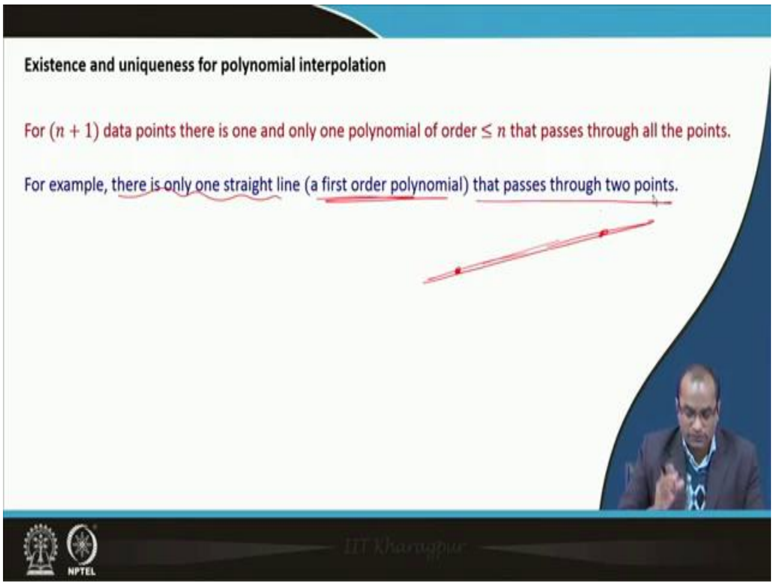

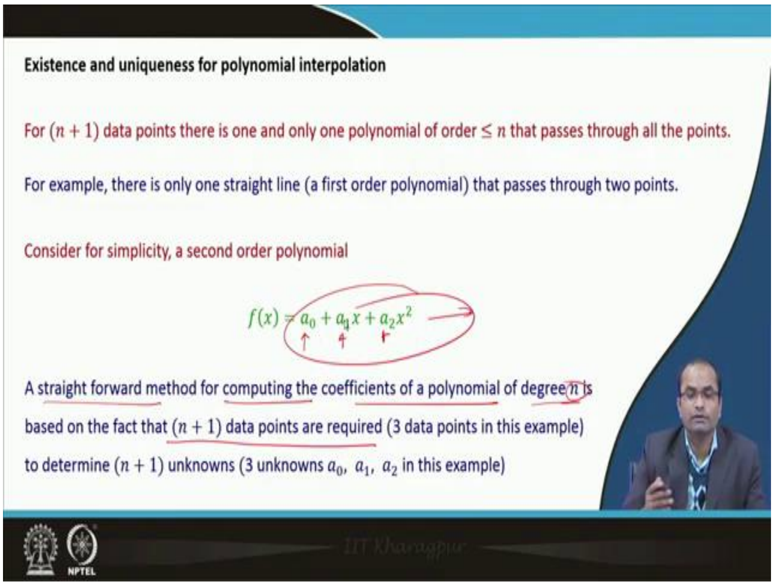

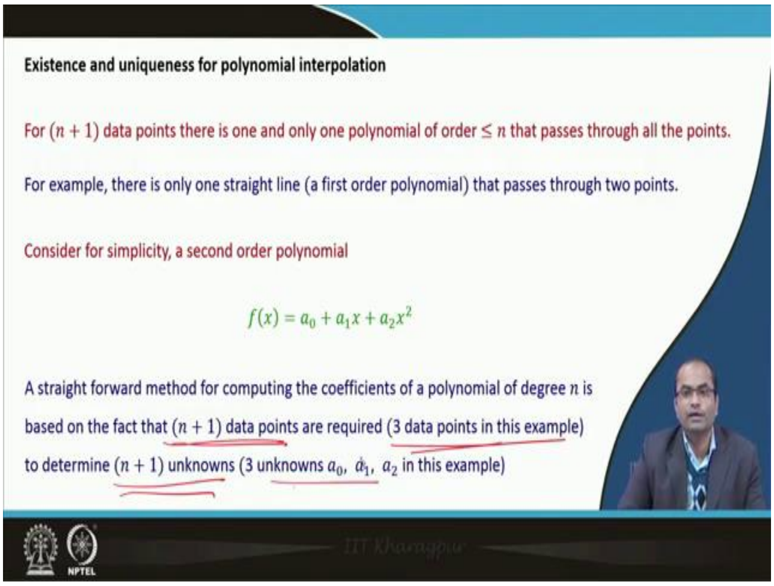

So, the first question is whether given data points the polynomial which we want to fit is it unique and does it always possible to construct such a polynomial. So, for n plus 1 data points there is one and only one polynomial of degree less than or equal to n that passes through all the points. That what we are going to proof here in this result so that is the main result that there it exist and it is unique so when n plus 1 data points are given there is only one and there is one only one polynomial of degree this less than or equal to n that passes through all the points.

A very simple example, which we can talk about that when we are talking about the two points and we know that, there is unique straight line. So, a polynomial of degree 1 which passes through these given two points they cannot be two lines, two different lines which passes through both the points. So, it is a uniqueness and there will always a line so there is existence so there is only one straight line a first order polynomial in this case when two data points are given that passes through the given two points.

So, now to proof this result which is a true for general n, we will consider a second order polynomial we will proof this for a second order polynomial and one can extend this for nth degree or order polynomial. So, suppose our function here with the polynomial a naught, a1 x, a2 x square a second order polynomial we have here.

So, how many unknowns do we have? We have a naught unknown, a 1 and a 2. So, there are three, there are three unknowns which will define this polynomial so to, in order to compute these three unknowns we basically need three data points. The point is a straight forward method for computing the coefficients of the polynomial of degree n.

So, if we talk in general n based on the fact n plus 1 data points are required like in this case it is second degree polynomial and we need three data points because there are three unknowns and for each data points we can construct one linear equation in a0, a1, a2 so we need three data points to have three equations and they can be solved and we will see in this particular case that there exist a unique solution. So, for this uniqueness we need n plus 1 data points and three data points in this example to determine n plus 1 unknowns and in this example we have three unknowns for instance.

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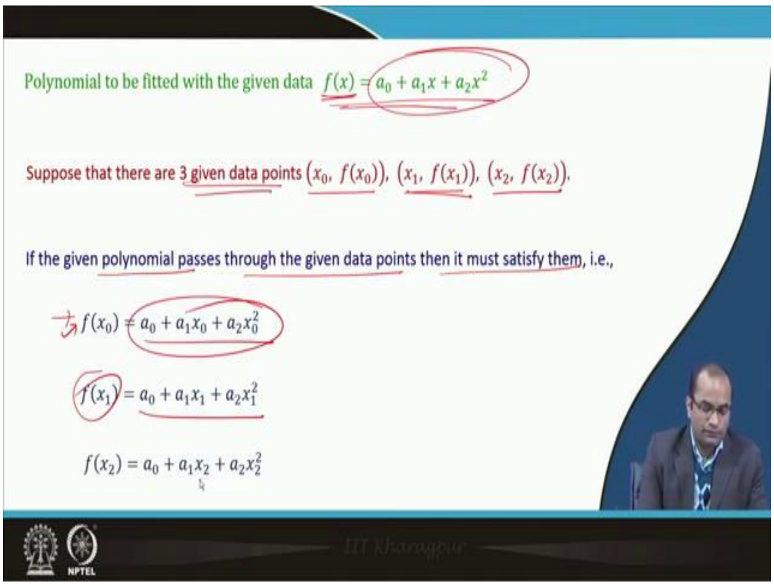

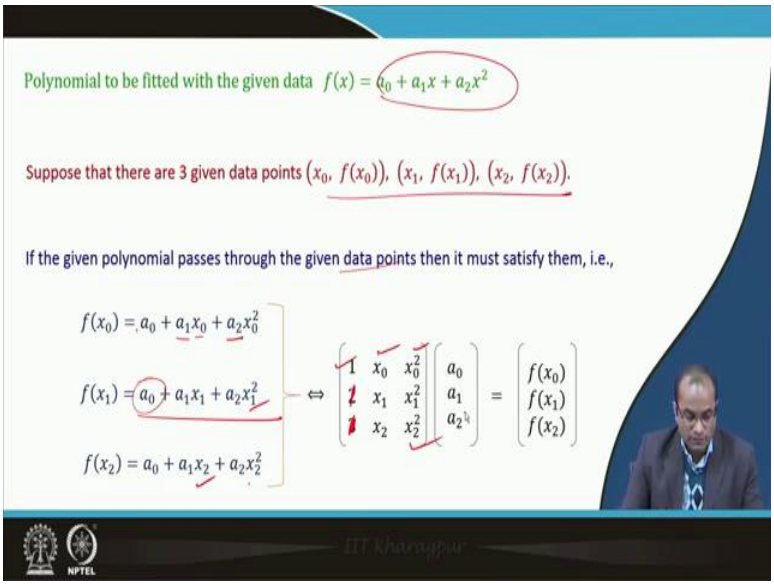

So, now this is the polynomial to be fitted with given data and suppose there are the 3 data points x naught fx naught, x1 f x1, and x2 f x2, so these 3 data points are given and we want to a construct a polynomial which actually passes through these 3 points. So, if the given polynomial passes through the given data points then it must satisfy that means the f x0 if we pass here x naught point so fx naught must be equal to a naught plus a1 x naught, plus a2 x naught square.

So, that is the first condition we should have that these polynomial should satisfy the given data point. The second data point is x1 f x1, so when we substitute in the polynomial x1 for x then we should get f x1 and the third one when we substitute tihs x2 we should get f x2. So, these are three equations we have from this given polynomial which passes through the three given points.

This we can also rewrite in the matrix vector form so here the matrix is then the coefficient of a naught that is 1 here, the coefficient of a1 that is x naught, the coefficient of a2 that is x naught square. Similarly, here we have the coefficient of this a naught that is 1, so it is not 2 it is 1 and then we have x1 there, we have x1 square and in this case again we have 1, we have x2 and we have x2 square. So, this is the system we have for corresponding to this given linear equations.

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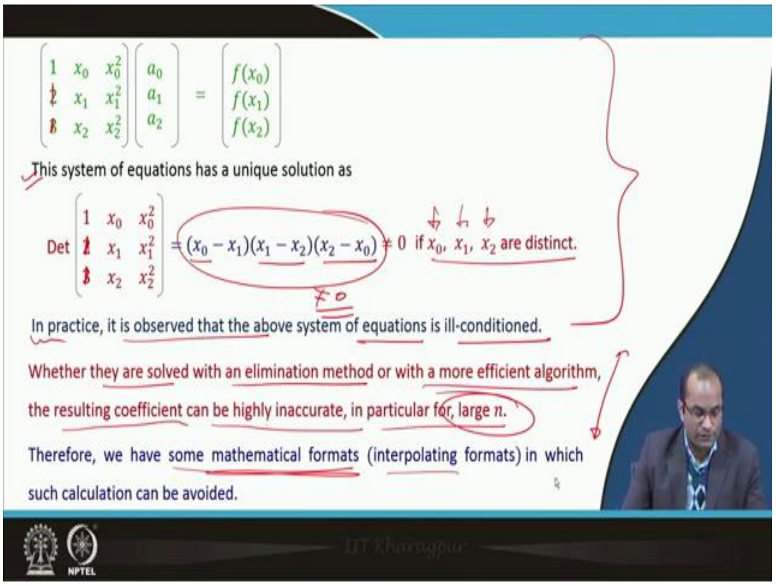

So, having this system now, what we will analyze now that this system of equations has a unique solution, this we will show now in a minute. And when we compute the determinant of this function here what we will realize that the value of the determinant is nothing but x naught minus x1, x1 minus x2 and x2 minus x0.

So, having this value what we realize that if these points x0, x1, x2 are different, if these point are different, what will happen that the value here will be non-zero so the value of this will be non-zero and once we have this determinant non-zero we know that there will be a unique solution.

So, if this are distincts and they are obviously distinct we are talking about x naught, x1, x2 they are distinct points then definitely we have this unique solution. So, in practice. What is the problem now? So, this is what we can, this is, this explains the method also the way we can construct a polynomial given data points.

So, if n plus 1 data points are given we can construct uniquely a polynomial of degree n. So, what is the problem why we are now discussing more about the polynomial, the reason is that in practice it is observed that the above system of equations is ill conditioned. What do we mean my ill conditioned here?

So, when, whether they are solved, so this system or the equation they are solved with an elimination method Gauss elimination et cetera or more they like iterative methods we have already discussed. So, any other efficient algorithm we take the resulting coefficient can be highly inaccurate.

So, this ill conditioned means very little error in the, in this matrix entries can lead to a very inaccurate, highly inaccurate values of its solution that means a naught, a1 and a2 so that means the ill conditioned so this is what observed we are not going much into detail of this part.

But for large n, one can understand that the system is actually difficult to solve because of this ill conditioned system. So, what we have then there are some mathematical formats which we will be talking about in the next lecture and those are the methods how to get this interpolating polynomial without following this approach that constructing the system of linear equations and then solving them. Rather we have a better ways and direct ways of constructing the polynomial without solving such a system of linear equations.

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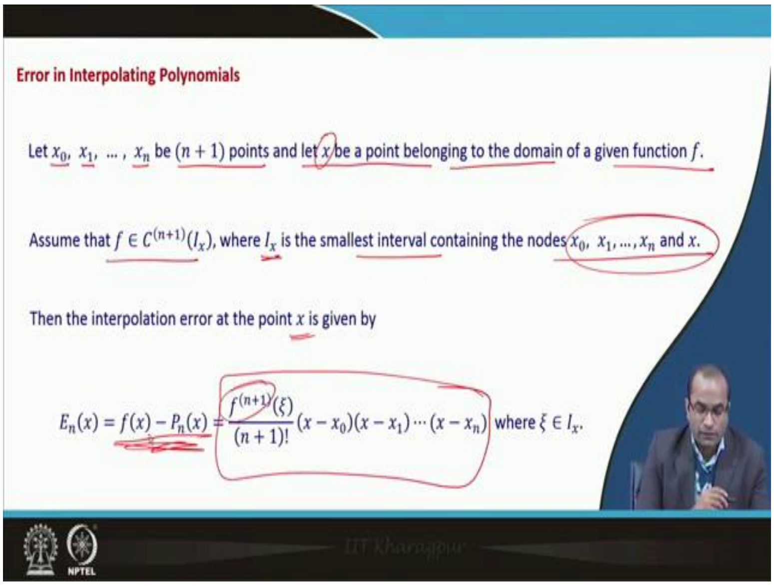

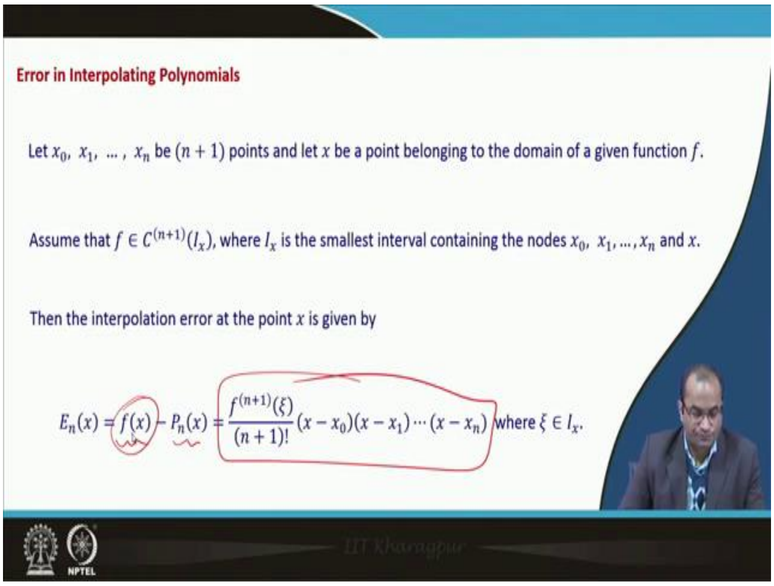

Well, so the another, the last portion of this lecture we will be talking about the error in the interpolating polynomials. Because the polynomial is unique and now we can talk about before we go for different methods how to construct that polynomial because the polynomial is unique so we can talk about what is actually the error in this approximation that is also an important point which we should address.

So, suppose x naught, x1, x2, x3, xn these are n plus 1 points. And we let that x be a point belonging to the domain of a given function. So, all these points x0, x1, x2 et cetera and also we have taken one more point x these all belongs to domain of the function. And we assume that f is n plus 1 times differentiable function where this I x is the domain here or the interval containing all these notes and this x is a smallest interval we can think of which contains all these points including this x. So, Ix is basically the domain of these function f.

So, then the interpolation error at the point x, so because we are talking about the polynomial which or the function which has its domain. So, at any point x difference between the polynomial and the function is given by so f x minus this Pn x one can compute by this formula here which has this n plus 1th order derivative so f the n plus 1th order derivative at point xi which belong to this interval itself factorial n plus 1. And there is a product here x minus x naught, x minus x1, x minus xn. So, this is the error we have when we interpolate this function, approximate this function by the polynomial taking these n plus 1 points.

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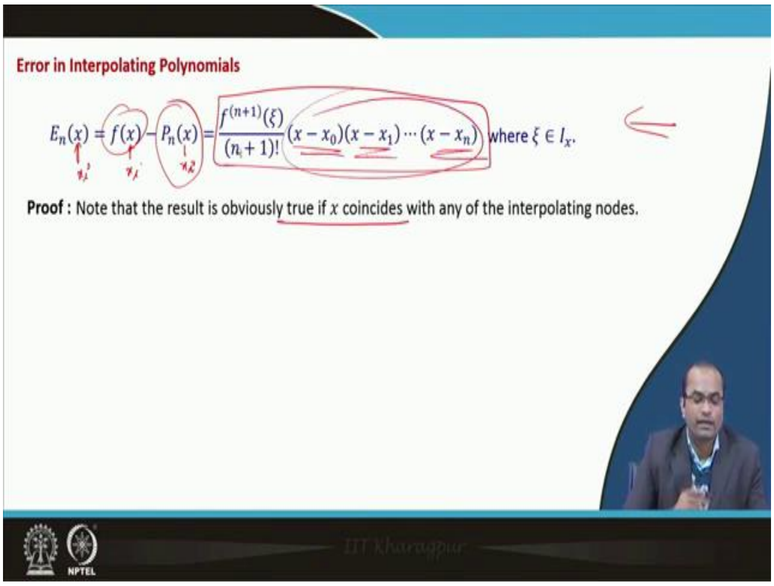

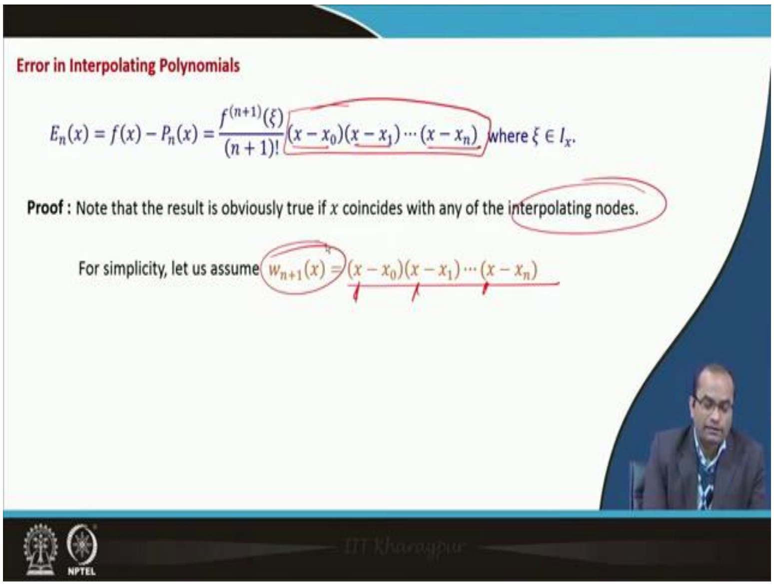

Error in Interpolating Polynomials

$$E_n(x) = f(x) - P_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x - x_0)(x - x_1)\cdots(x - x_n) \text{ where } \xi \in I_x.$$

Proof : Note that the result is obviously true if x coincides with any of the interpolating nodes.

For simplicity, let us assume $w_{n+1}(x) = (x - x_0)(x - x_1)\cdots(x - x_n)$

Now, define for any $x \in I_x$, the function

$$G(t) = E_n(t) - \frac{w_{n+1}(t)E_n(x)}{w_{n+1}(x)};\ t \in I_x$$

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So, we want to derive now how to get this formula here for the error so the proof we will just go for the outline, so the result is obviously true when x coincides with the any of the interpolating nodes. Why? So, if we are talking about that this x coincides with x i then we have here x i and this also x i and naturally this polynomial passes through all these points fx i so this will be 0 here, so then the error is 0 and then here this side also this formula will give because when x is one of this nodes so this will make everything 0 so we have 0 equal 0.

So, that is obviously true when this x coincide with the interpolating nodes meaning this x naught, x1, x2, x3 and so on. And now for simplicity we assume because there was a product here lengthy product x minus x naught, x minus x1, x minus xn. So, this product here we are denoting with the Wn plus 1, n plus 1 is actually the degree of this polynomial again because this product is nothing but a polynomial and that degree of this polynomial is n plus 1 so we are we are denoting this by W n plus 1.

And now we define that for any x we take in the domain of this function where we want to calculate this difference between the two, we define this function and this function will to help us prove exactly this formula for the error which is the aim this lecture now. So, we define this function Gt as En t, En is defined already that is the difference between this fx and the polynomial Px.

So, here we have En t then use here this Wn t which is already define here as wn x, we have again En and we have again Wn so in terms of this En and Wn which is also Wn is defined here and En is defined there, so in terms of this Wn, En we have defined a function Gt. And now we will discuss what are the properties of this Gt and how we can get out of with the help of this function the error formula.

So, the first thing we should notice that since this f is n plus 1 times differentiable function this G is also n plus one times differentiable function and why? Because Gt is nothing but the Et, En t and Wn plus 1 t, En x and this Wn x they are constants because for any but fixed x here we are talking about this formula here t is only the variable which we are considering now.

So, what is En t? En t is ft minus this Pn t. So, again this is a polynomial and f is a n plus 1th time differentiable function so En we have here the n plus time differentiable so there is no problem for the differentiability this En for, sorry for this Gt as long as this first term is concerned and the second one here this is polynomial these are some fixed values so this is fixed for given x.

And Wn t is a polynomial so here we have this polynomial, here we have that function which is again n plus 1 time differentiable so everything here we can differentiate n plus 1 times that means the G function is n plus 1 times differentiable. That is the first property of this function G.

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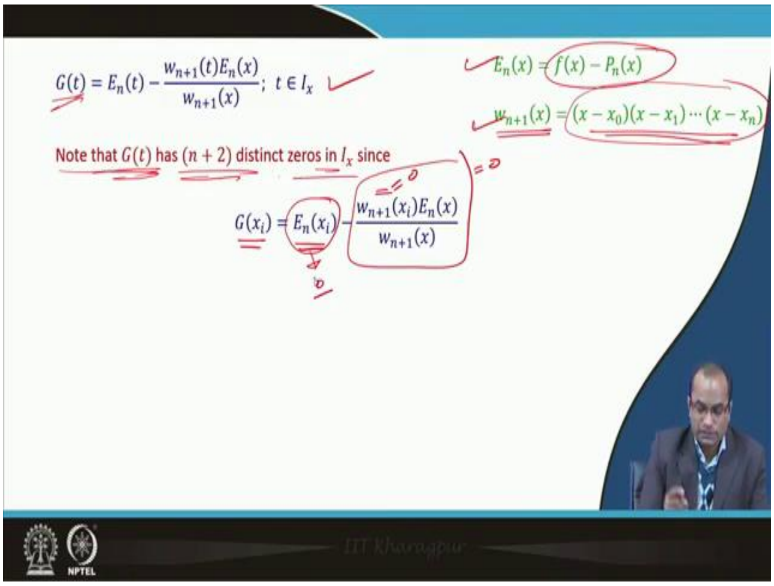

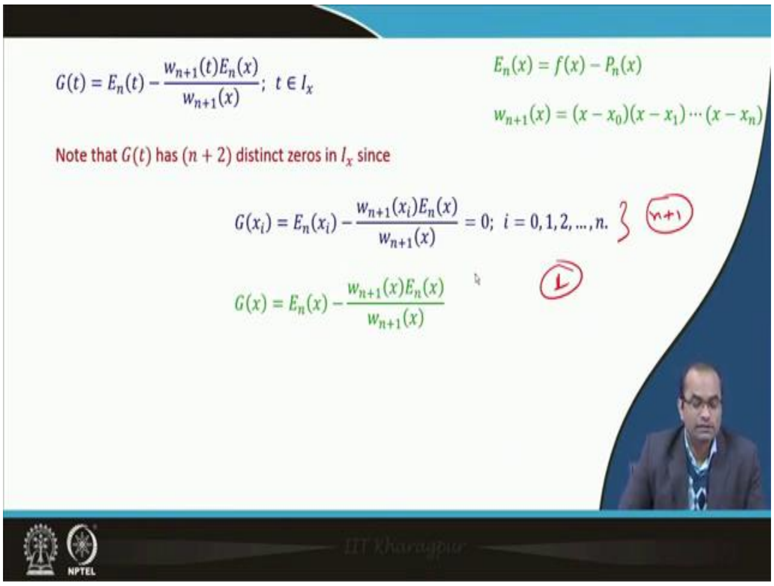

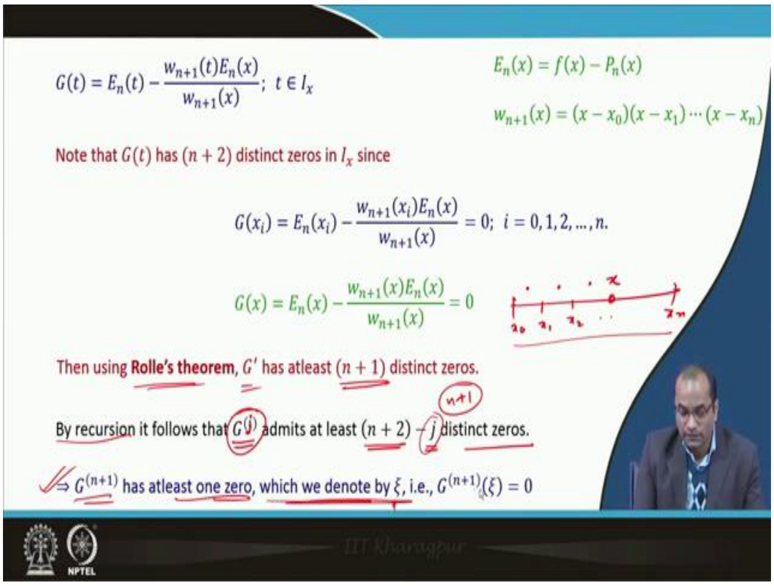

So, having this G we have En we have Wn already and what we will notice now that Gt this function has n plus 2 distinct zeros in this Ix. How these, how this function has n plus 2 distinct zeros we will notice now, so, just consider that G xi so we have En xi we have here also Wn plus 1 xi. Now, this Wn plus 1 xi this is a Wn plus 1, when we put here for x equal to xi, this will become 0, so this part is 0, and hence this part is 0 so the second part is 0.

Now, coming to the first one, when En xi this error here at xi naturally that will be 0 because this xi is the narrow point. So, this is also 0 and 0 minus 0 then everything is 0 here for all i is 0, 1, 2, 3 and so on n. So, there are here n plus 1 points, so here we have n plus 1 points where this G xi vanishes, G vanishes so there are n plus 1 distinct points we have discuss and one more point that is the x which we have already fixed at the beginning its arbitrarily point but fixed point here so this at Gx also notice we will notice that this is equal to 0 because we have now En x, Wn and so on, these two are the same so cancel out we have En x minus En x and that is 0.

So, here the Gx is also 0, so we have n plus 1 points here, n plus 1, n plus 1 so the total there are n plus 2 distinct zeros in I x of this G. So, having this property now, we can use the Rolle’s theorem which will say that G prime its derivative will have at least n plus 1 distinct points because whenever we have a function for instance it has these two zeros here so if Fa is 0 and Fb is 0, then the Rolle’s theorem says that there will be a point in between where its derivative will be also 0.

So, between the two zeros here, its derivative will have one zeros and the same argument we can have here so the G function G is 0 at x0 and x1, x2 and so on and also xn and also there is a point x somewhere. So, there are n plus 2 points there and between each these two zeros there will be 1 one point where G prime will be 0 and so on and we count this will be just a 1 less than the number of these zeros.

So, there were n plus 2 zeros, so G prime will have n plus 2 minus 1 that is n plus 1 distinct zeros. By recursion the we can continue this process that G double prime will have n distinct zeros and so on or ith derivative of this G this is ith derivative of G it will admit n plus 2 and minus the here if it is jth so let us say jth derivative then n plus 2 minus jth distinct zeros this will have.

So, n plus now we are continuing and now let us say if j is n plus 1 so if we put here n plus 1 so this will remain just 1 there, so Gn plus 1 will have at least zeros this is the implication of this Rolle's theorem now and which denotes and so we are denoting by this Xi this n plus the Gn plus 1th derivative has one zero and that is we are consider that is z Xi that means that G n plus 1 Xi equal to 0 so this result we have.

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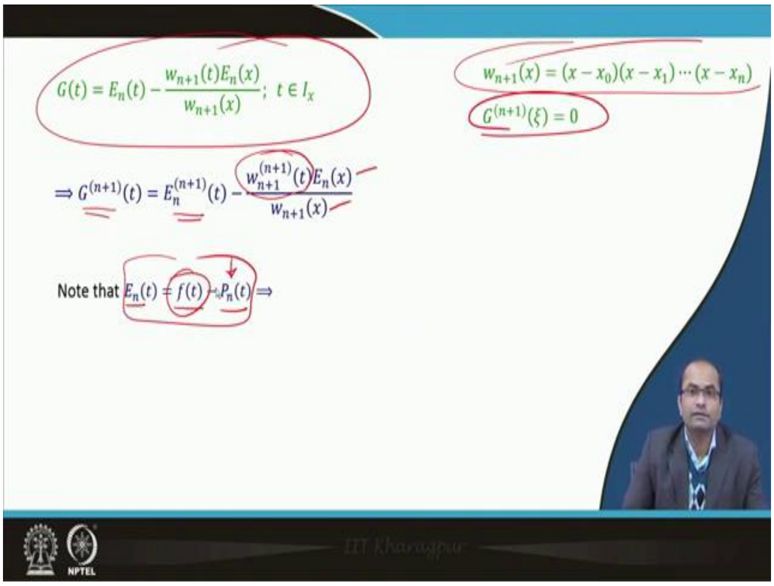

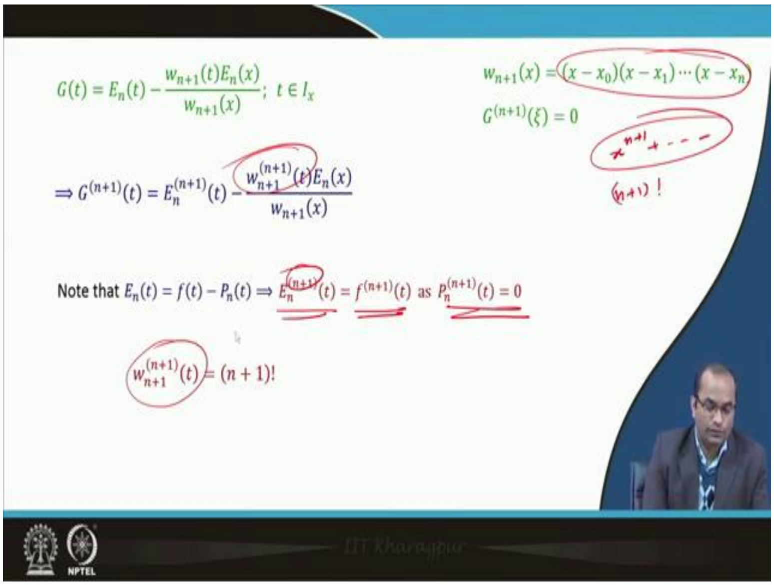
$G(t) = E_n(t) - \frac{w_{n+1}(t)E_n(x)}{w_{n+1}(x)};\ t \in I_x$
$w_{n+1}(x) = (x - x_0)(x - x_1)\cdots(x - x_n)$
$G^{(n+1)}(\xi) = 0$
$x^{n+1} + - - -$
$(n+1)!$
$\Rightarrow G^{(n+1)}(t) = E_n^{(n+1)}(t) - \frac{w_{n+1}^{(n+1)}(t)E_n(x)}{w_{n+1}(x)}$
Note that $E_n(t) = f(t) - P_n(t) \Rightarrow E_n^{(n+1)}(t) = f^{(n+1)}(t)$ as $P_n^{(n+1)}(t) = 0$
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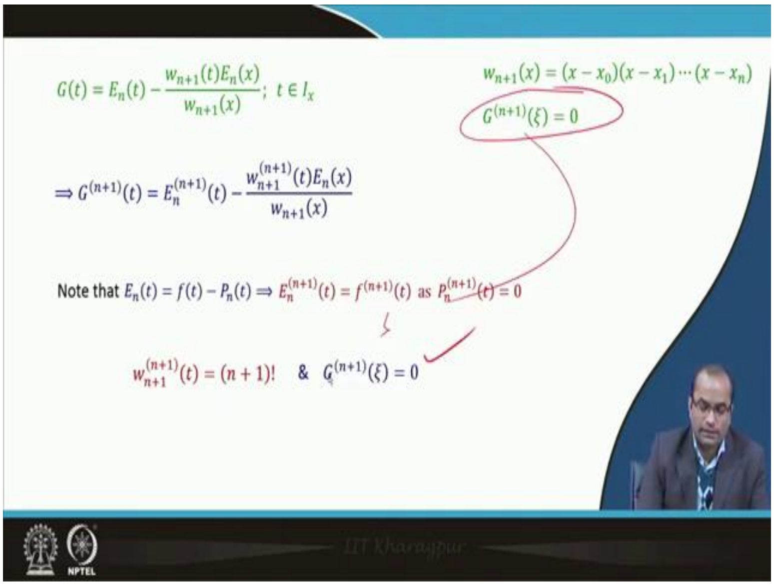
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$\Rightarrow G^{(n+1)}(t) = E_n^{(n+1)}(t) - \frac{w_{n+1}^{(n+1)}(t)E_n(x)}{w_{n+1}(x)}$
Note that $E_n(t) = f(t) - P_n(t) \Rightarrow E_n^{(n+1)}(t) = f^{(n+1)}(t)$ as $P_n^{(n+1)}(t) = 0$
$w_{n+1}^{(n+1)}(t) = (n+1)!$ & $G^{(n+1)}(\xi) = 0$
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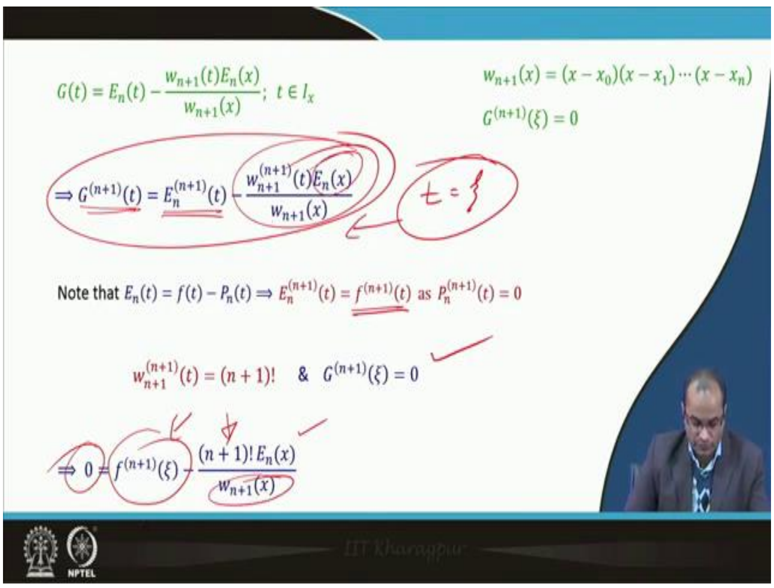
$G(t) = E_n(t) - \frac{w_{n+1}(t)E_n(x)}{w_{n+1}(x)};\ t \in I_x$
$w_{n+1}(x) = (x - x_0)(x - x_1)\cdots(x - x_n)$
$G^{(n+1)}(\xi) = 0$
$\Rightarrow G^{(n+1)}(t) = E_n^{(n+1)}(t) - \frac{w_{n+1}^{(n+1)}(t)E_n(x)}{w_{n+1}(x)}$
$t = \xi$
Note that $E_n(t) = f(t) - P_n(t) \Rightarrow E_n^{(n+1)}(t) = f^{(n+1)}(t)$ as $P_n^{(n+1)}(t) = 0$
$w_{n+1}^{(n+1)}(t) = (n+1)!$ & $G^{(n+1)}(\xi) = 0$
$\Rightarrow 0 = f^{(n+1)}(\xi) - \frac{(n+1)!\,E_n(x)}{w_{n+1}(x)}$
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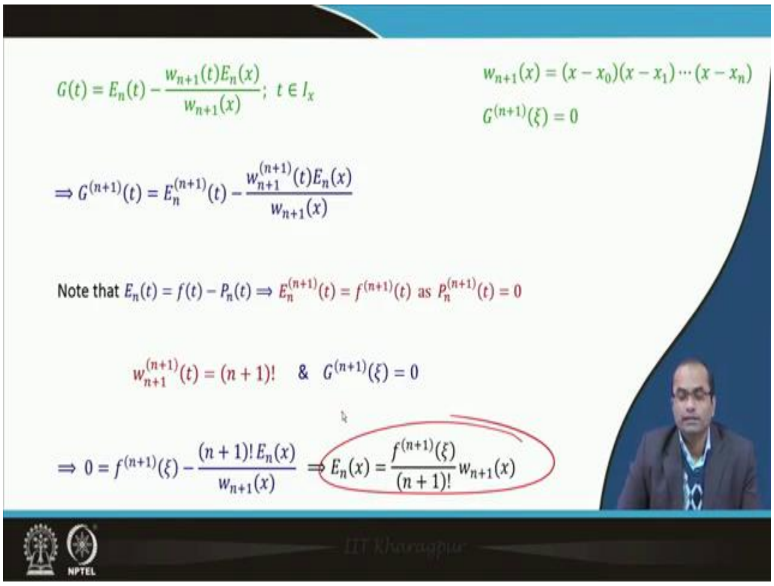

So, what do we have? We Gn plus 1 Xi equal to 0, Wn plus 1 was defined by this product and we have this function which we have started with. Now, we can talk about the n plus 1th derivative of this that is En n plus 1th derivative and the n plus 1 derivative of this are Wt the rest here they are just fixed values.

Now, we should note that talking about this En. So, En t was f t minus Pn t and when we take the nth order derivative so nth order derivative will go to f and this is a n degree polynomial, n plus 1th degree, n plus 1th at derivative will be 0 of this polynomial so what we have that En the n plus 1th derivative will be simply the n plus 1th derivative of f because this is equal to 0.

What else we have the n plus 1th derivative here of this W, W is again this is n plus 1th degree polynomial so here we have something x n plus 1 plus anything else there with lower degree terms and when we take the n plus 1th order derivative so simply we will get n plus 1 factorial which is written here so that is done and obviously we have this result n plus 1th derivative of G at Xi is equal to 0.

So, having all this result now we will substitute here for t is equal to Xi so if we put if put here t is equal to Xi what will happen G n plus 1th Xi will become 0. Then here we have this n plus 1th derivative of E that is nothing but the n plus 1th derivative of f so we have here f n plus 1 Xi and then here also this n plus 1 factorial, then we have here E n plus 1 over Wn plus 1.

So, this is what we have now from this given function which we have chosen and now just from this we can in conclude that En x so this is En x, En x will be equal to this one and this is exactly the error formula which we want to derive. So, now in this lecture we have also derived the error formula because this polynomial is unique so irrespective how do we calculate but we know that will be the error.

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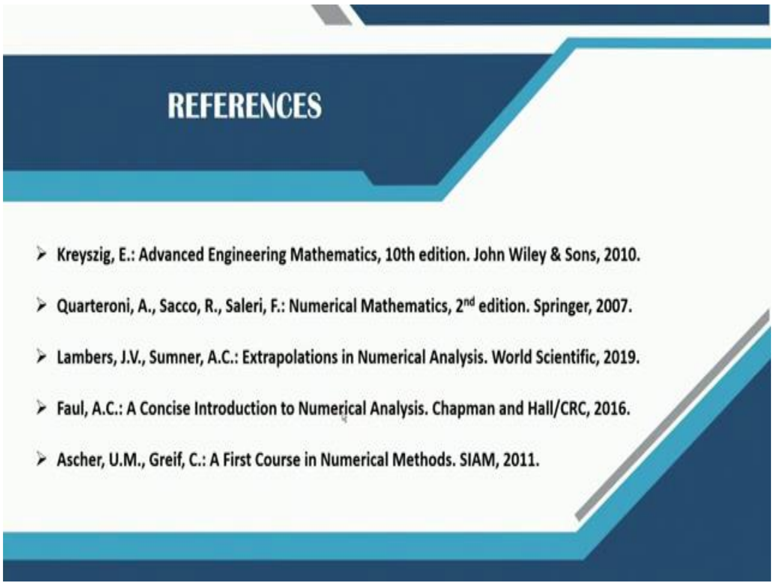

So, these are the references we have used for preparing this lecture

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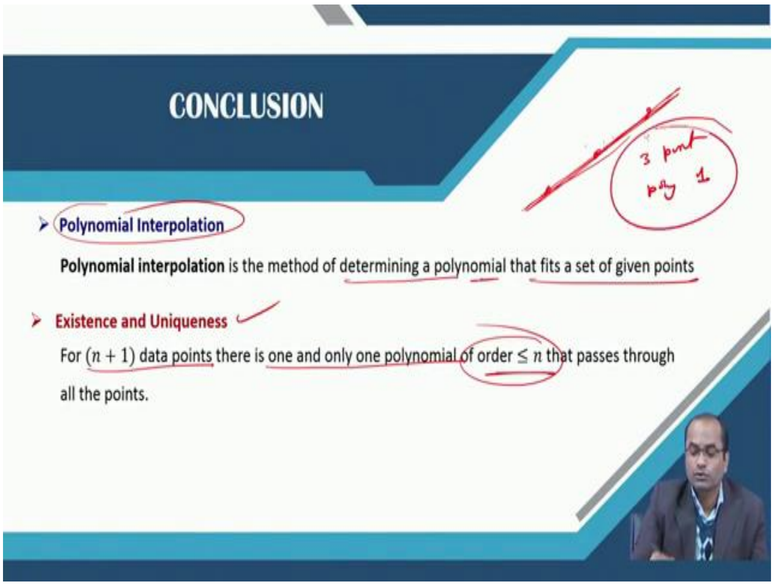

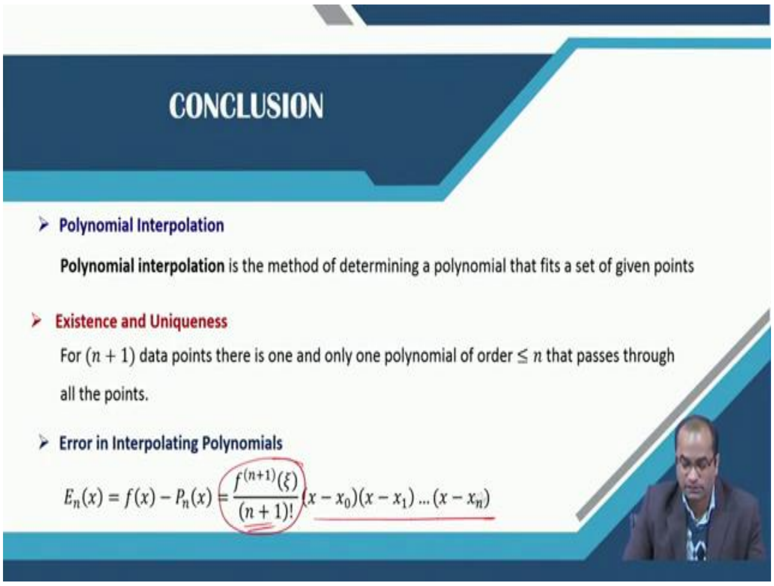

And just to conclude that we introduced the polynomial interpolation which is the method of determining polynomial that fits a given set of data points. And we have also proved the existence and the uniqueness for n plus 1 data points there is only one and only one polynomial of order less than equal to n. Why less than equal n we are talking about?

Because for instance if we take, if we take three points, so according to our discussion we should have a polynomial of degree 2. But in this particular case because they are points in one line so will get only linear equation that means only the line because they were in one line co-linear points.

So, therefore, so here again having three points but we are getting a polynomial of degree 1. So, that what we have written here that is always possible that when n plus 1 data points are given there is one and only one so uniqueness is always there. And there would be polynomial of degree less than or equal to n that passes through all the points.

And we have also discussed the error in the interpolating polynomial which is given by this formula f n plus 1 at point some Xi and then we have n plus 1th factorial and this product x minus x naught, x minus x1 and x minus xn. So, that is all for this lecture and I thank you very much for your attention.