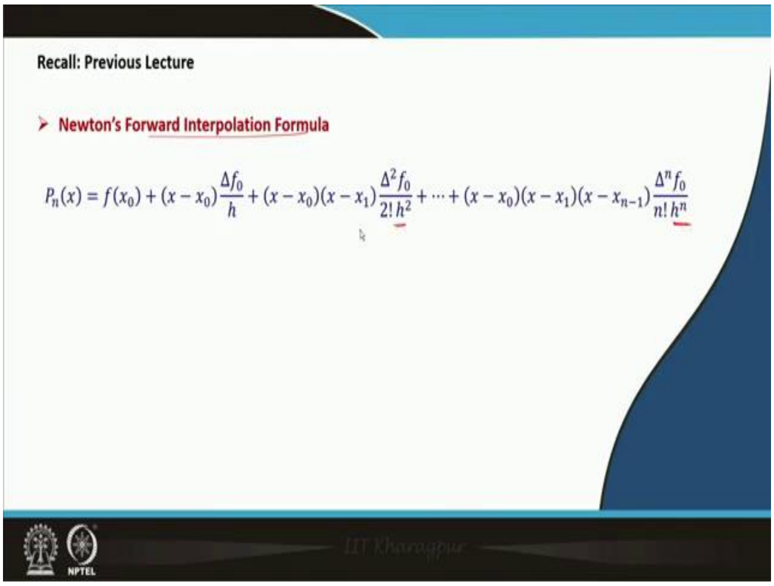
Engineering Mathematics -II
Professor Dr. Jitendra Kumar
Department of Mathematics
Indian Institute of Technology, Kharagpur
Lecture 29
Polynomial Interpolation (Contd.)

So, welcome back to lectures on engineering mathematics two and this is lecture number 29 and we will continue our discussion on polynomial interpolation.

(Refer Slide Time: 0:23)

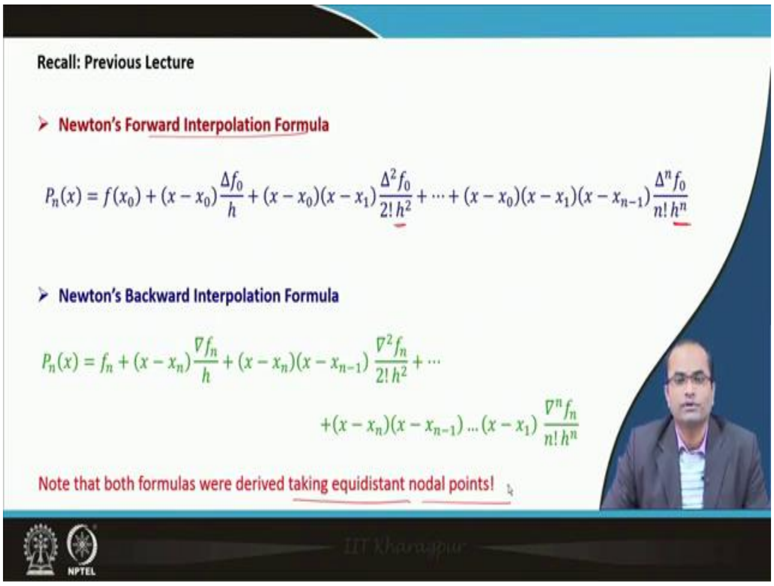
So, in this lecture we will be talking about two different techniques two different methods for getting interpolating polynomial. So, the one is Newton's Divided difference which is quite similar to what we have done Newton's forward and backward interpolation formulas and the other one we will be talking about the Lagrange interpolation formula and these both are for unequal intervals. So, the grid points or the nodal points are not just equidistant in this case.

(Refer Slide Time: 0:56)



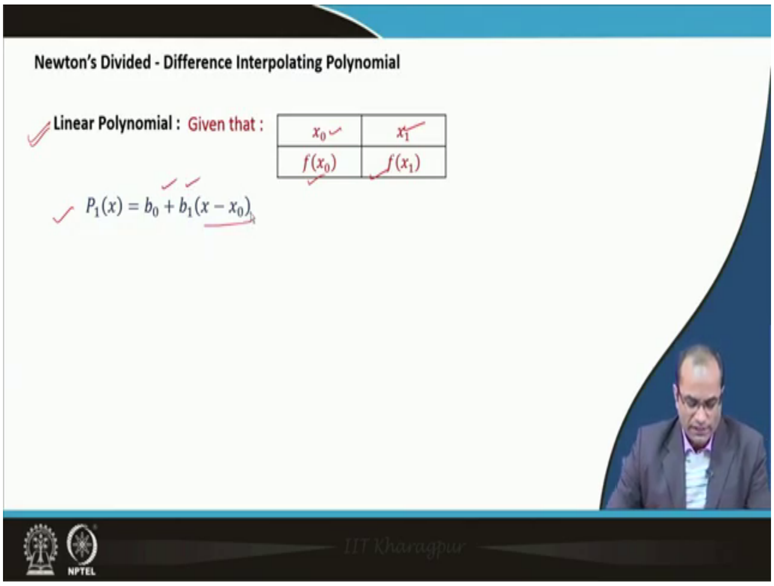
So far we have discussed in previous lectures the Newton's forward interpolation formula where the formula looks like this one, where we have somewhere there in each term this h, which was the distance between the nodal points x naught x 1 or x 1 x 2, and it was assumed that they are equidistant.

(Refer Slide Time: 1:20)



Similarly, for the Newton's backward formula as well we have assumed that these nodal points are equidistant. So, this is a note here also that in both the formulas were derived taking equidistant nodal points and now, we will be talking about two more formulations which are more general in the sense that we can have non equidistant nodal points as well.

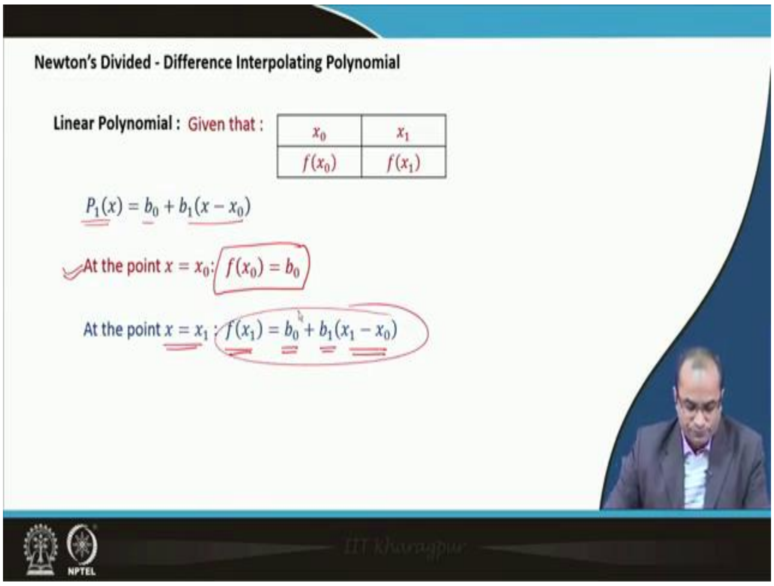
(Refer Slide Time: 1:52)



So, the first one is the Newton's Divided difference formula which is very similar to what we have already done. So, let us go back to what we have started with for the derivation of Newton's forward or the backward difference formula. So, given that we have two points x naught and x 1 and their corresponding values f x naught and f x 1 are given.

So, basically two points are given and as we know that we can fit a straight line which will pass through these two points. So, we take this linear polynomial as b naught and b 1 x minus x naught as usual we have taken earlier also for the sake of simplicity of the evaluation of these coefficients we have taken this special format.

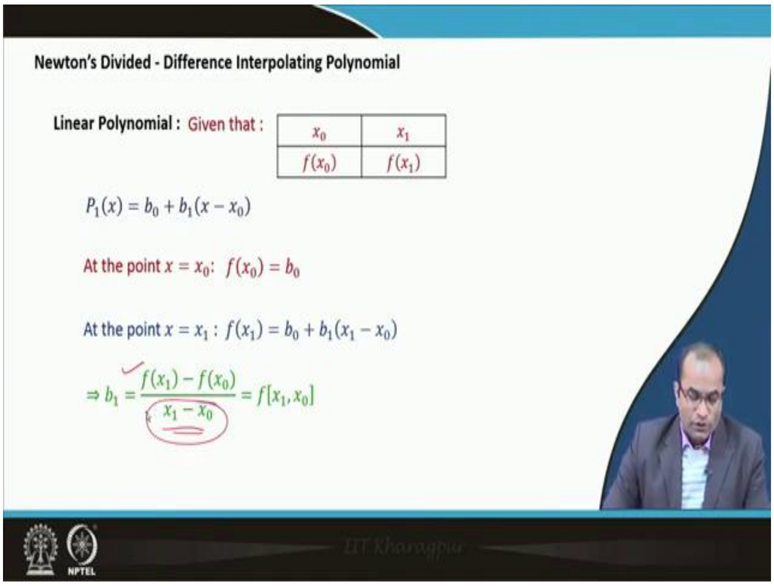
(Refer Slide Time: 2:40)



And immediately when we substitute x is equal to x naught so, this polynomial has to give the value f x naught an equal to b naught and this term will be 0 so, the b naught we will get as f x naught and then we can go for the other point which is x 1.

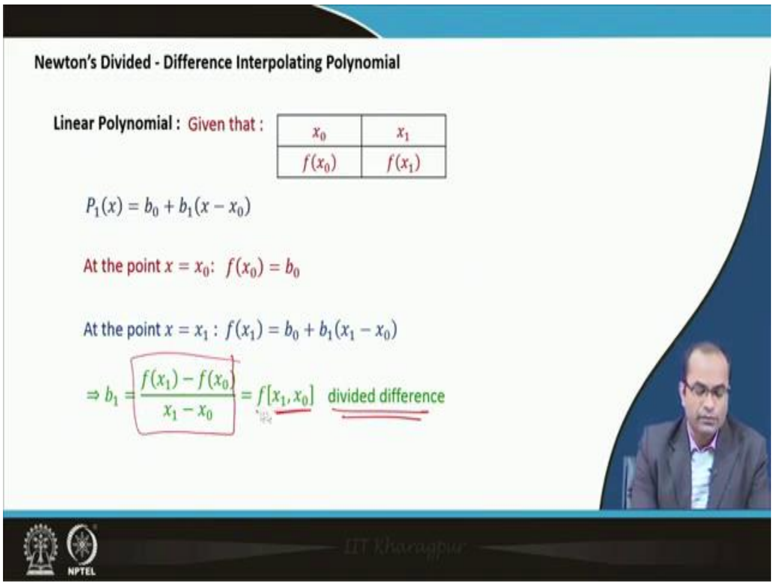
So, at the point x is equal to x 1 when we substitute we will get f x 1 is equal to b naught plus this b 1 and x 1 minus x naught.

(Refer Slide Time: 3:15)



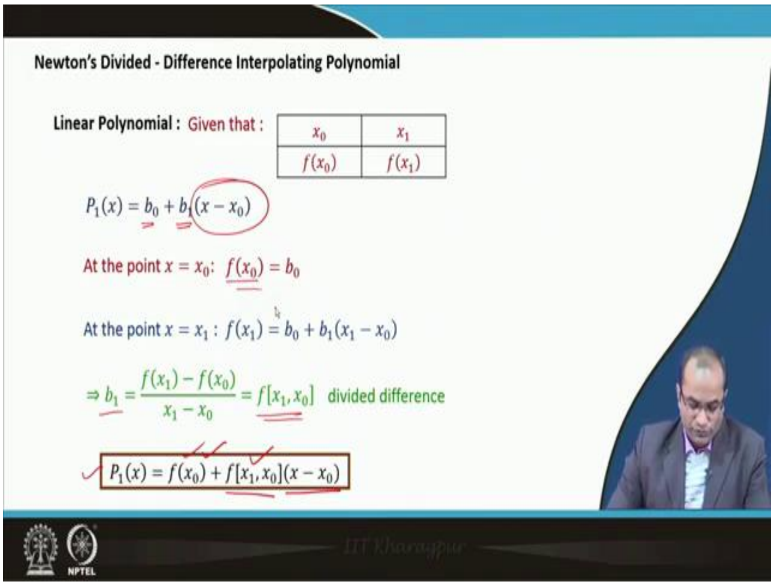
So, from here we can get b 1 as usual we have done before and the b 1 will be f x 1 minus f x 0 over x 1 minus x naught. And earlier we have written this b 1 in terms of the forward difference formula assuming that they are equidistant and this x 1 minus x naught was taken as h.

(Refer Slide Time: 3:38)



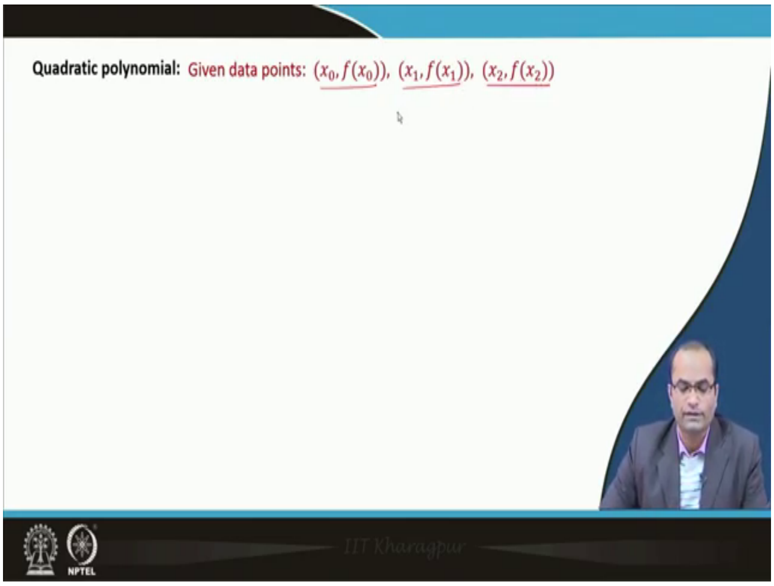
So, now to make it more general this expression here which is f x 1 minus this f x naught over x 1 minus x naught, we are defining this as f x 1 x naught with this symbol which is called the Divided difference formula so, this is the divided difference taking these x 1 and x naught to nodal points. We are defining this ratio by this symbol f x 1 x naught with this square bracket. So, this is what we call the divided difference.

(Refer Slide Time: 4:13)



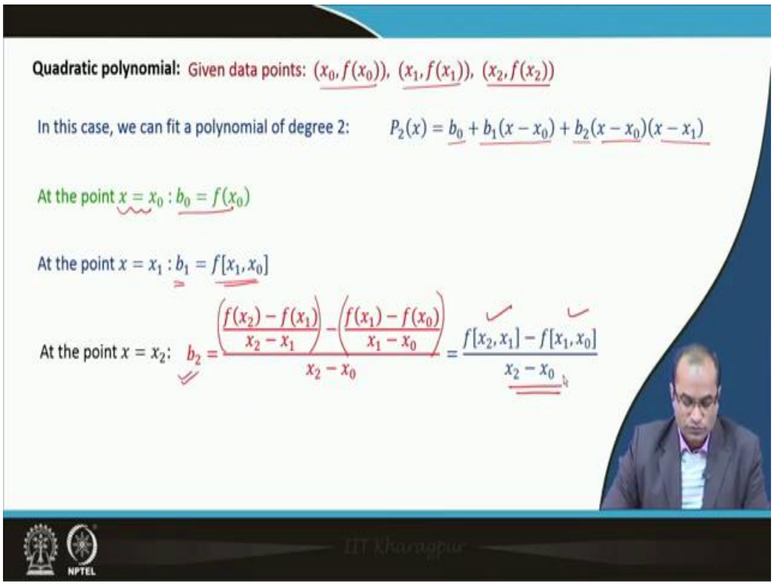
So, having this b 1 in terms of this f x 1 x naught, we got the polynomial as f x not plus this divided difference here with two points and then we have here x minus x naught which was there already in the polynomial so, the b 1 we have substituted as this divided difference and the b naught was f x naught. So, we got this polynomial of degree 1 in terms of the divided difference.

(Refer Slide Time: 4:40)



Similarly, we can continue for the quadratic polynomial as well and in this case we need three points. So, x naught x 1 and x 2 and corresponding values f x naught f x 1 and f x 2.

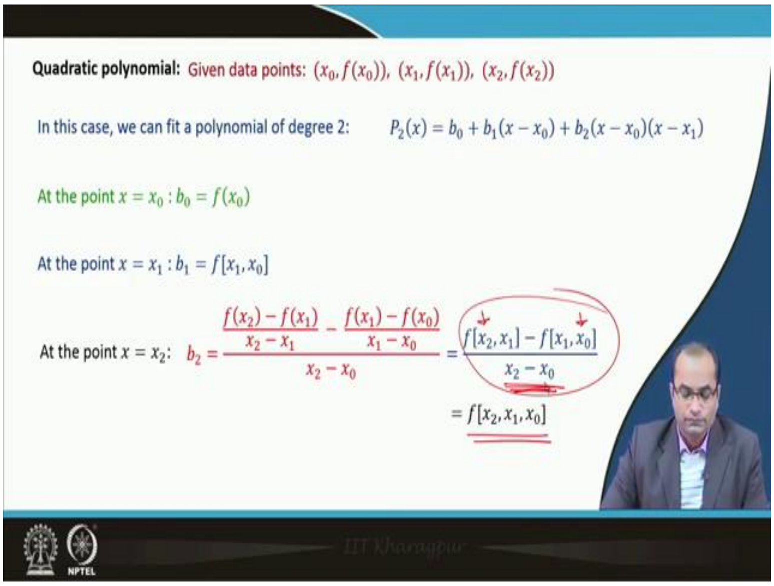
(Refer Slide Time: 4:54)



So, in this case naturally we can fit a polynomial of degree 2 and again we will take this structure b naught b 1 x minus x naught and b 2 x minus x naught x minus x 1 and we will again go for the evaluation of these unknown these coefficients that is b naught b 1 and b 2. So, for b naught we when we substitute x is equal to x naught we will get immediately b naught and similarly b 1 which we have already evaluated before it will be the same value. So, f x 1 x naught and then substituting x is equal to x 2, we can also get b 2.

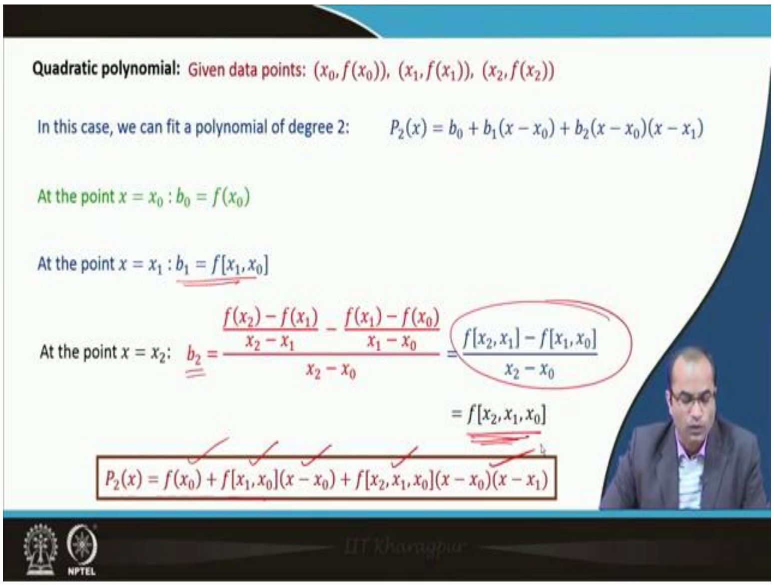
So, b 2 will have this format and these two expressions here we can again write in terms of the divided difference which we have already defined for two points earlier. That means this the first one is f square bracket x 2 x 1 and the second one this is the divided difference, which is denoted by f and the square bracket x 1 x naught and divided by this x 2 minus x naught.

(Refer Slide Time: 6:08)



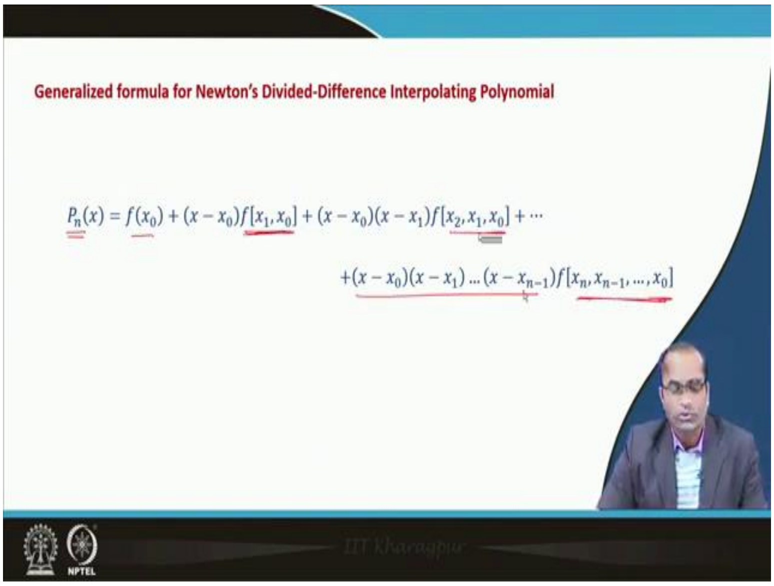
So, here we have another this divided difference of this higher order, which we can denote, because this is again the difference of those divided difference which we have computed first and then divided with this the maximum difference between these nodal points we have, so x 2 was the highest value x 1 the minimum value, so, that difference is coming in the denominator.

(Refer Slide Time: 6:29)



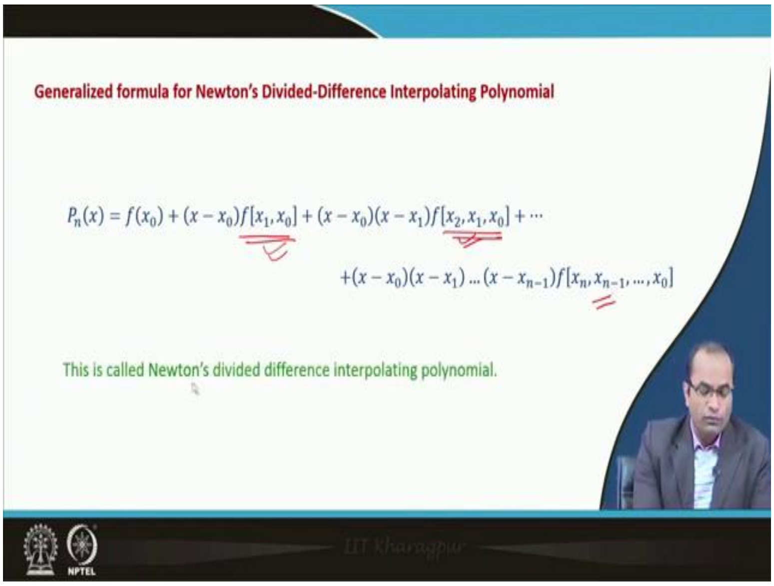
So, here this ratio we are defining as this divided difference again with these three points, that is x 2 x 1 and x naught. So, having this b 1 in terms of this divided difference b 2 also in terms of this divided difference, we can now write down the polynomial which is second degree polynomial now, so f x naught then f x 1 x naught this divided difference multiplied by x minus x naught and then we have this another divided difference and multiplied by this product x minus x naught and x minus x 1.

(Refer Slide Time: 7:07)



So, now, we can continue this for the third order polynomial or the fourth order polynomial. And in general we have such a formulation see nth degree polynomial for given n plus 1 points, we have f x naught this is the divided difference taking x 1 and x naught here again we have the three points and so on we have here n plus 1 points and the corresponding products.

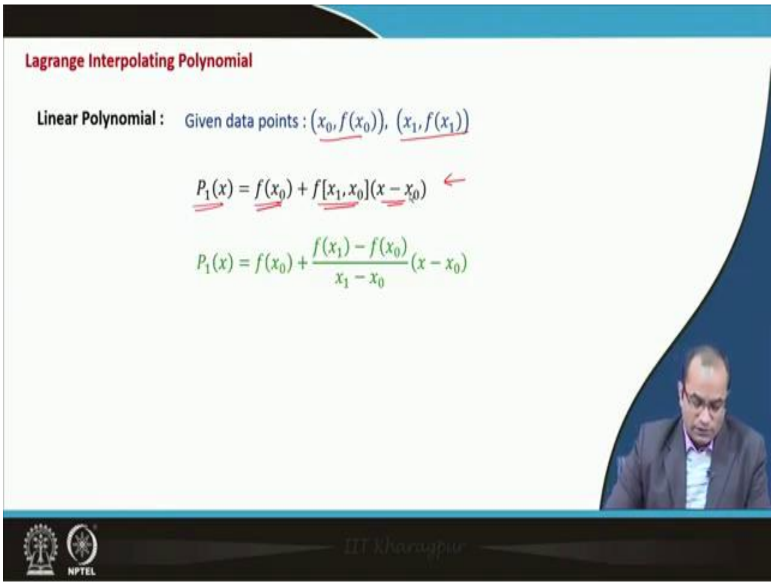
(Refer Slide Time: 7:38)



So, this is the formula which is more general than for instance the Newton's forward difference formula and this can be used when we have non equidistant nodal points given in our data. And the formula looks much simpler than the divided difference because in this divided difference already that h, I mean the distance between the nodal points are hidden.

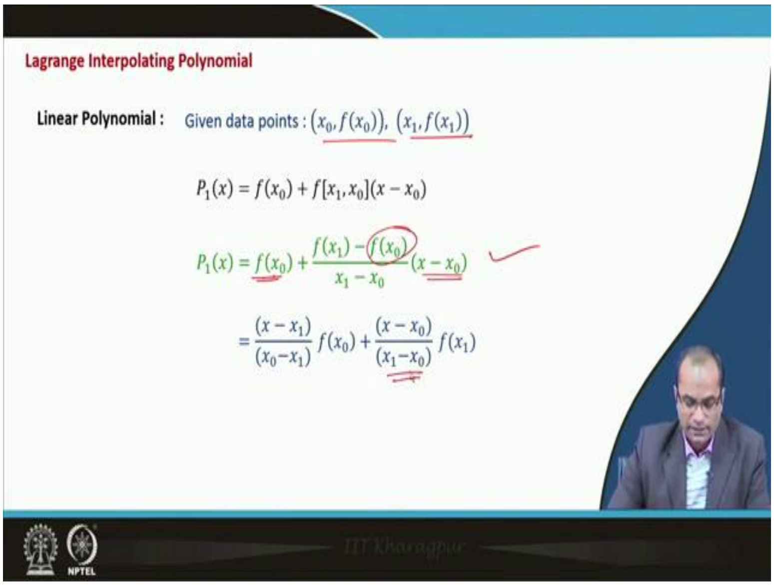
So, we have much simplified formula in terms of this and the interesting point is that these divided difference again we can get with the help of table as we got the differences the finite differences backward and the forward here also these divided difference we can directly obtain from the, from the table and substitute here and we can get the desired polynomial. Well, so this formula is called Newton's Divided difference interpolating polynomial.

(Refer Slide Time: 8:34)



Now, coming to the other one, which we will also discuss in this lecture, that is the Lagrange interpolation formula. So, in this case again, we will go back to the what we are doing with the linear polynomial for instance, so, taking two points, x naught f x naught and x 1 f x 1, what we have just seen, we have seen this divided difference formula, so, the polynomial of degree 1 was f x naught and this first order is divided difference and then we have x minus x naught.

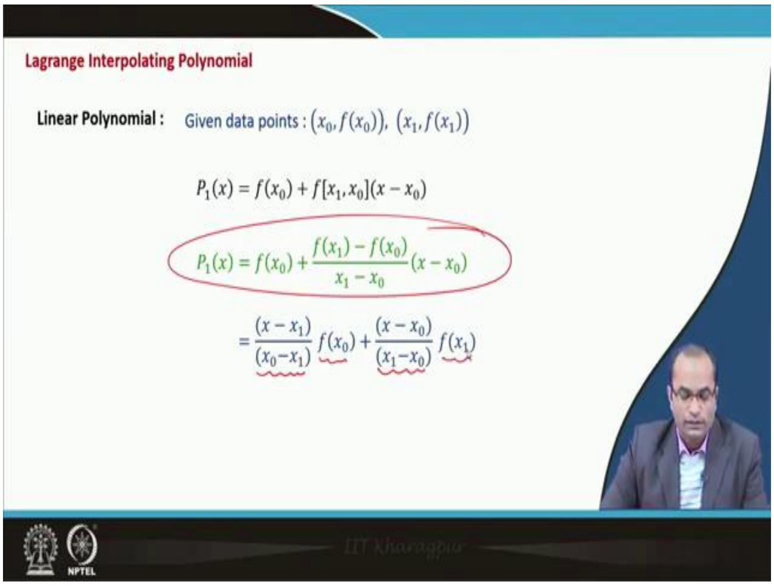
(Refer Slide Time: 9:09)



Or if we substitute here this divided difference so, that was actually this f x 1 minus f x naught over x 1 minus x naught. So, having this polynomial of degree 1 which passes through these two points and now what we are doing we are just rewriting this and that will be another format which is called the Lagrange interpolating polynomial.

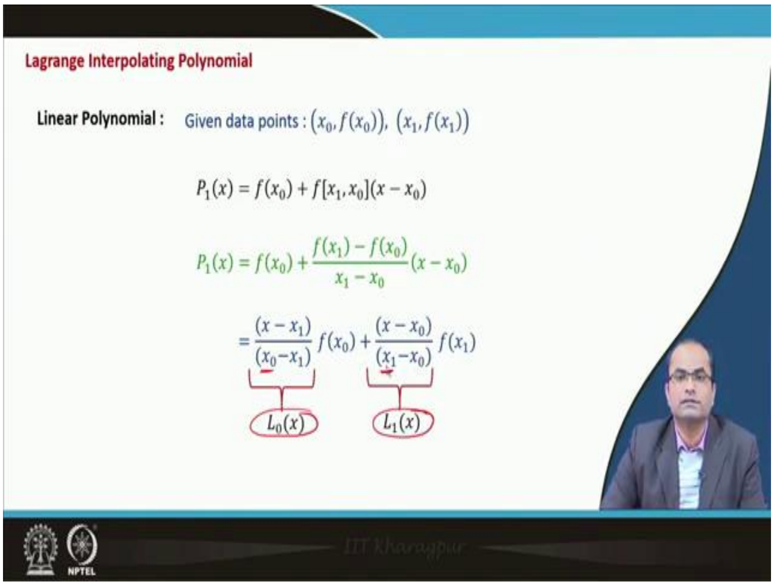
Well, so, we have written now f x naught we have taken common because f x naught f x here also and f x naught there and then we have f x 1 which will just appear with x minus x naught over x 1 minus x naught. And with the f x naught this expression will come x minus x 1 over x naught minus x 1.

(Refer Slide Time: 09:49)



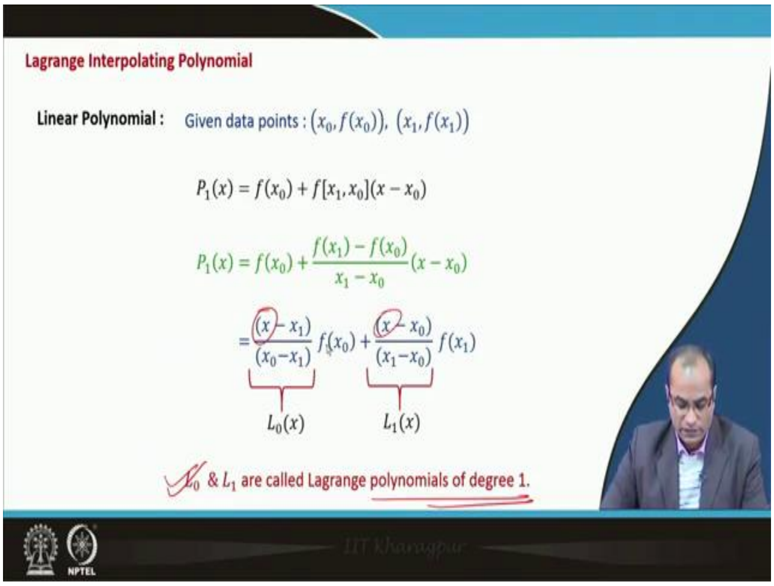
So, we have rewritten this polynomial of degree 1 in this form where we have some polynomial here the value of the function again this polynomial of degree 1 and then f x 1.

(Refer Slide Time: 10:07)



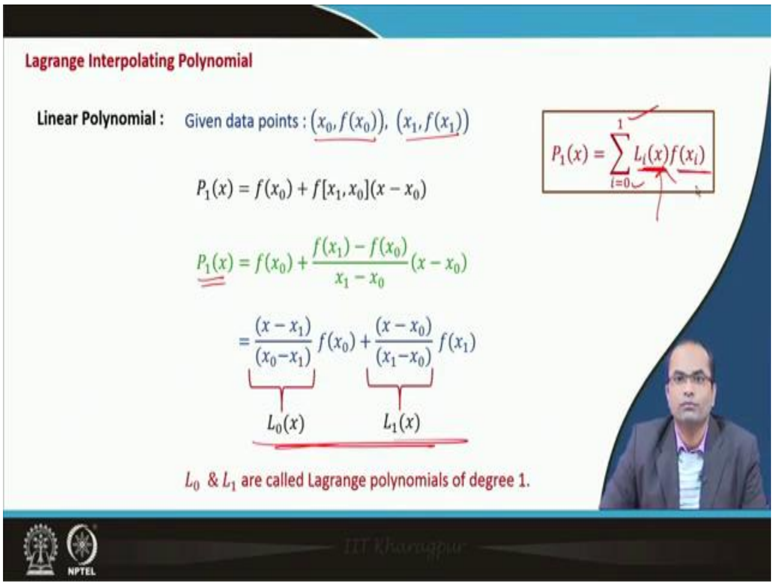
So, these polynomials here for example, this first one this is called L 0 x so the Lagrange polynomial and this is the Lagrange polynomial L 1. So, this 0 means, we have x naught here minus the other points here L 1 means we have subtraction here with x 1 minus. So, we will generalize this in the next slide and then it will be more clear.

(Refer Slide Time: 10:30)



So, these L 0 and L 1 are called polynomials of degree 1 because these are degree 1 polynomial we have only x there and we have only x there.

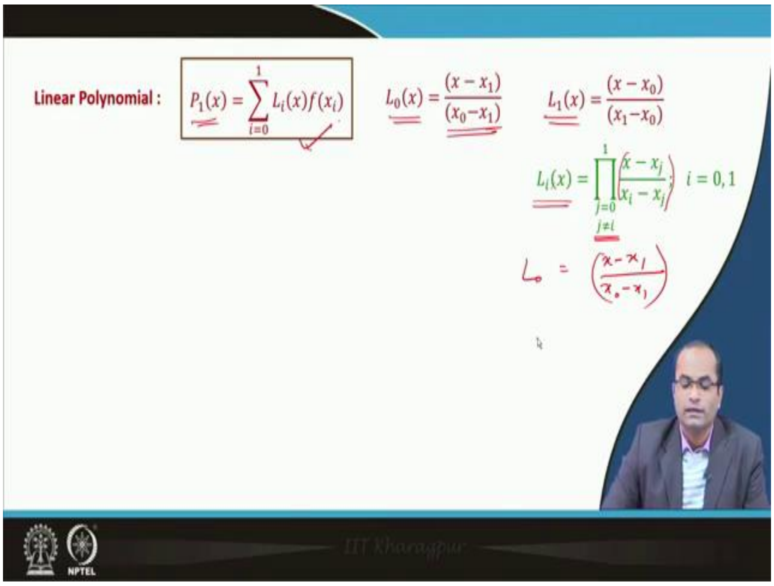
(Refer Slide Time: 10:44)



So, having this format what is the beauty now that we can write down in the form of a sum that this P 1 here is i goes from 0 to 1 we have this so called the Lagrange polynomial of degree 1 and then we have f x i the value of the function at x i. So, these Lagrange polynomial of degree 1 we can define Lagrange polynomial of degree 2, degree 3, etcetera and depending on the given number of points we can generalize this formula in this case we have only two points there.

So, these Lagrange polynomial of degree 1 are appearing here Li x, but when we have for instance three points, those polynomials will be of degree 2 polynomial again the same structure of the formula will continue.

(Refer Slide Time: 11:31)

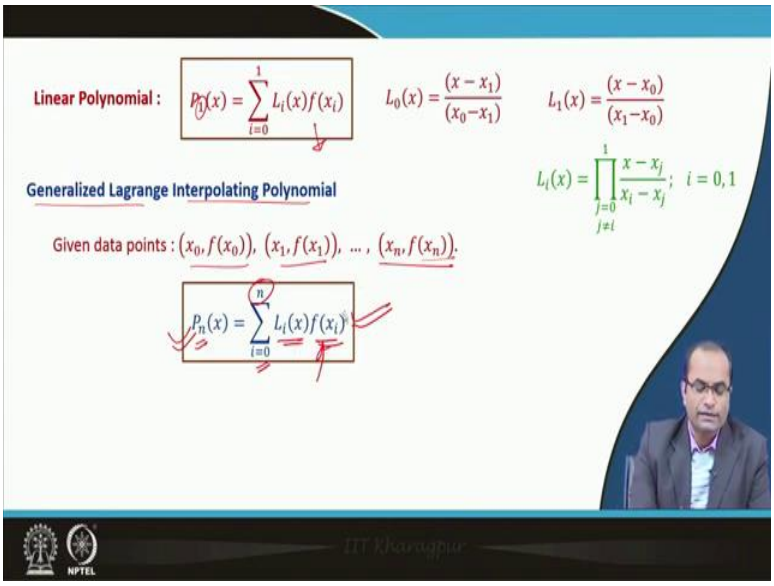


So, for the linear case we have seen that the P 1 x can be written in this summation format and then we have L 0 and the L 1 written in this form. Indeed this L 1 L 0 we can write in a more compact form or more general form that Li is nothing but this product here because when we derive the second order, we can see the structure easily so, we are avoiding here, just to save some time.

And here we have just the product so, this with this term here and then keep on doing this product, but in this case since this is going from j 0 to 1 and then we have j not equal to i, so, when we are talking about L 0, so, j will not be 0, j will be just 1 and we have x minus x 1 and this i is 0.

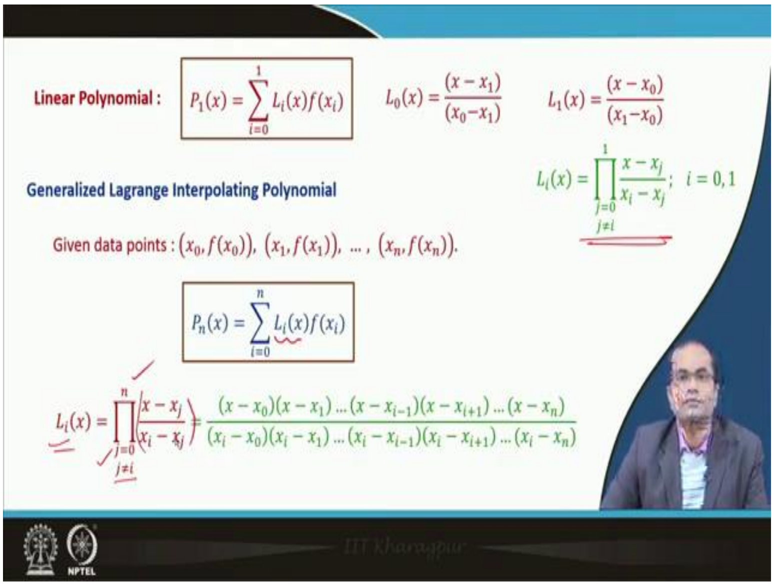
So, x minus x 1 only this there is no product there. And similarly, with L 1 we will get this expression. So, this is more general but when we are talking about the second order polynomial or third order polynomial then this product will come such terms will come in product to make the second order or the third order polynomials.

(Refer Slide Time: 12:45)



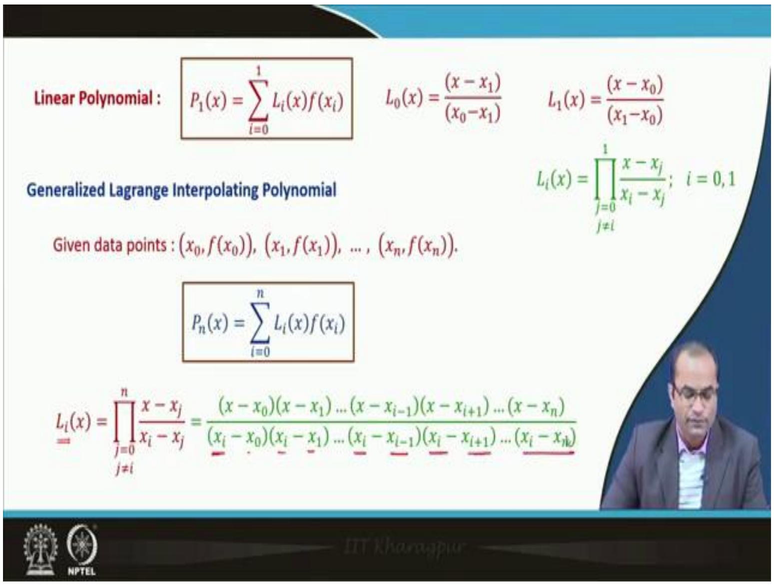
Well, so, the generalized Lagrange interpolating polynomial, suppose there are n points are given x 0 f x 0 or n plus 1. So, x 1 f x 1 and so on x n f x n, so there are n plus 1 data points are given, then we can construct a polynomial of degree n. That means, now generalizing just this polynomial which we have just done for degree 1, but we can do for 2 and then generalization will be much easier. So, here this Pn that was just 1 there so, we have n there and therefore, this n will come here to i goes from 0 to n Li x and f x i. Because there are this f x i are also n plus 1 in numbers every all f x i will come here and these are the Lagrange polynomial of degree n now because n plus 1 points are given.

(Refer Slide Time: 13:45)



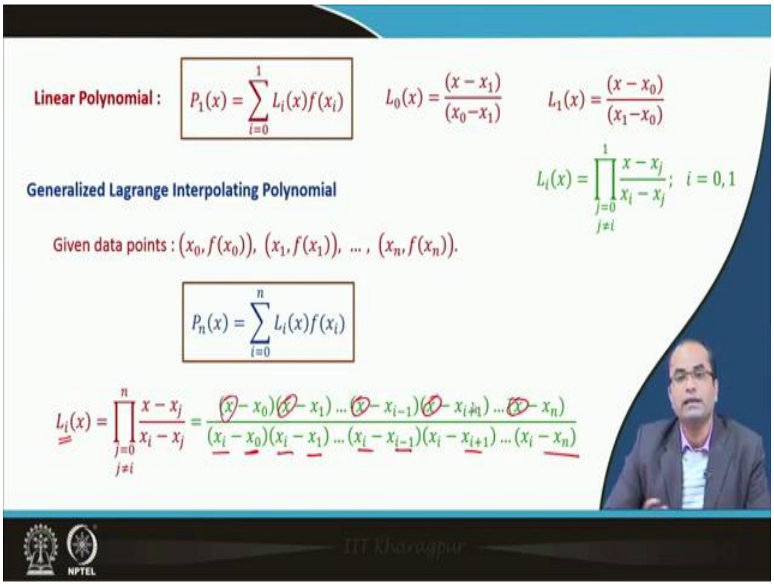
So, these Lagrange polynomial are defined exactly what we have done for this 1, j naught equal to i and j goes from 0 to n and then such term will be multiplied.

(Refer Slide Time: 13:58)



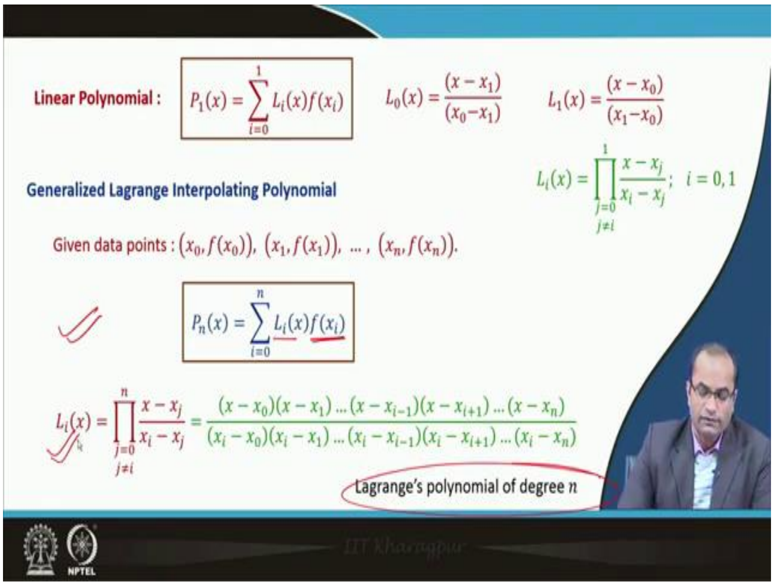
So, for instance, this one if we write in the expanded form, so, for ith, so, if we are talking about Li, there will be a missing term here, because we are doing with xi. So, xi minus x naught xi minus x 1 x i minus x i minus 1 and naturally x i minus x i will not come otherwise, this does not make any sense. So, except that i from this xi all other points will be subtracted in the denominator that is what it is easy to remember.

(Refer Slide Time: 14:36)



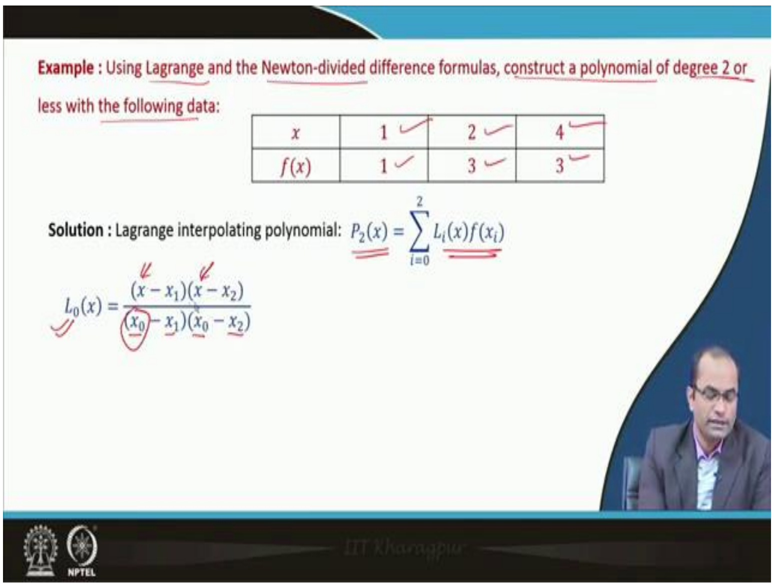
So, the first let me just write down this denominator from this if we are talking about the ith, this polynomial then we have xi, xi, xi everywhere and all other points other than xi has to be subtracted there. And now we go to the numerator this xi will be just replaced by x everywhere and the rest everything will remain the same. So, this is what we can easily remember this Lagrange polynomials of degree n indeed.

(Refer Slide Time: 15:02)



So, these are called the Lagrange polynomials of degree n. And so it is a very simple formula to remember we have just Li x f x i that is the formulation here and we know already these are the standard results that what are the Lagrange polynomials of degree n.

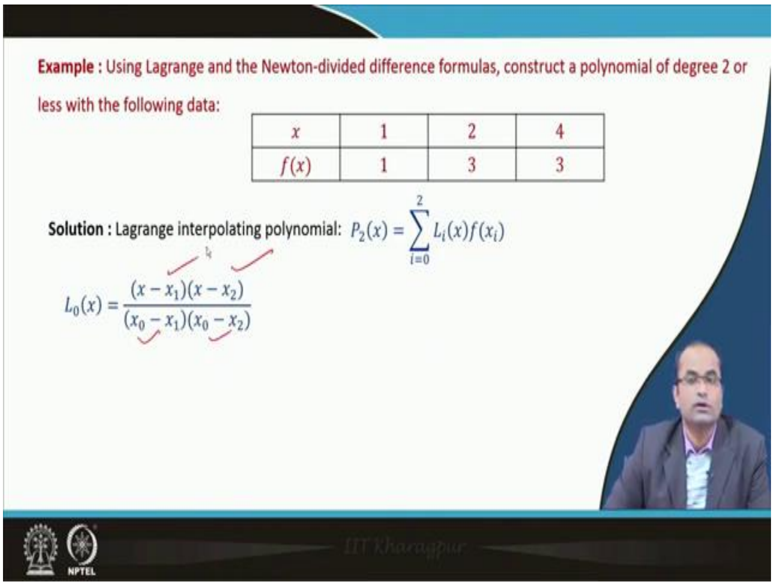
(Refer Slide Time: 15:21)



So, now we can go to the numerical example. So, the first we are taking here that using Lagrange, and the Newton's Divided difference formula, construct a polynomial of degree 2 or less with the following data, so we have the three data points, so we can expect a polynomial of degree 2 or less in some cases, the value of f is also given here, 1, 3, and 3.

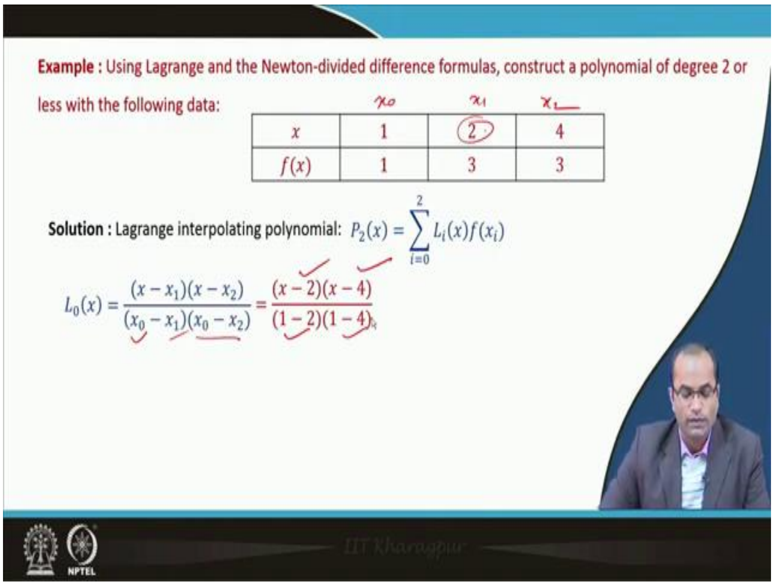
So, first, let us go through the Lagrange interpolating polynomial. So, here we have three data points, so we will get Lagrange polynomial of degree 2. So, polynomial can be written again, this Li f xi and L naught now here we note L naught L1 and L2. So, L naught will be x naught x naught so, x naught minus x 1 x naught minus x 2 and this x naught we will replace by this x both the places and that will become the numerator.

(Refer Slide Time: 16:23)



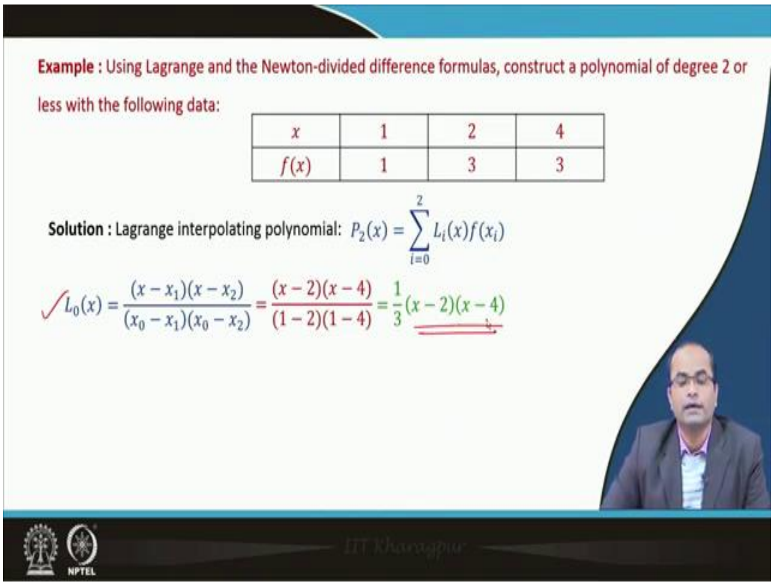
So, we have x minus x 1 x minus x 2 x naught minus x 1 and x naught minus x 2.

(Refer Slide Time: 16:30)



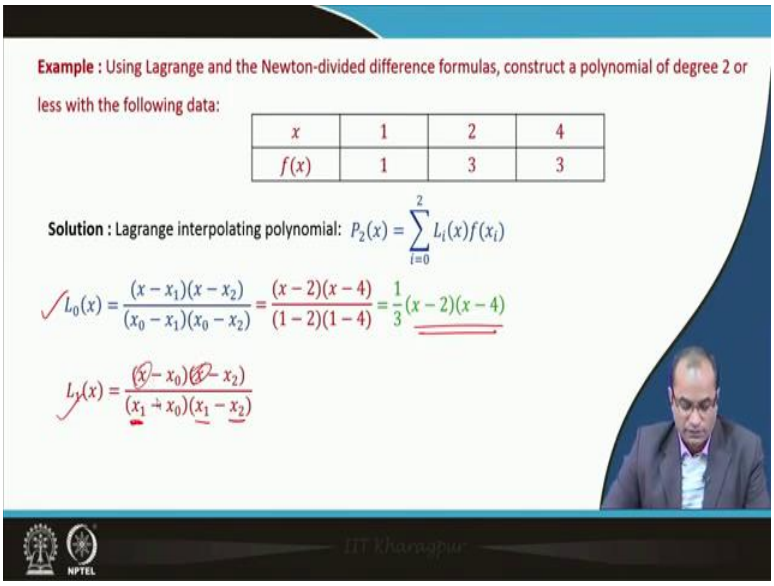
So, x minus x 1 is so, this is x naught the first one this is x 1 and this is x 2. So, x minus x 1 that is 2 x minus x 2 that is 4 there, then we have x naught that is 1 minus the x 1 that is 2 and here we have x naught minus x 2 that is 1 minus 4.

(Refer Slide Time: 16:50)



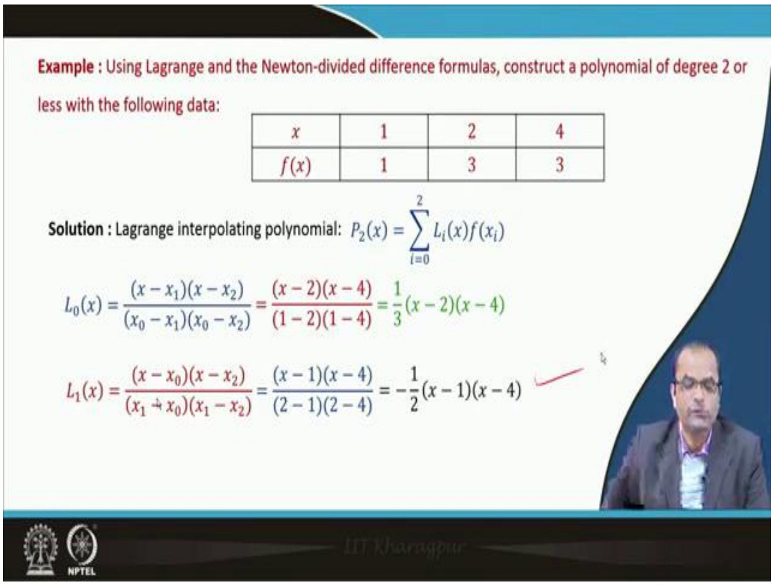
So, this Lagrange interpolation, the Lagrange polynomials of degree this 2 the L0 we have x minus 2 x minus 4 by 3.

(Refer Slide Time: 16:54)



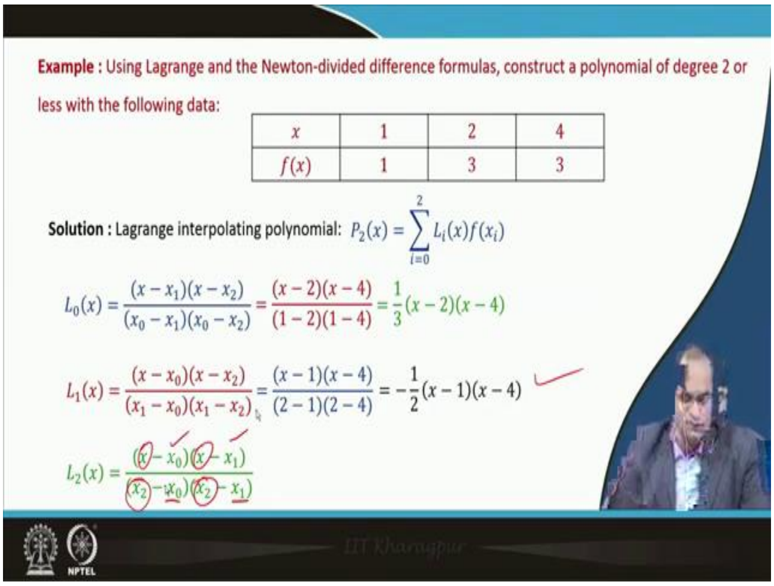
Similarly, for L 1 we can compute again. So, here the only difference is now we have x 1 and then all other points will be subtracted. So, first we have x 1 minus x naught, then we have x 1 minus this x 2, and then we have in the numerator, this x 1 will be replaced by the x there and we have the formula.

(Refer Slide Time: 17:30)



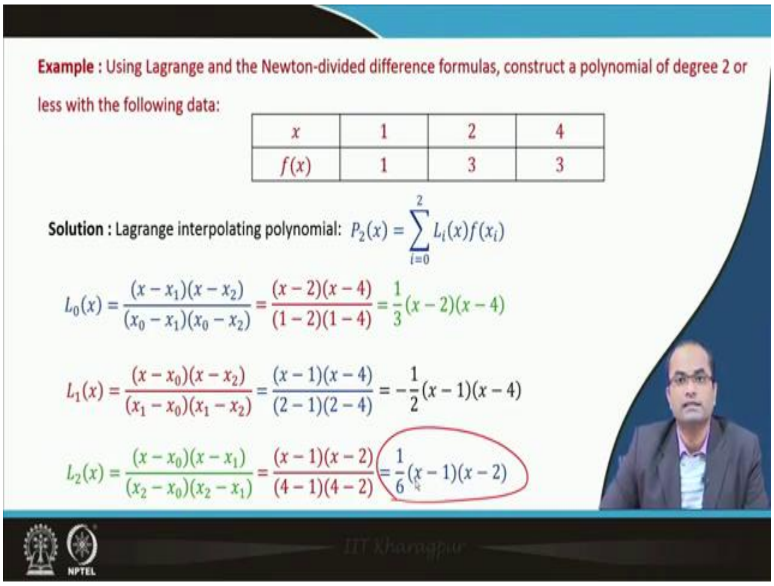
So, again putting all these values of x naught and x 1 x 2 we can get the second order polynomial this L 1.

(Refer Slide Time: 17:37)



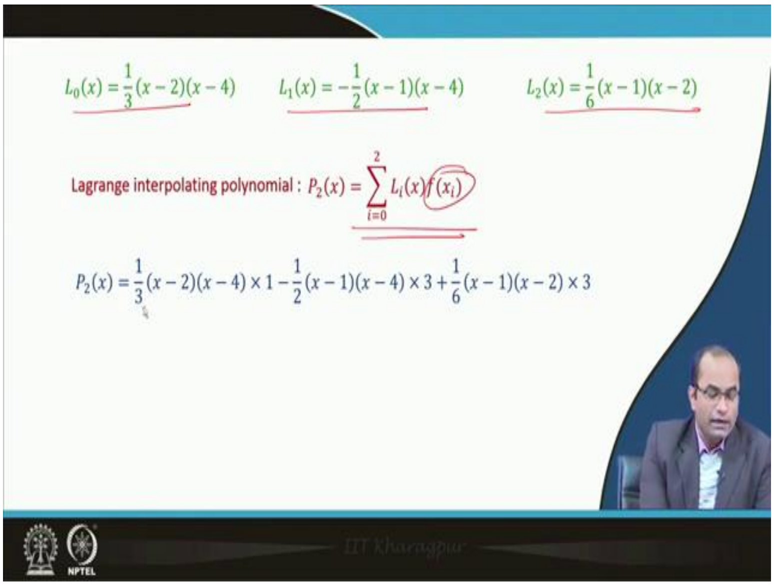
And then we have the L2 there for L2, we have to have here x 2 x 2, so, x 2 minus x 0 x 2 minus x 1 all other points will be subtracted and then we have there x 2 will be replaced by this x 1. So, we have x minus x naught and x minus x 1.

(Refer Slide Time: 18:00)



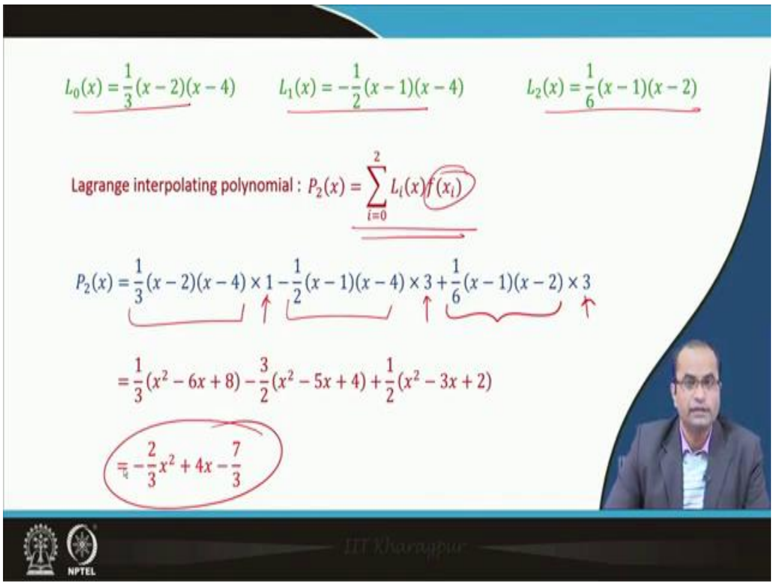
So, putting all these values of x naught and x 1 we will get the second order polynomial, second degree polynomial which is written here, 1 by 6 x minus 1 and x minus 2.

(Refer Slide Time: 18:10)



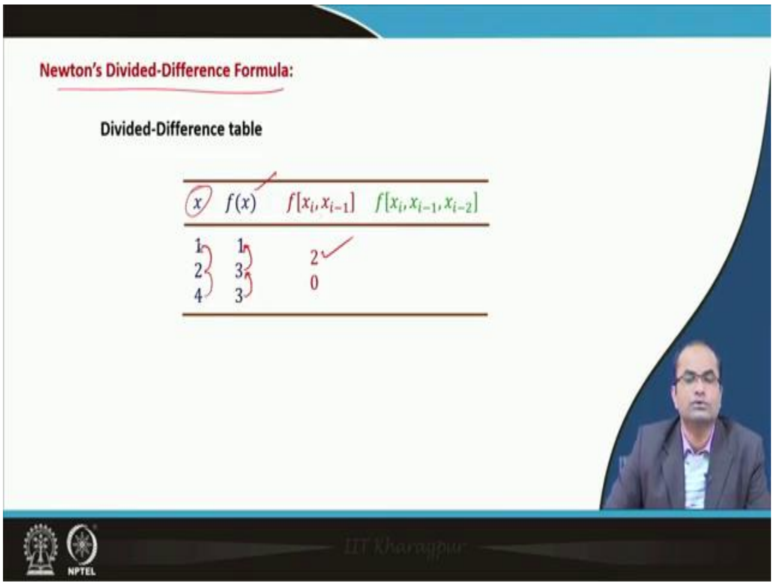
So, having these all desired polynomials of degree 2, L0 L1 and L2, we can now write down this Lagrange polynomial only thing we have to multiply just by the function value to each of these L.

(Refer Slide Time: 18:28)



So ,here we have this first L naught multiplied by this value of f x naught, this was 1, then we have this L 1 there and the value at this x 1 point it was 3 and then again here it was 3, so, this is L 2. So, when we can simplify this and we get minus 2 by 3 x square plus 4 x minus 7 by 3 so, that was using Lagrange interpolation formula.

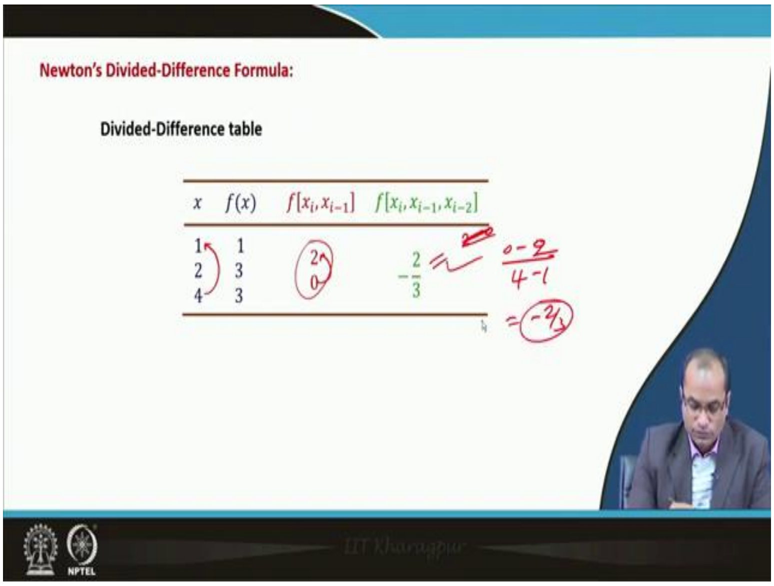
(Refer Slide Time: 18:57)



Now, coming to the Newton's Divided Difference formula, we have to construct the difference table to use this formula. So, first we will get this divided difference table. So, we will write down in the first column as x. So, 1, 2, and 4 in the f x we have 1, 3, and 3. And then the first this divided difference how do we get is exactly similar to getting these finite differences which we are doing in Newton's forward and backward difference formula, just subtract this one here.

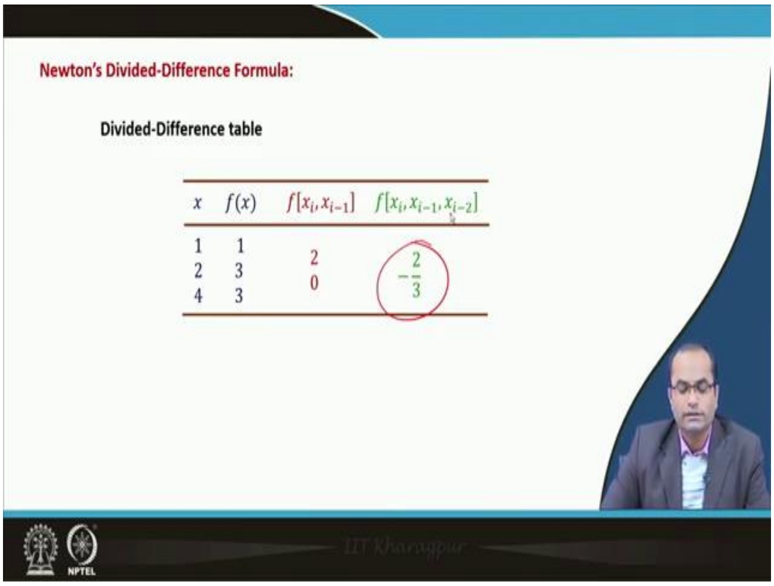
So, 3 minus 1 that is 2 and we have to also divide by the corresponding values of x that is 2 minus 1, 1. So, 3 minus 1, 2 divide by 1 that is 2 here, and in this case 3 minus 3 that is 0 and then 4 minus 2 that is 2 so 0 by 2 that is 0 again.

(Refer Slide Time: 19:53)



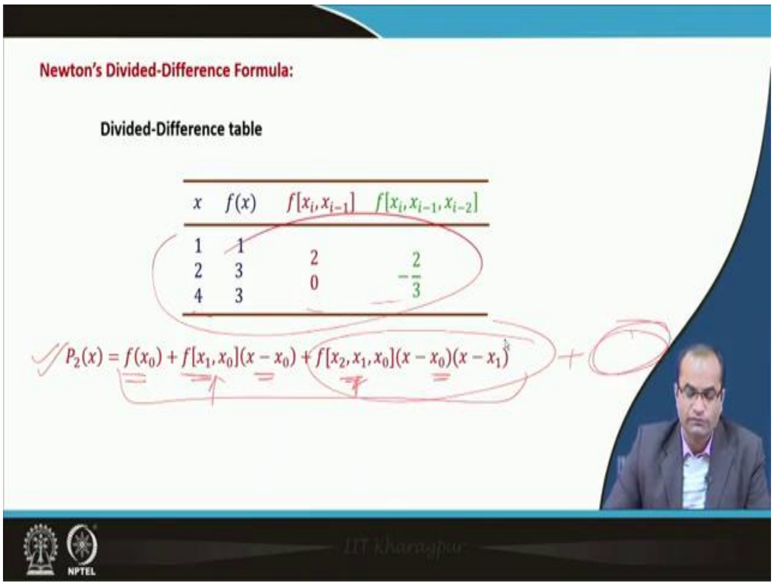
So, we have this now in this the last one, we have to subtract this 0 minus 2 that is minus 2 and then we have to subtract 4 minus 1. So, this is nothing but this 2 minus 0, sorry 0 minus 2, so, 0 minus 2 and divided by this 4 minus 1, 4 minus 1. So, we have minus 2 by 3.

(Refer Slide Time: 20:19)



So, that is the value here for this divided difference having these three points.

(Refer Slide Time: 20:25)

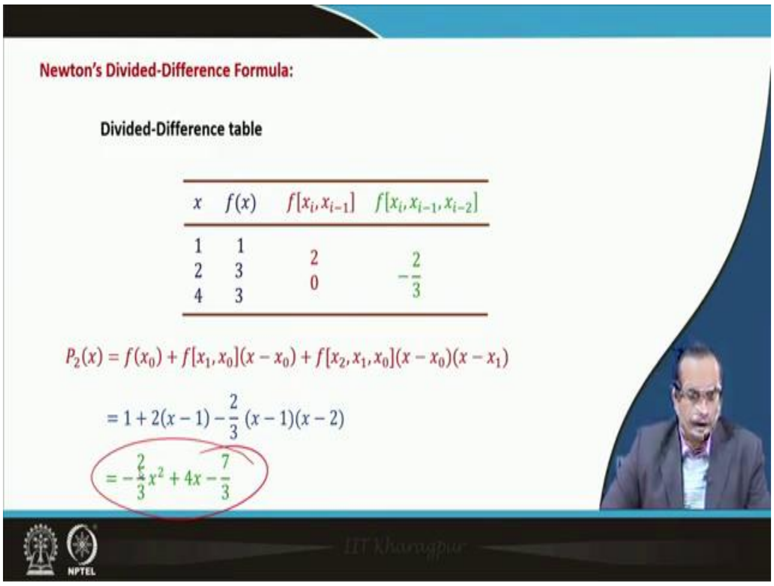


And then we can use simply in this formula where we have f x naught then this divided difference x minus x naught divided difference x minus x naught. So, in this case, we can substitute now these divided difference one more benefit here, similar to what we have for

Newton's forward and backward difference formula. If we want to add one more point for instance, then one extra term will be coming in this formulation, but the rest all the calculation will remain as it is and we can continue just having one more term there and we will get a new polynomial.

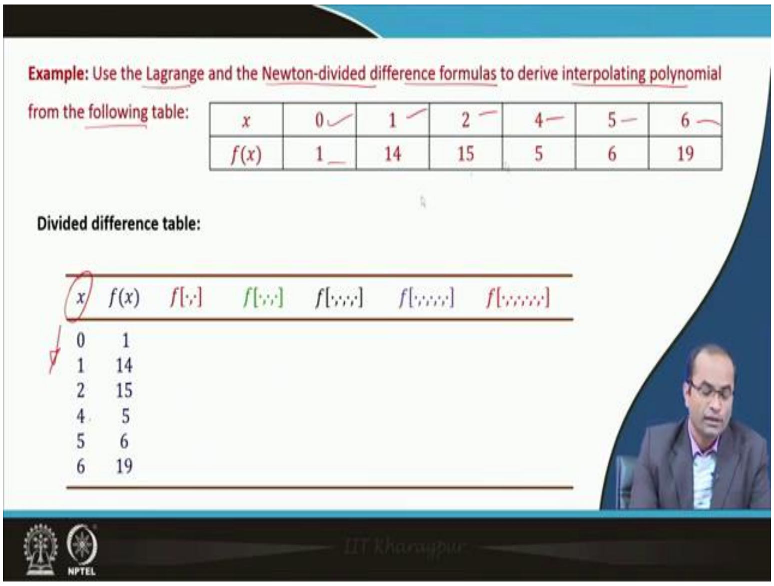
Whereas in this Lagrange interpolation formula what we have, we have to compute again all these Lagrange interpolating polynomials, the Lagrange polynomials, because the degree of the Lagrange polynomial will change. So, there will be again a fresh calculations, but here in this case if you want to add one more point here, the table has to be updated and then just one extra term will be appearing besides this existing one.

(Refer Slide Time: 21:32)



So, here again we when we substitute these finite differences, so, we will get the result which is naturally the same polynomial which we have just derived using Lagrange interpolation.

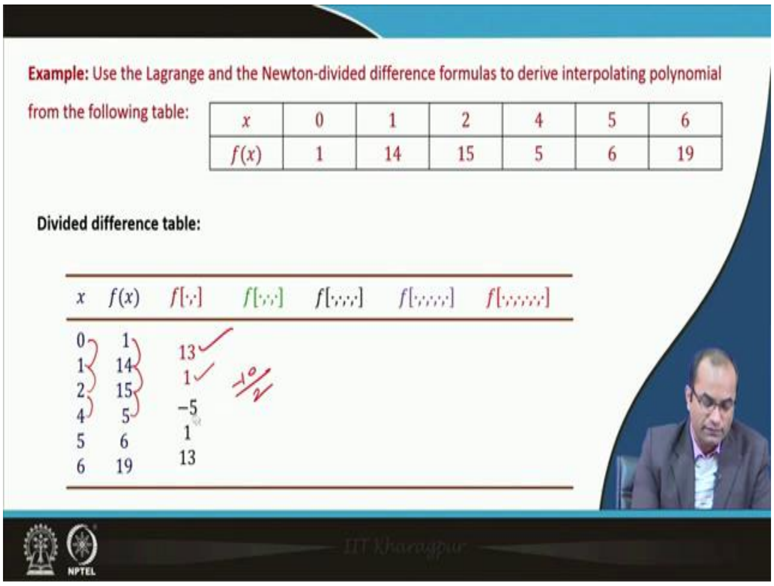
(Refer Slide Time: 21:46)

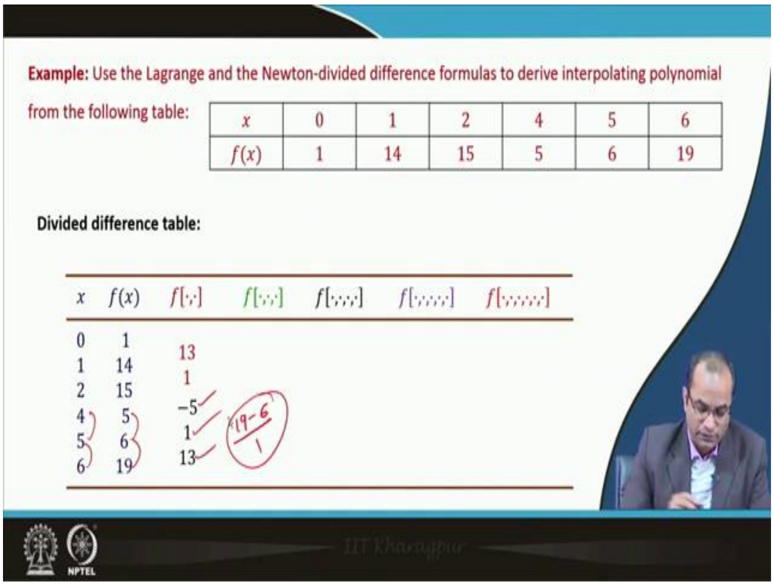


One more example that is the last example now in this lecture, so, we use the Lagrange and the Newton's divided difference formula to derive interpolating polynomial again from this following table. So, now we have more data. So, we have six points 0, 1, 2 and the 4, 5, 6 they are also not equidistant and then corresponding values are given.

So, in this case, we will go first with the Newton's divided difference formula, and we will realize indeed, when we have this much data and in particular in this example, you will see that the Newton's divided difference formula it is much easier to derive this interpolating polynomial, whereas the Lagrange interpolating formula is much more complicated. So, coming to the divided difference table we have to write all the values of these x here 0, 1, 2, 4, 5 and 6 and then the corresponding value of f which are 1, 14, 15, 5, 6, and 19.

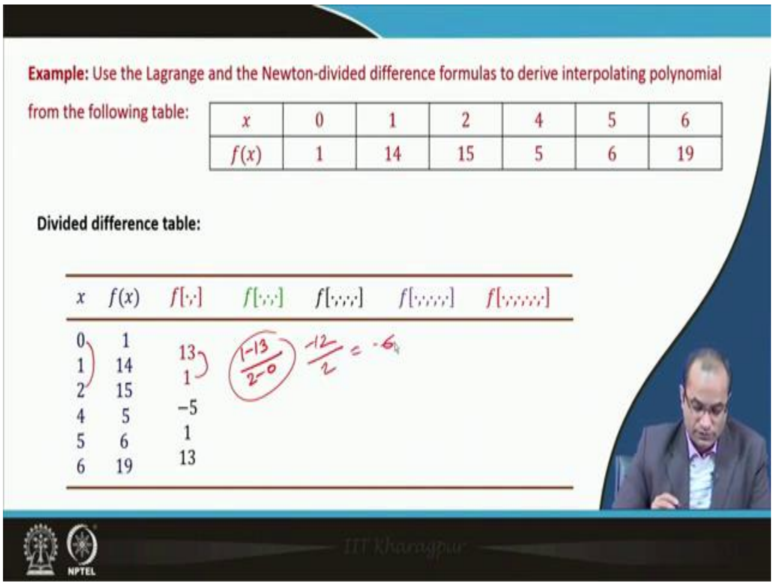
(Refer Slide Time: 22:50)





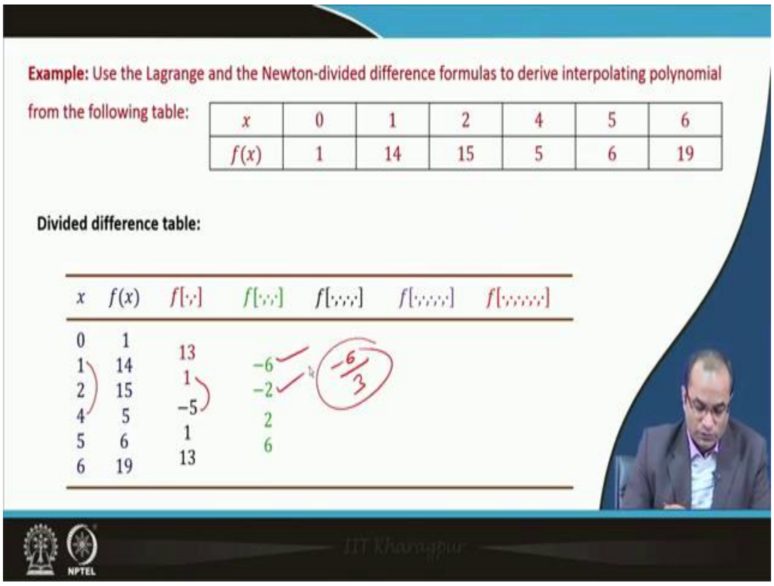
And then we have to construct this divided difference table as we have done before. So, this 14 minus 1 that is 13 and divided by 1, so this will be 13 here, again 15 minus 14 that is 1 and 2 minus 1 1, so division with 1, so we have 1, then 5 minus 15 that is minus 10 and then we will divide by this 2, so we will get minus 5 that is true here, then we will get 6 minus 5 1 and so 1 divide by 1 1, then we get 19 and minus 6, 19 minus 6 divided by this 5 minus 6 that is 1 so we have 13, which is there.

(Refer Slide Time: 23:52)



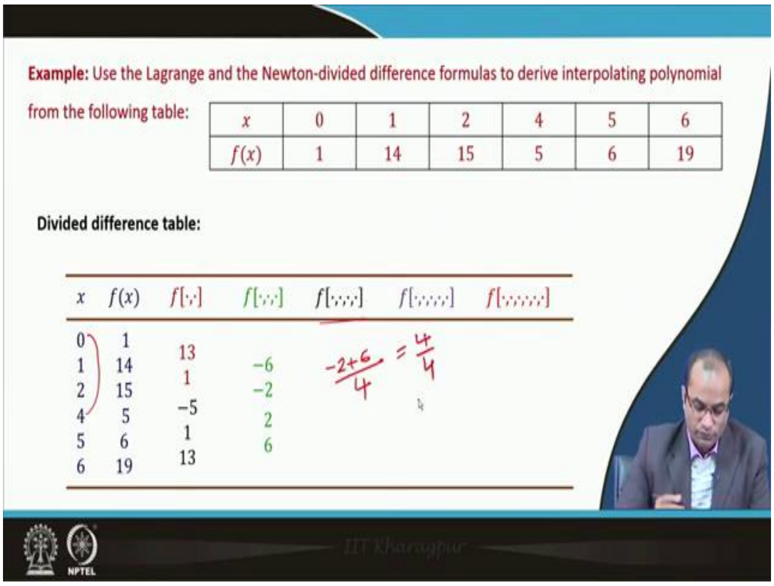
So, similarly, we have to continue with this process for this case, now we have to make this difference 1 minus 12 so 1 minus 13. So, will be minus 12. And then we have to divide now with 2 minus 0, so this will be 1 minus 13 and divided by this 2 minus 0 because now the difference here three points will be coming together. So, this is minus 12 and then 2, so it is a minus six there. So, here we have minus 6.

(Refer Slide Time: 24:07)



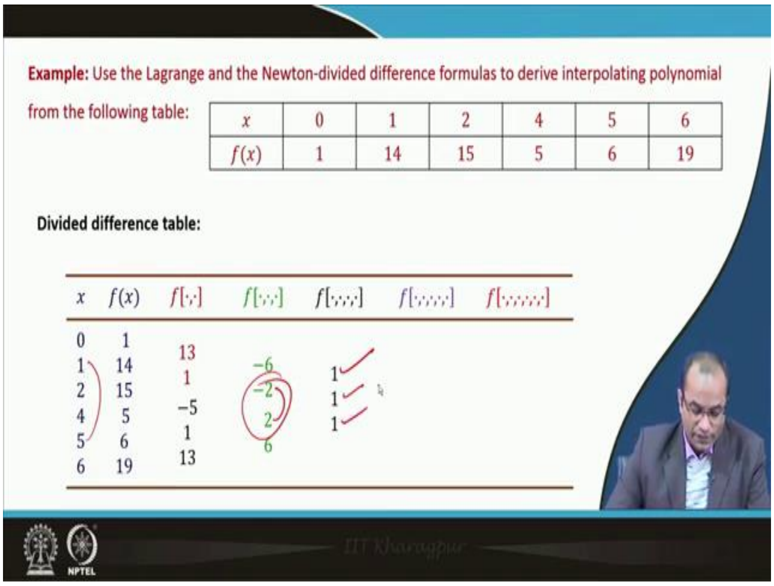
And then similarly, we have 5 minus 1, minus 5 minus 1 so minus 6, and then we have here 2, so minus 6, minus 5 minus 1, so minus 6 and divided by 3 there 4 minus 1, 3, so we will get minus 2, so and so on, we have to continue this would be 2 and then we will have a 6 there.

(Refer Slide Time: 24:42)



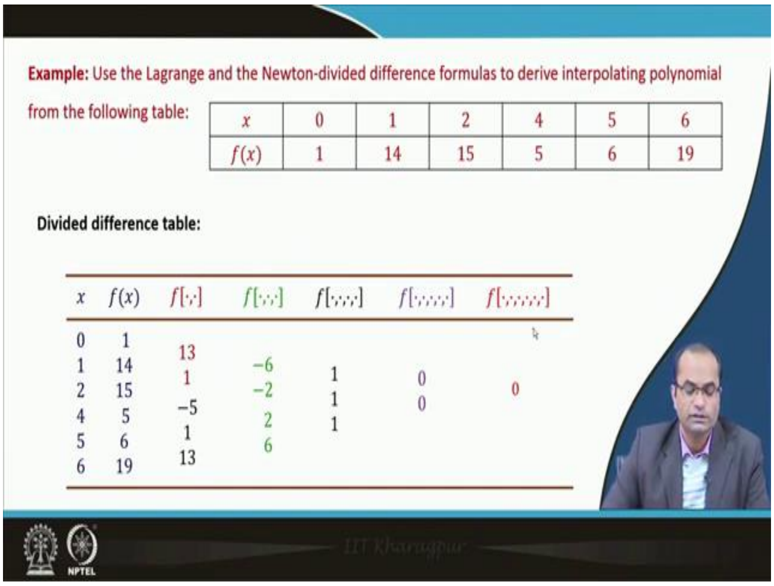
And further, we have to go with this next difference so here will be like minus 2 and minus, minus 6, so there will be 6 there and divided by now we have to have a difference of these 4 points. So, here 4 minus 0, so that will be 4. So, here 6 minus 2, that is 4 and divide by 4 so we will be having 1 there.

(Refer Slide Time: 24:59)



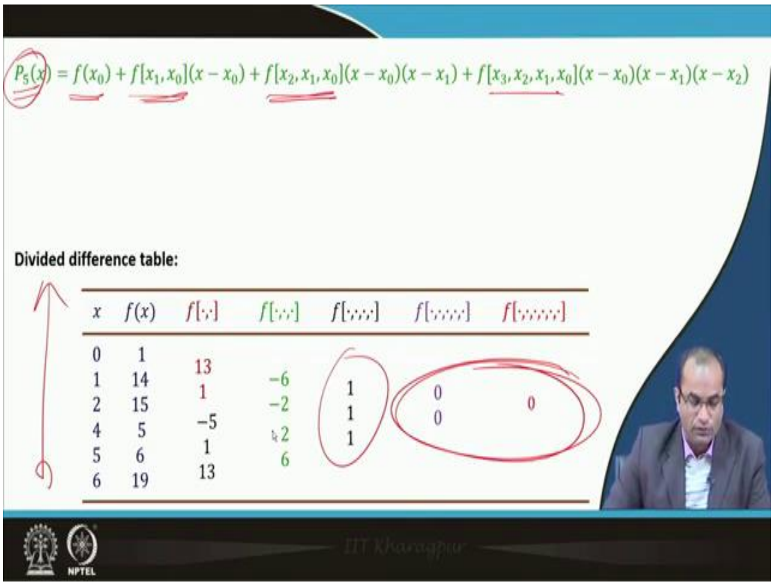
So we have 1 and interesting here is that when we do here also we have for 2 minus, minus 2 that is 4 and from here also we will getting 4. So, 4 divided by 4 again 1 here also will get 1.

(Refer Slide Time: 25:18)



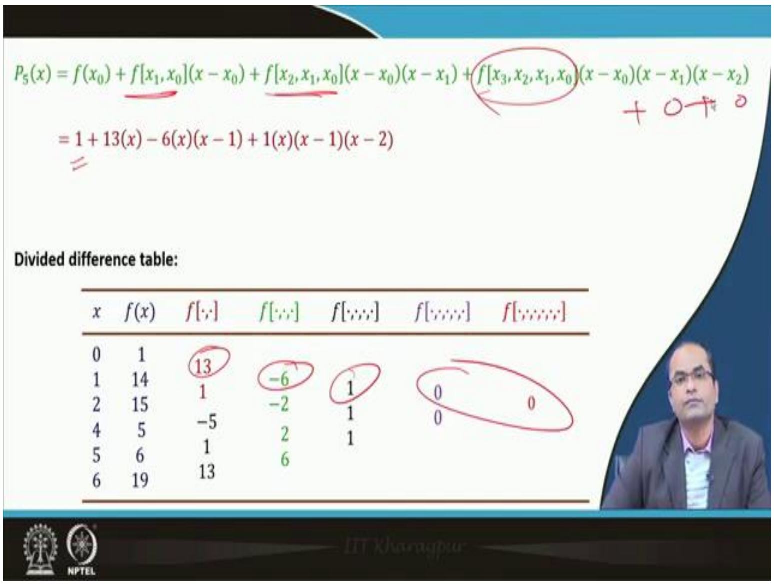
So, all our 1, 1, 1 and once we get such a situation rest all going to be 0. So, we have 0 there and we have 0 there.

(Refer Slide Time: 25:27)



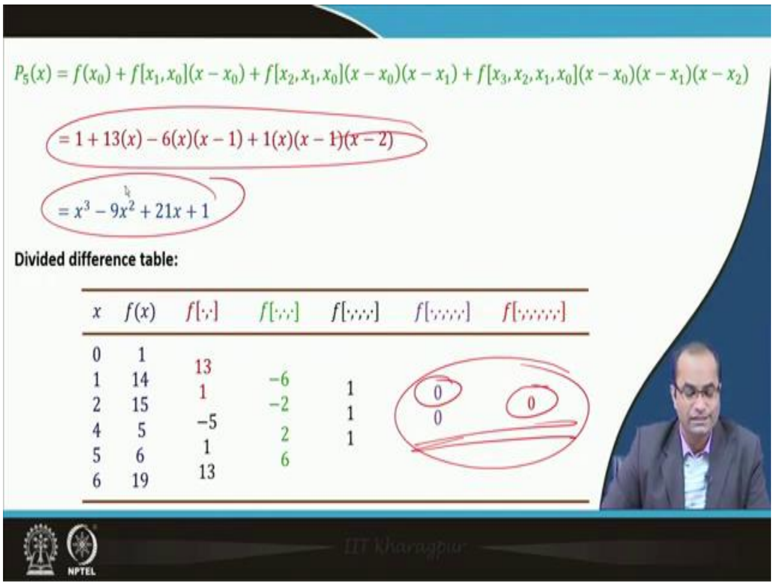
So, this is the finite difference table we have, having this finite difference table, we can now go for the formula the divided difference formula which has here f x naught we have f x 1, x naught and then here and then and so on, we are not writing now this further because these are going to be 0. So, though we are expecting when 6 points are given the fifth degree polynomial, but in this case, we will not get we will get only the third order, third degree polynomial.

(Refer Slide Time: 26:02)



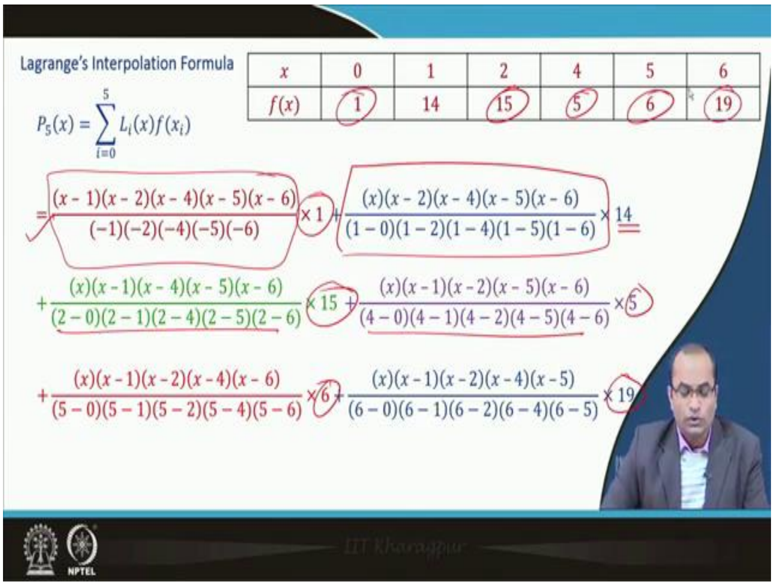
So, here substituting all these values. So, this f x 1 x naught is 13, then we have these three points, so we have minus 6 when we have these 4 points is 1, and these are 0, so, two more terms will come, but they will become 0, so we are not writing them.

(Refer Slide Time: 26:21)



And now it is much simpler to work with. So, we got this third degree polynomial. So, what is our observation on these divided difference formula that once the table is ready, and now for instance, in this case, we got these zeros at two places there, so there was no higher order terms coming automatically and we have less calculation and finally we got this third degree polynomial easily.

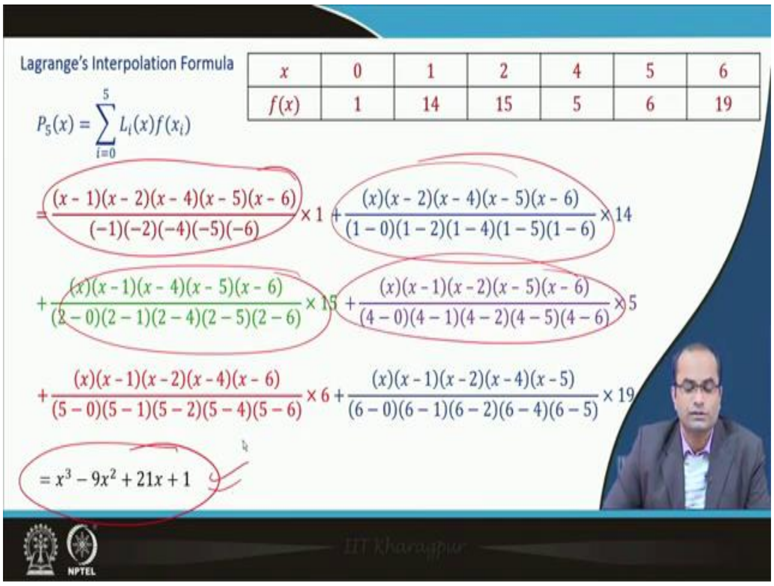
(Refer Slide Time: 26:50)



Whereas if we want to go with the Lagrange interpolation formula, what will happen, given the data we have to now compute all L1, L0, L1, L2 L3 L4 and L5. And then only after combining all these with a lot of calculations we can get, we will get again this third degree polynomial, but here the calculations will be more because we have to compute these 6 terms.

So, this is the first time we have this L 0 here fifth, 5 degree polynomial multiplied by this value f x 0, then we have to have this again L 1, the five degree polynomial, then will be multiplied by this 14 we have again five degree polynomial multiply by 15 and then here again five degree polynomial which will multiply it by 5, this will be multiplied by 6 here and then multiplied by 19.

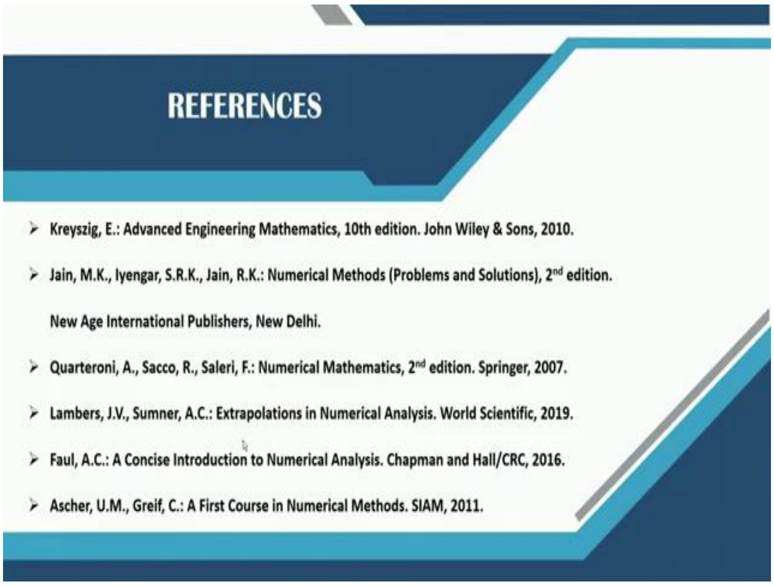
(Refer Slide Time: 27:58)



So, we have to now simplify all these here each having five degree polynomial and there will be cancellation of degree 5 terms degree 4 terms and finally, we will get the polynomial which is just third degree polynomial, but what is the difference between this finite difference divided difference formula and this one, that here we have to do a lot of calculations because each Lagrange polynomial is a fifth degree polynomial and that has to be simplified to have this lower degree polynomial which is the case here.

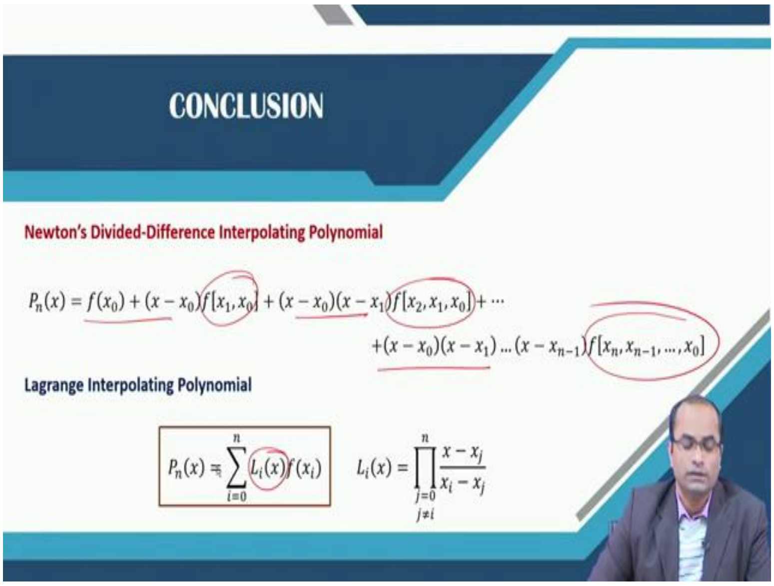
So, in many cases this Lagrange interpolation usually when it is a higher order data is given and more points are given. So, this Lagrange interpolation formula becomes really tedious. Whereas, for instance in this case where the there was a simplification the finite difference systems become 0. So, the evaluation of the lower degree polynomial was easier in the finite difference formula.

(Refer Slide Time: 28:56)



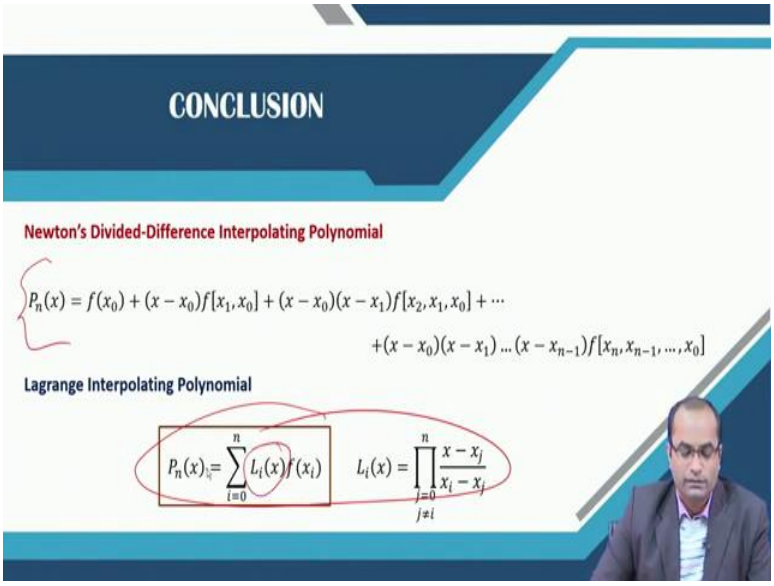
Well, so, these are the references used for preparing this lecture.

(Refer Slide Time: 29:04)



And just to conclude again, so, we have discussed two formats here for computing the interpolating polynomial, one was the Newton's Divided Difference formula, where we have to just compute these divided differences usually from the table and then we can get this polynomial, whereas in the Lagrange interpolation formula it is slightly different, we have to compute the Lagrange polynomial L0, L1, L2 and so on multiplied by this f x i

(Refer Slide Time: 29:30)



And the last example, we have seen that, using this Newton's divided difference formula it was much easier to compute the polynomial which was actually lower degree polynomial though more values were given there. And whereas, in the Lagrange interpolation formula we have to compute all the Lagrange polynomials and then we have to combine in this formula and the calculations were really tedious. Well, so that is all for this lecture and I thank you for your attention.