Engineering Mathematics - II
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Lecture 37
Half Range Fourier Expansions

So, welcome back to lectures on Engineering Mathematics - II. And this is lecture number 37 on Half Range Fourier Expansions.

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So, in this lecture, we will see the half range fourier sine series expansion and also the half range fourier cosine series expansion.

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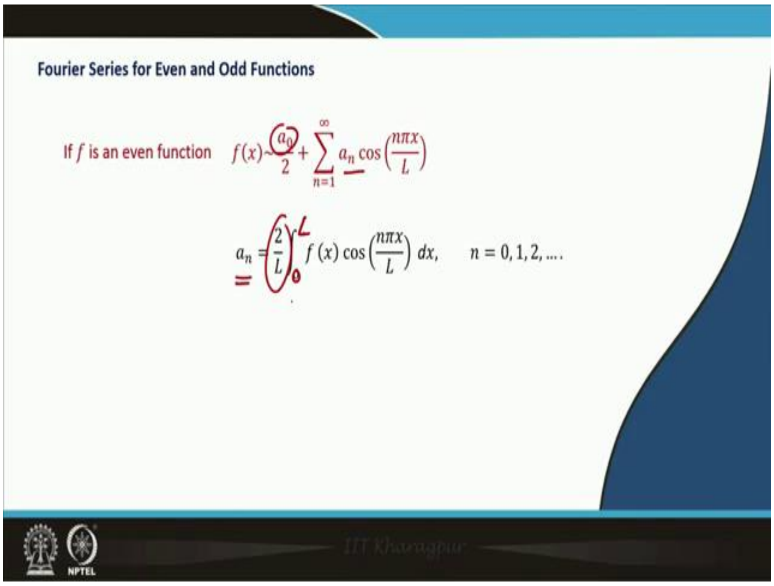

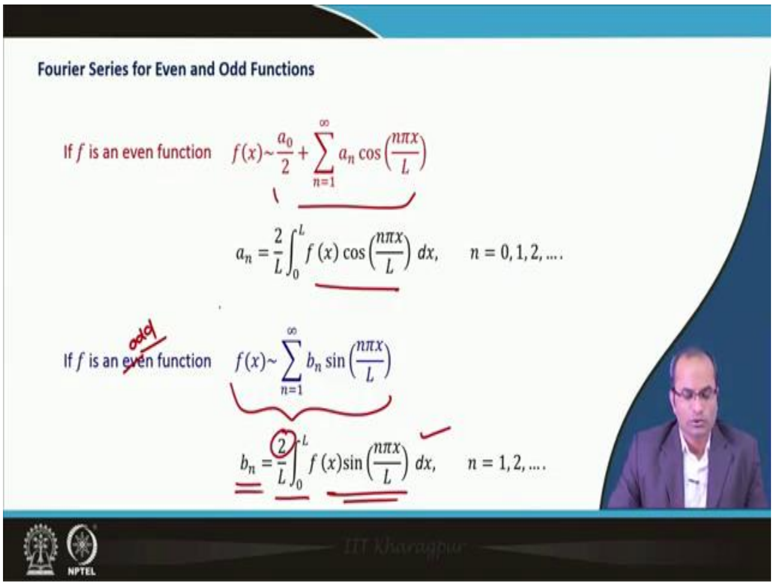

So in the last lecture, we have already seen that the Fourier series for even and odd functions can be simplified. So, if f is an even function, then f x can be expressed in terms of just the cosine functions. And these coefficients, an can also be simplified just, they can be evaluated from the 0 to L and the 2 times because the originally it was minus L to L. And then we have this f x cos n pi x over L dx and n is for 0, 1, 2, etc.

So, because the functions is even, we have observed that b n coefficients become 0 and we got only the cosine series. Similarly, when f is an even function, we will get only the sine terms in the series and the coefficient b n can be evaluated with the help of the simplified integral again, 0 to L. So, we have again consider this 2 times the integral 0 to L f x sine, sine and pi x over L dx and n is again here 1, 2, etc.

So, this was the simplified form depending on the function. If it is an even function then we will get only the cosine terms and if we have the odd function then we will get, so here it is odd function. So, if we have odd function, we will get only the sine terms in the expansion.

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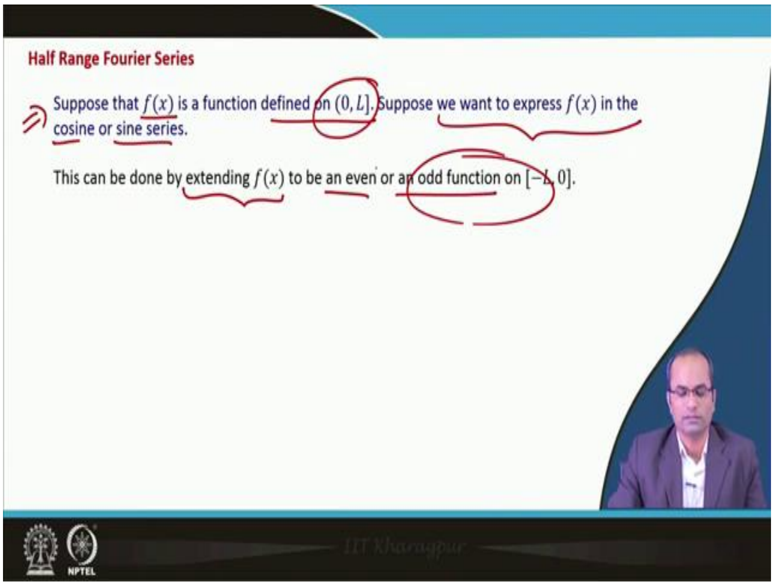

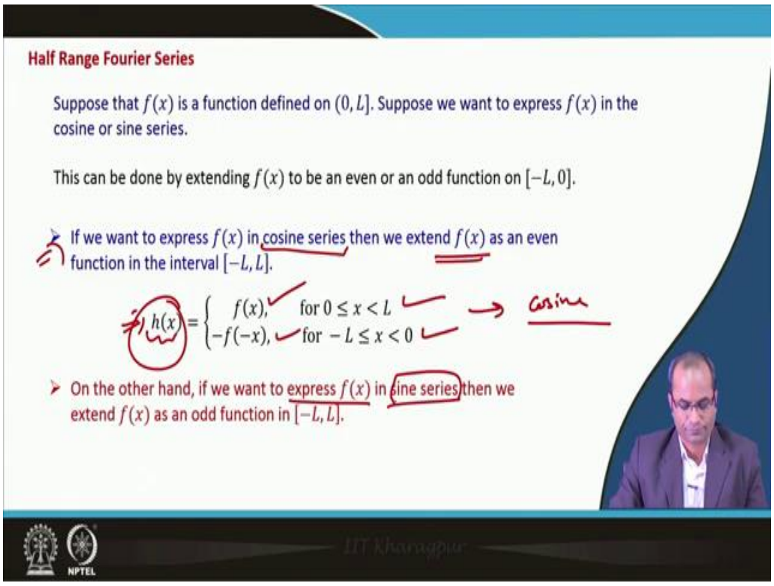

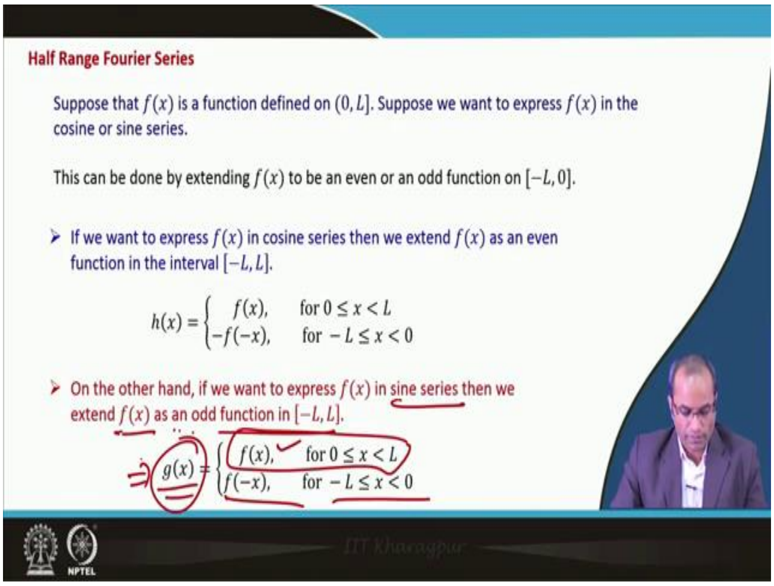

So, now what we are going to discuss here the half range fourier series, the idea is exactly what we have discussed for the odd and the even functions. So, let me just explain here. Suppose that f x is a function defined only in the interval 0 to this peak L. And suppose, we want to express this f x in the cosine or sine series. Now here we are putting our choice that we want to express this f x function which is defined in some interval into a sine or a cosine series.

In the earlier lecture, for instance, we have seen the fourier series. And naturally, when the function was odd function, it was coming sine series. When the function was even function, it was automatically coming as cosine series. But now the situation is different. So, the function is again defined in some interval. We have basically a choice of getting the fourier series which will have sine and cosine both the terms. But here the aim is not to have the fourier series but aim is to have either fourier sine series or cosine series, meaning the series which has only sine terms or cosine terms.

So how to do that, so this can be done by extending this f x to an even or an odd function. And this is exactly the idea which will be used now from the previous lecture that once we have a function which is an even function then it's fourier series will have only cosine terms and when we have an odd function, then its series will have only the sine terms.

So, if you want to express f x in a sine series, the objective is to express this f x in cosine series. Then we will extend naturally this f x as an even function in the whole interval minus L to L. So, 0 to L the function was already defined and we will now define in the interval minus L to 0, so that the whole function in the interval minus L to L becomes an even function. And once we have the even function, so naturally we will have only the cosine series of that new function.

So here we will define, we will redefine the function. Let us say, the new function is h x such that in the 0 to L we have exactly f x and in the minus L to 0 we have minus f minus x, so that this h x now becomes the even function. And when we write the fourier series of this, it will have only the cosine terms. So, we will get automatically the cosine series. And on the other hand, if you what to express for example this f x in sine series, so if you want to express now in sine series, then we have to extend this function as an odd function in the interval minus L to 0, so that in the whole interval minus L to L the function becomes an odd function.

So, this is the idea again which was also explained in the previous lecture but now we will go into more details. So, the g x is f x, this is given already and 0 to L that the function f x is given. And in the interval minus L to 0 we have extended it as f minus x, so that this h x becomes an odd function overall.

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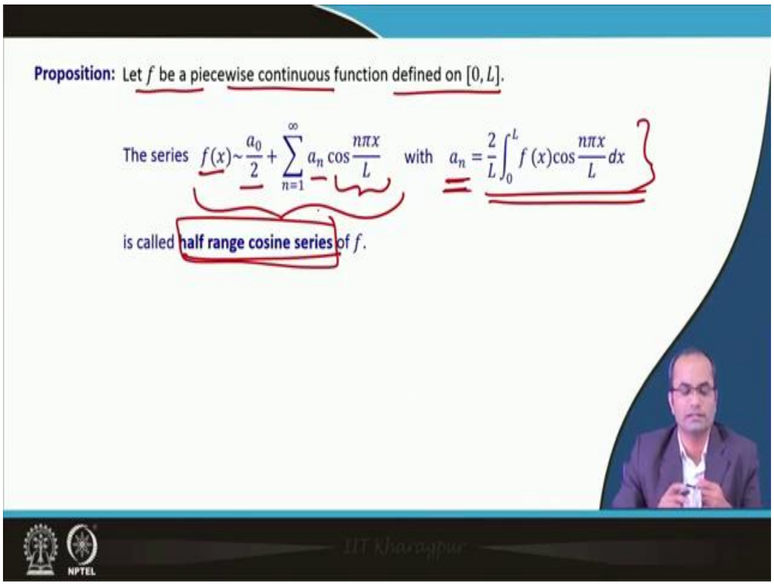

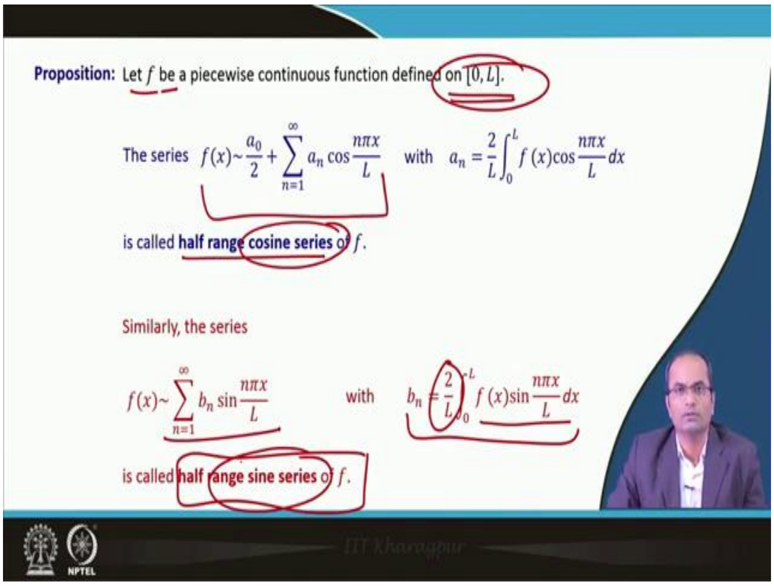

So well, so having this now, we have this result. So for example, f is a piecewise continuous function defined in the interval, in some interval 0 to L. The series f x with a by 2 and we have an there and then cos n pi x over L. This series with this an is called half range cosine series of this f. And we know now how do we get this cosine series, we have this extended function f in the range minus L to L, minus L to 0 so that the whole function becomes an even function and then we can write its fourier series which will be automatically a cosine series, given here whose an's we can compute with this formula and bn's naturally will be 0.

So, this is called half range cosine series. So, there is a difference then the chapter, then the lecture which we have discussed in the previous lecture. So there the function was an even function or an odd function and then we were computing its fourier series or we were evaluating its fourier series and then automatically it was coming as a sine series or cosine series.

Here the situation is different. The f is given in the interval 0 to L and we can have both the possibilities. We can extend it as, so that the whole function the range minus L to L becomes an odd function or an even function. Accordingly, we will get either this half range cosine series or half range this sine series, where the b n we can compute with this formula 0 to L f x sine and pi x over L and this factor 2 over L. And this is called, half range sine series.

So, for a function given in some interval we can have both the expansion, the half range cosine expansion or half range sine expansion. On the other hand, for a function defined in 0 to L we can also have its fourier series which may have both the terms, sine and cosine. So, to get only the cosine terms, we have to extend the function to have an even extension or to have an odd extension, so that we get only the cosine or the sine series.

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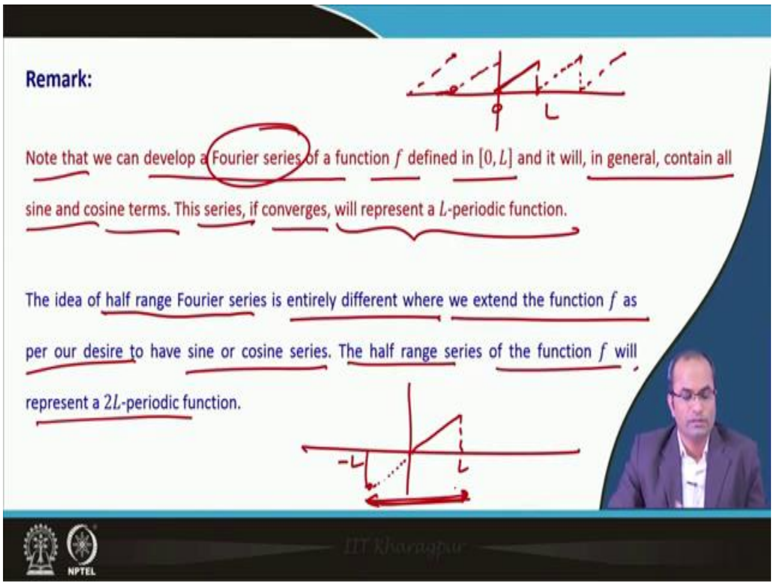

So, just a short remark. Note that, we develop a fourier series of a function f defined in the interval, so we can develop the fourier series, not the half range the fourier series and it will in general contain all sine and cosine terms and this series if converges, we know that, that it will represent a L periodic function because the function was defined in this 0 to L, L region. So, suppose this is the function in 0 to L and when we write its fourier series then it will have just an, that series will converge to such a function which is a periodic with period this 0 to L.

But now, the idea of this half range series is entirely different from the fourier series which we have discussed so far. So here extend the function as per our desire to have sine or the cosine series. And the half range series of the function f will represent a 2 pi periodic function. So for instance, if I consider the same function defined in 0 to L and suppose I want to have a sine series of this function, so what I will do, I will make an extension to this function in this region minus L to 0, so that the function in this range minus L to 0 now becomes an odd function.

And now I can write down its fourier series and that fourier series will converge to a function which will be 2 L periodic, not just the L periodic. So, I think the idea is clear with this remark. So, we can proceed further.

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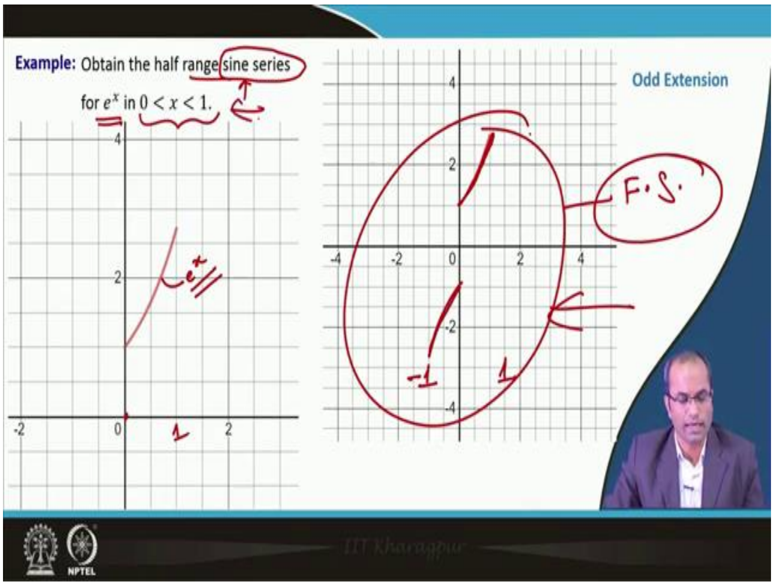

We want to obtain for instance the half range sine series. So here the objective is to have sine series. We could also get the cosine series. But let us go with the sine series first. So, exponential function e power x given in this region 0 to 1, and we want to write its sine series, that means we have to extend this function in the interval minus 1 to 0, so that the whole function minus 1 to 1 it becomes as an odd function.

So if this is for example, this exponential function x in the interval 0 to this 1, so what we do, we want to have this extension here. So, this was the given function, the exponential function in 0 to 1. What we have done, we have extended this, so that the function in this range minus 1 to this 1, it has become now an odd function. And now we will write its fourier series. So, if we write the fourier series of this function, then naturally this will have only the sine series.

And we have the direct formula now, we do not have to write the fourier series of this function we have already in the previous slide, we have discussed that what is exactly the formula to compute sine series when the function is given in some interval 0 to L. So, we can compute its half range sine series.

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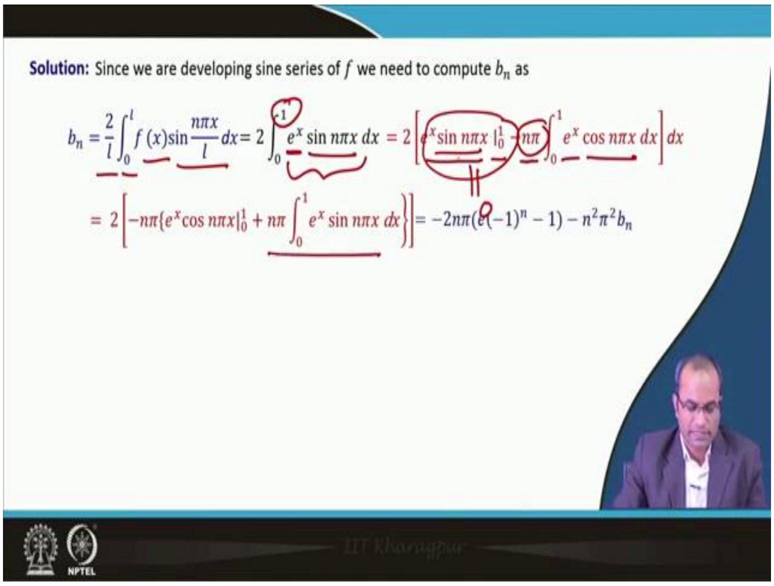

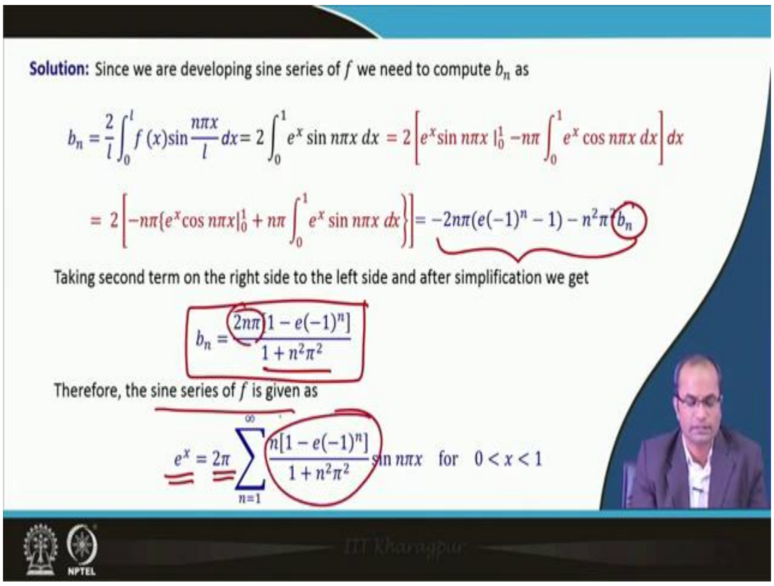

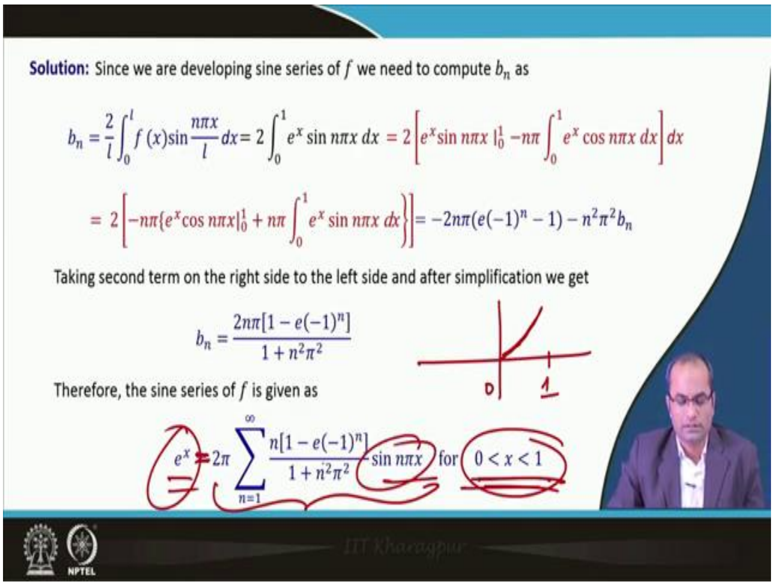

So, for half range sine series we have to compute this bn the fourier coefficients bn 2 over L and 0 to L f x sine and pi x over L. So, the L is 1 there, so we have 0 to 1, the function e power x and then we have sine and pi x and the L is 1 again. So, this we can integrate by parts, so we have e power x sine n pi x 0 to 1. And then this sine has become this cosine and e power x will remain again e power x, when integrated.

So, this n pi factor has come now and then we can simplify this, so we can, in this case for instance, here the sine 0 will become 0 and sine n pi will be also 0. So, this term will become 0 and the second term again we have to integrate and we will get this result here, which can be simplified to get in term of this bn again and if we take this to the other side and we can simplify for bn. So, after simplification we are getting this bn as 2 n pi 1 minus e minus 1 power n and 1 plus n square pi square.

So, having this fourier coefficients now, what we can do, we can write down the sine series because here the objective was to have sine series. So, the sine series for e power x will be this 2 pi, this 2 pi there and then the bn this coefficient is substituted here. And then we have the sine nx term and this equality here, we have written this equality, which is true in this region 0 to 1 because we have the continuous function.

So, our function was this e power x in this region 0 to 1. So, this function is continuous in this range and naturally by this (())(14:03) theorem, this fourier series will converge in this region 0 to 1. We will have a problem at 0, we cannot make this equality at 0 because it is an odd extension and at 0, at one end the value is 1, the other end it is minus 1 and we have to take the average. So it will become 0, the fourier series of this whole function with the extended portion will converge to the 0, to 0 at 0 only.

So that is, but in 0 to 1 we can certainly write this equality. So, this series here is equal to e power x and it has only the sine terms, so this is, this sine series.

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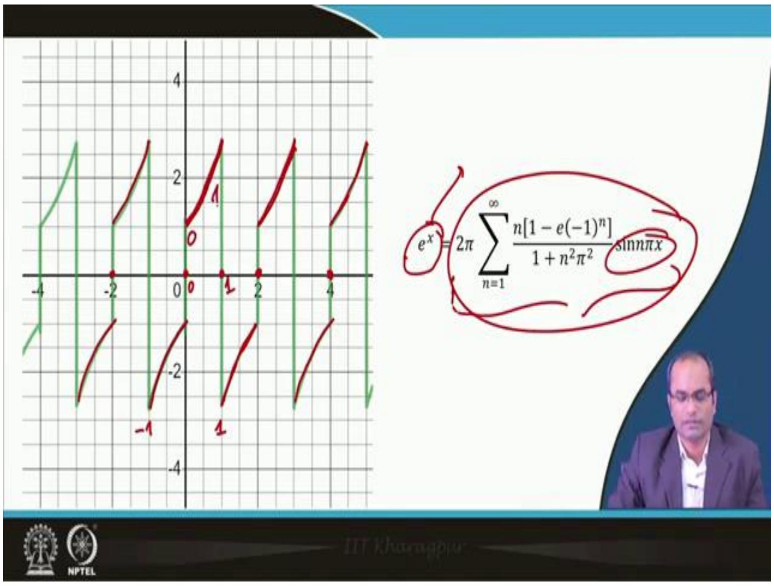

Just to plot this, so this is the sine series we have just evaluated, constructed. So, this was the function, this was our function ex and you remember this, the extension was then such that as a whole from minus 1 to 1 the function becomes the odd function. And now, if you plot this sine series for the whole range of r, so we will have this structure again, this one and it will converge to something like this. So here again, here and this one and so on. So that will be the fourier series and this middle this will go to, will always pass to 0.

So, at the point of this discontinuity, it will converge to the average value. Otherwise, it is a, it will converge to for any other value of x, to exactly this exponential function. So, what is interesting now that, if instead of this sine series we could have obtained for the same function e power x as a cosine series, so we can have the other representation of e power x in terms of cosine functions.

And what is interesting that in this region 0 to 1, so that in terms of the cosine we will have a completely different convergent function but in the region 0 to 1 it will again match the e power x. So, we can have 2 different expansions here. One is purely in terms of sine, other we can obtain the cosine series in terms of cosine. Both the series will converge in 0 to 1 to the same function e power x.

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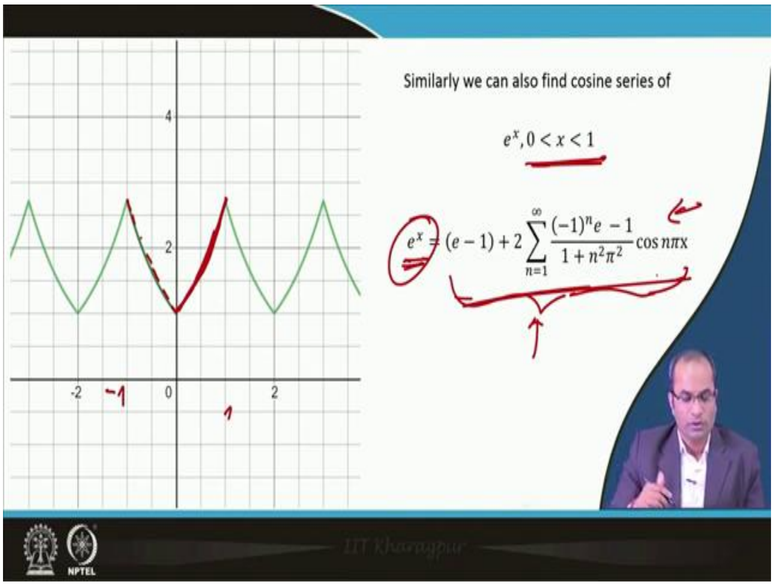

So we have also, for instance, plotted here the cosine series. So, if we write its cosine series with the similar steps, we have to follow what we have done before. So, in the region 0 to 1 this will be equal to this exponential function which was extended now this time, so that it has become the even function in the interval 0 to 1 and then we have written the fourier series which is now the cosine series.

And so, the same function here in this region e power x and now it is represented by the pure cosine terms but it is also converging to the same exponential function. So, we have e power x completely in terms of the cosine. In the earlier slide, in the previous slide we have e power x equal, it was written in purely sine terms.

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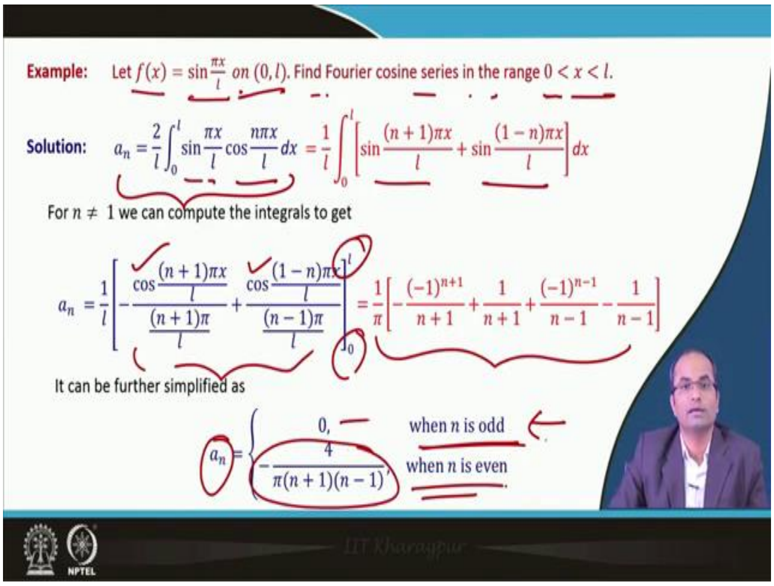

Well, so we can proceed now to another example. Here we will consider, f x as sine pi x over L in the interval 0 to L and we will write down its cosine series in this range 0 to L. So again, the similar steps, so for the cosine series we have to compute this an which is 2 over L and 0 to L and with the sine pi x and then cosine and pi x over L. So, we can use this 2 sine a cos b formula and then we have to integrate it.

So, I believe this we can proceed further. So, with the sine we have the cosine term, here also we have cosine term and then, so this was the integral and then we have to put the limits 0 to L, so which comes out to be such an expression which can be simplified to get these values of a. So, an will be 0 when n is odd and when n is even, this is simplified by 4 over minus this pi and n plus 1 and n minus 1.

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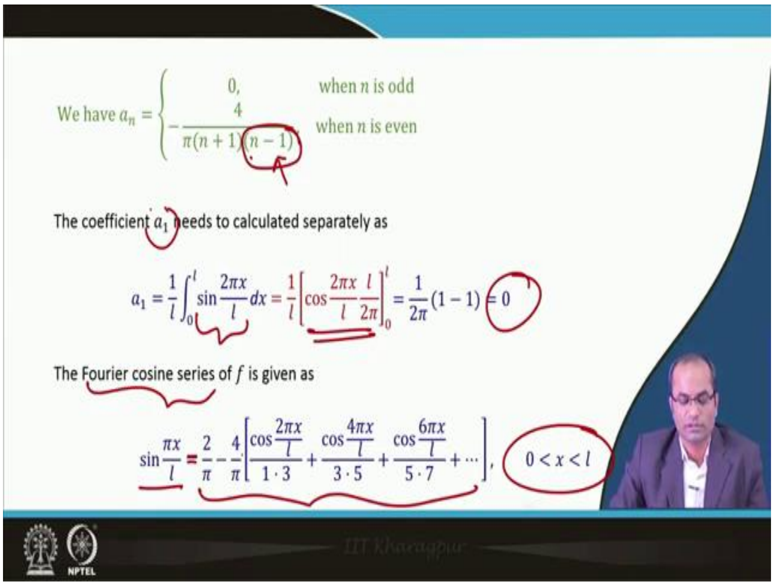

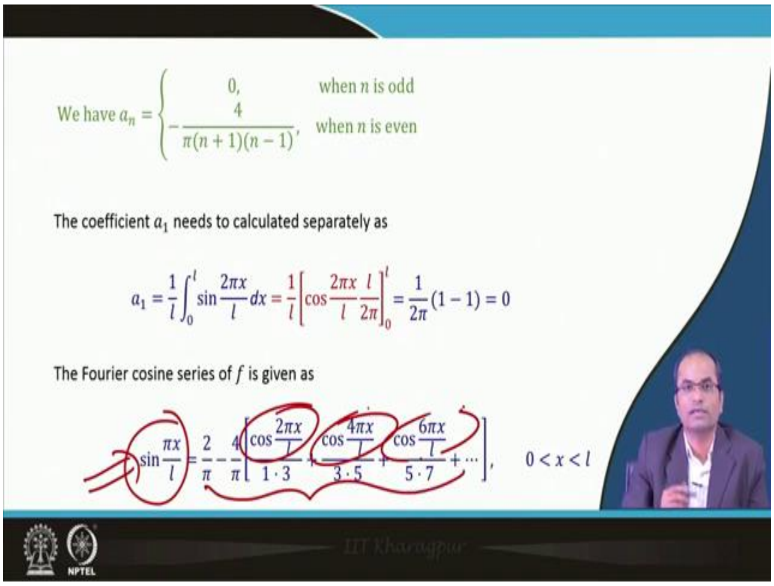

So, we have this an and then we have to also compute this a1 because this was, the n equal to 1 was excluded in the computation, otherwise we cannot have this n minus 1 in the denominator. So, this a1 we can calculate separately. So that is the result for a1, when we put n equal to 1 there. And that can be again integrated and what we realize that the a1 is 0 in this case.

So, the fourier cosine series of this f is given as, then in terms of the cosine functions which is varied again in the region 0 to L and we have equality because of the continuity of this sine function in the interval 0 to L. So, what is interesting here that this sine function, sine pi x over L is written in terms of purely the cosine functions.

So that is how we, I mean given a function in a range, now if you want to compute the fourier sine series, we have to just extend this function as an odd extension or if we want to have purely cosine terms, we have to extend this as an even function. So here the sine function is written in terms of the cosine functions.

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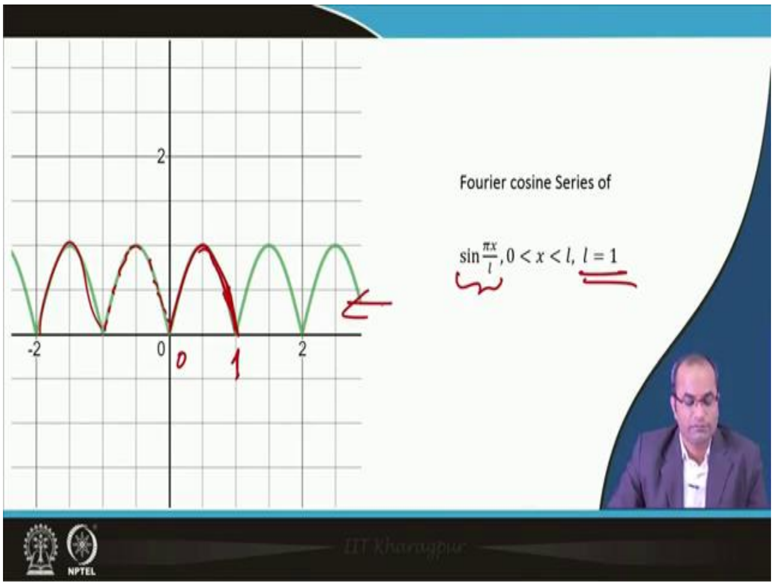

Well, so just to look at again what is happening here for the cosine series of this sine pi x over L. So, we had this function here sine pi x over L in the interval 0 to, this L equal to 1 we have taken here, so 0 to 1. And then, we have written this cosine series making this extension as the even extension. And then as a result of this fourier series is having this cosine terms and we got the result which was just given in the previous slide.

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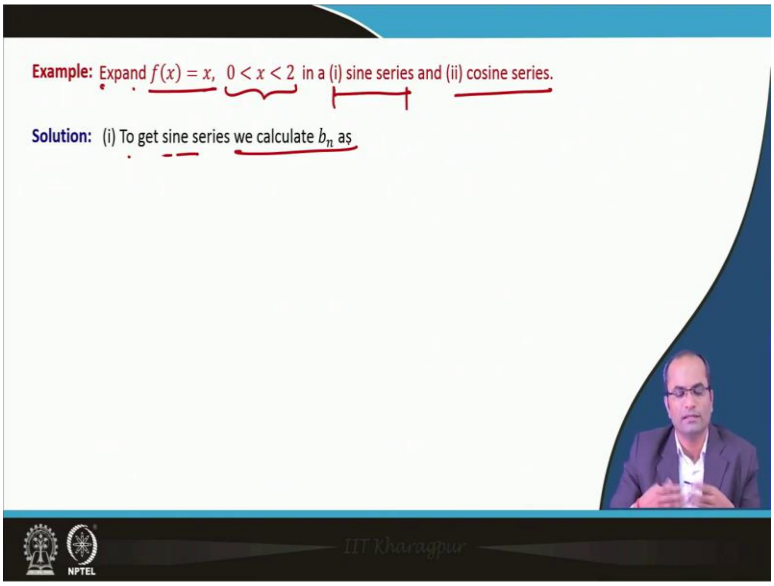

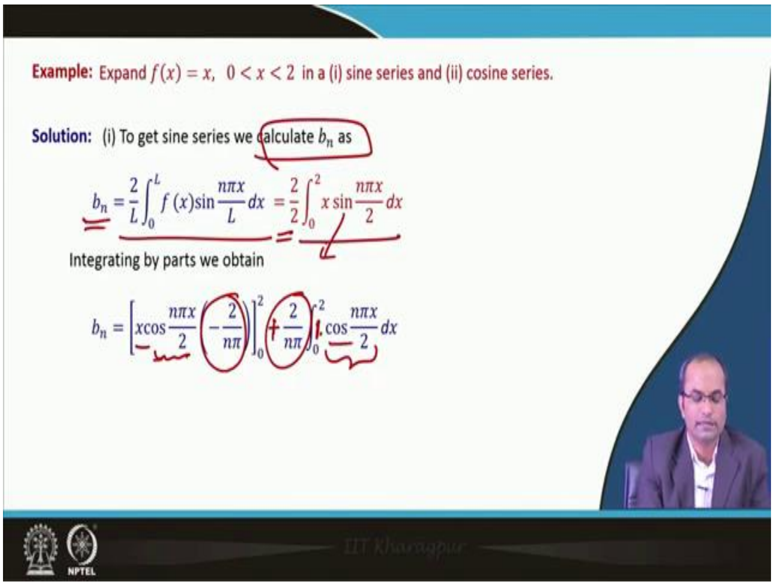

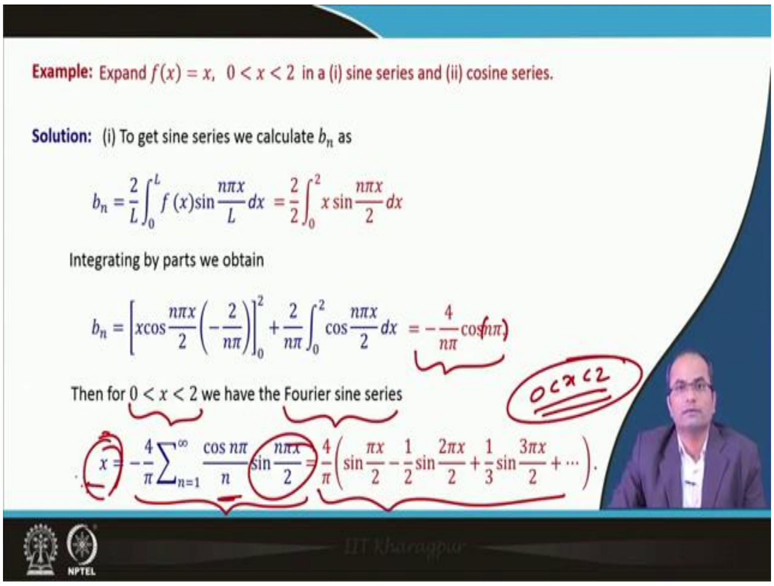

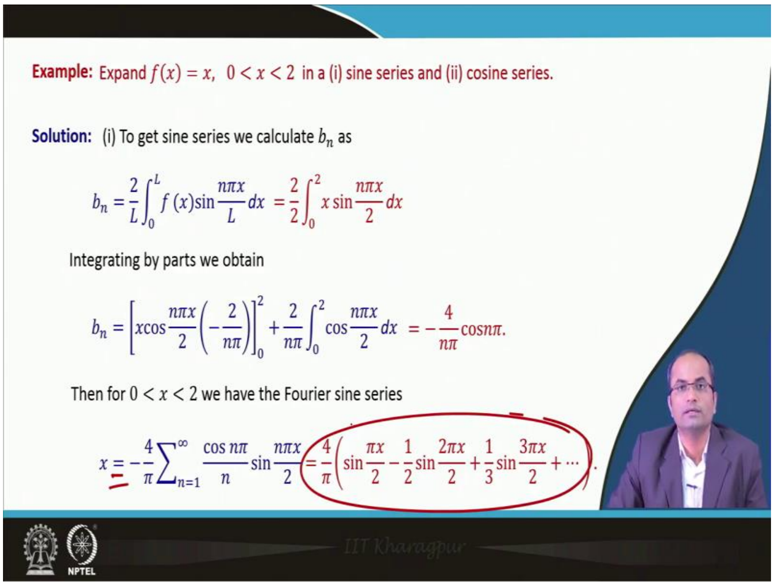

Well, so now here we want to expand for instance, this f x is equal to x given or defined in this 0 to 2 in a sine series and as well as in the cosine series. So we can do, so because we have to just expand this function in the interval minus 2 to 0 accordingly to have sine series or cosine series. So, to get the sine series we need to compute just bn. Indeed, we do not have to extend it, I mean we, but the idea should be clear that what is happening here but with the help of already discussed this formulas, we will just calculate bn and write down it is a sine series.

For the cosine one, we will just have the an’s and then we can write down its fourier series. So, we do not have to basically expand it, extend the function as an odd extension or even extension and then we have to write the fourier series. That is not needed. So, we will just now compute this bn. So, bn is 2 over L and then 0 to L f x sine n pi x over L, the general formula to compute this bn.

So here, L is 2 and then we have 0 to 2 x sine n pi x over 2, so which we have to integrate by parts. So the first is x as it is, then we have the, the cos for the sine and then just to have this integration we have to also have minus sine there because sine is integrated. And then we have cosine with the minus sign and this 2 over n pi factor will be appearing there. So again, the same thing there 2 over n pi factor with the minus sign and minus was there, so it is adjusted. And then we have the differentiation of this x which is 1 here and then again, we are getting this cos n pi over L.

So, which can be further integrated, so first here also we will have, when we substitute this 2 there, we will have cos and pi with this 2 there and then 0 we will 0 there. So, all we have to simplify this to get this minus 4 over n pi and then we will have this cos n pi term. So, in this region, 0 to 2 we can have this fourier sine series, because we have the bn's ready now. So, the x can be written in terms of the sine series now, with the coefficient here 4 over n pi and this cos n pi.

Or we can expand it, so we are getting these only the sine terms in the series and this is the representation of this x. We have the equality there. So, in the interval this 0 to 2, we have this equality. So, this series will converge because we know this result from the, from the Dirichlet theorem that at the points where it is continuous it will be equal to exactly the function, the function was x there. So, we have the x equal to this series which is having purely sine terms.

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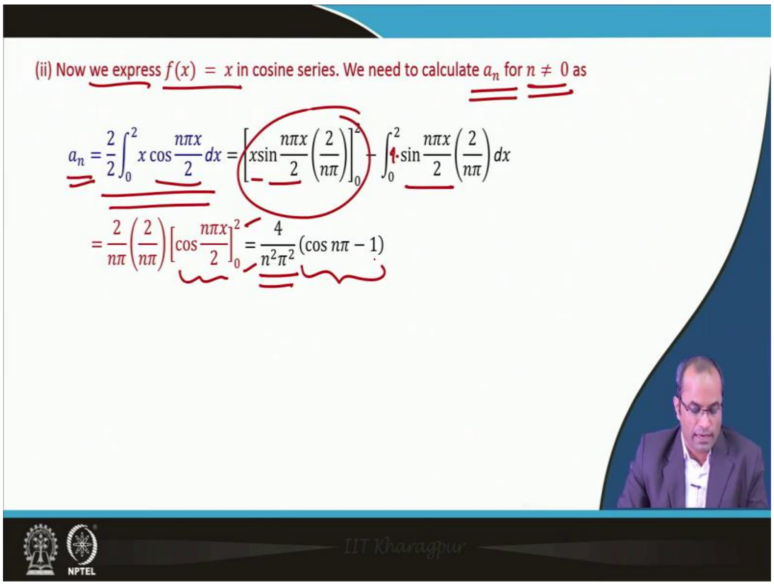

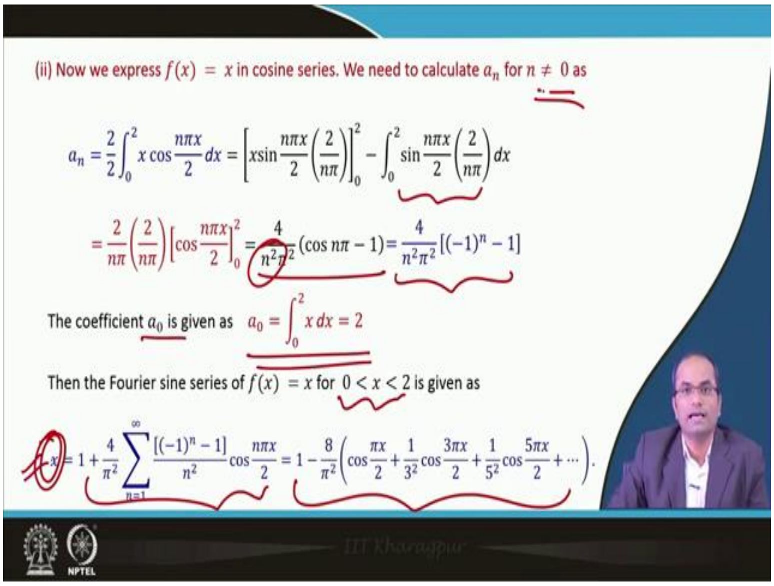

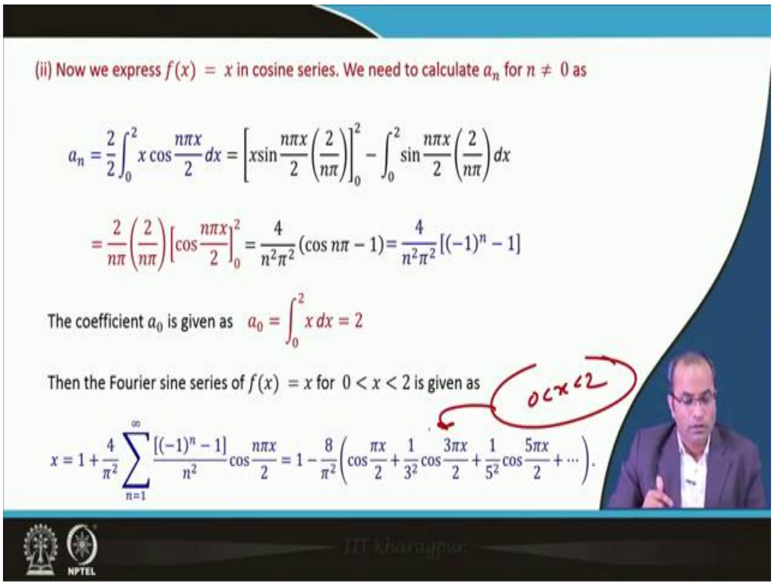

Well, we can also express this f x is equal to x in the cosine series and for that we need to calculate now an's, for n not equal to 0. For a0 we will do the separate calculation. So here we have an, again the same formula 2 over L 0 to L. We have this x cos n pi x over L and we can integrate this by parts again. So, x and then we will get sine there, and again sine and the differentiation of x will be 1. And then we have to differentiate again because this will become 0 and there, we have to differentiate to get this cos term and the limits from 0 to 2 again.

After substituting this limit 4 over n pi square we are getting cos n pi and this minus 1. So, we have the coefficient for an as well, so we can write down it is a, it is a series here with this n. So, the coefficient is 0, we have compute separately because this was not possible directly with the an because of this denominator n. So, a0 we can separately compute which is coming 2 there. And now we can write down the fourier series for x which is x is given in the interval 0 to 2.

So, we have x equal to completely the cosine series. So the same x, the same function which was given in the region 0 to 2, at once we have purely the cosine series and now the earlier one we had just the sine series representation of this cos x for this x and now we have written this x with the equality sign, with the equality sign in the region this 0 to 2. We have written now in terms of the cosine. So that is the beauty of this half range sine or cosine series, that the given function we can write down either in the sine series or in the cosine series.

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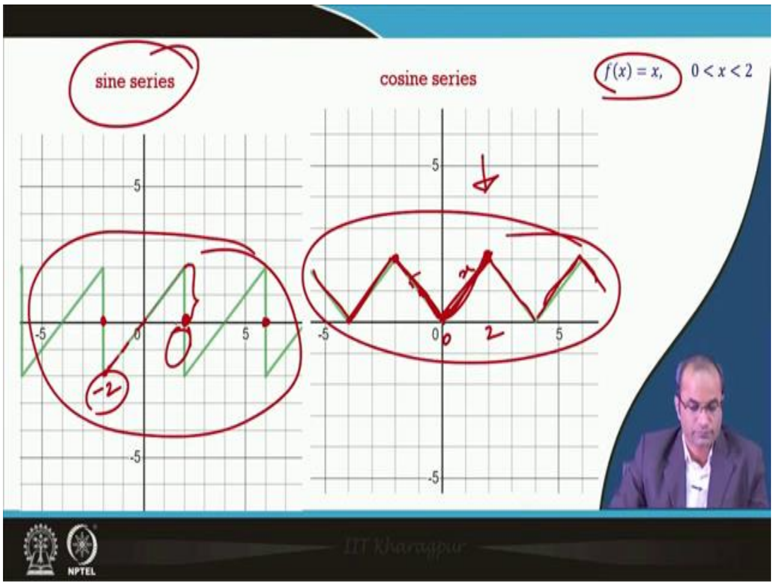

Well, just to see that what is happening here, so this was the function given in 0 to 2, so in 0 to 2. And then the fourier series will converge to periodic function with period 4 because this is the extension which has to be done there, though we have not explicitly done there and got the (())(27:13) we have applied the direct formula. But we should know that what is exactly happening. So, the one period of this function is going to be from this minus 2 to 2 and this was an odd extension of this function.

So naturally, we will get the sine series in this case. For the cosine series, for this f x function we have, so this is the function given, x in the interval 0 to 2 and now we have to have this odd extension and, even extension and then the fourier series will converge to exactly this 1 here. This is, will be the period now, which will be repeated here to have this periodic function. So, the fourier series now it is converging to this function in the case of cosine series, in the case of sine series, which is converging to this function.

So, for the cosine case for instance, you have the continuity everywhere and so the value of this fourier series, the fourier cosine series will be exactly the value of this function at any point. But in this case, for example, we have the problem at these points there but then it will converge the fourier series to the middle one, so to the 0, to the average value. So, that we have already discussed with this convergence results earlier.

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Well, so these are the references we have used for preparing these lectures.

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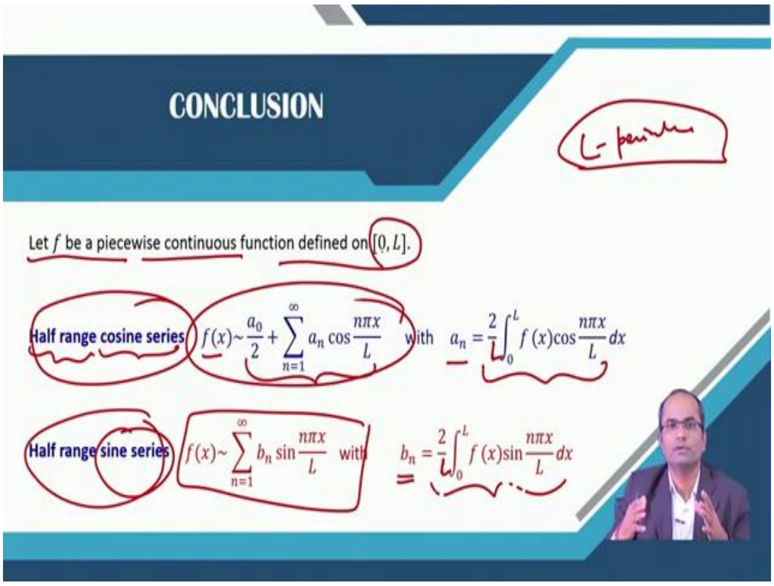

And now to conclude, that what we have discussed in this lecture. Suppose this f is a piecewise continuous function which is defined in the interval 0 to L. Then it is our wish to write down the cosine series or we call it half range cosine series, where f x can be represented by purely in terms of the cosine functions, where the an is given by 2 over L and 0 to L f x cos n pi x over L.

Or if we wish to expand this in the sine series, then we have to compute this bn with this 2 over L 0 to L f x and sine and pi x over L. And then we have the pure sine expansion of the function. So, there are 3 possibilities now, at least 3 possibilities we have discussed. Given a function in the range 0 to L we can construct its fourier series, which in general will have sine and cosine both the terms and the fourier series it will converge to L periodic function, L periodic function. And exactly this period will be defined by the given function 0 to L because we have written its fourier series.

Now for the half range series, if we want to have this cosine series, then the cosine series will converge to the periodic function having period 2L because now the in one period the function will be defined from minus L to L, and minus L to 0 it will have just even extension. Then only we will get this half range cosine series. Similarly, for the half range sine series, the fourier series, this half range fourier sine series will converge to the function having period 2L and in one period will be defined from minus L to L and from minus L to 0 we will have the odd extension of the function which is defined in this 0 to L.

So, I hope this is, the idea is clear of the half range which is entirely different from the fourier series, this is based on this odd and even extension and to have the desired fourier series either in terms of sine functions or in terms of cosine functions. Well, so that is all for this lecture and I thank you for your attention.