**Engineering Mathematics 2**
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**Lecture 42**
**Fourier sin and Cosine Integrals**

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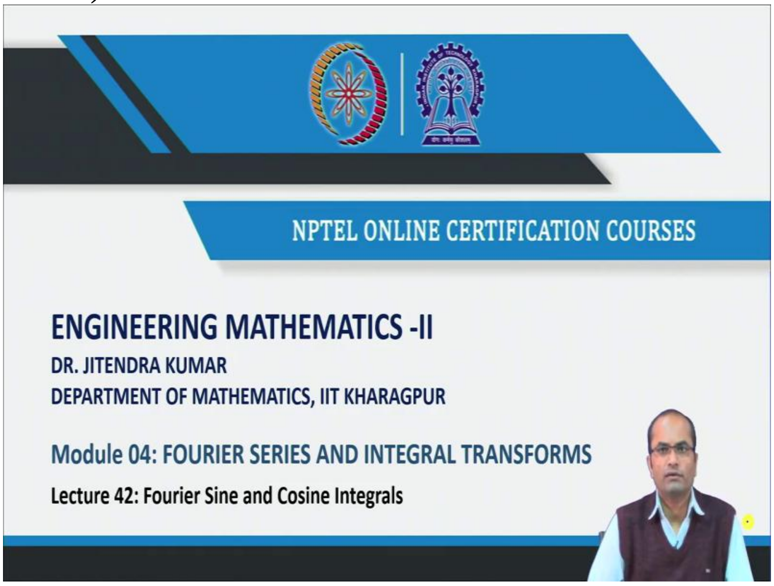


So, welcome back to lectures on Engineering Mathematic 2 and this is lecture no 42 on Fourier sin and cosine integrals.

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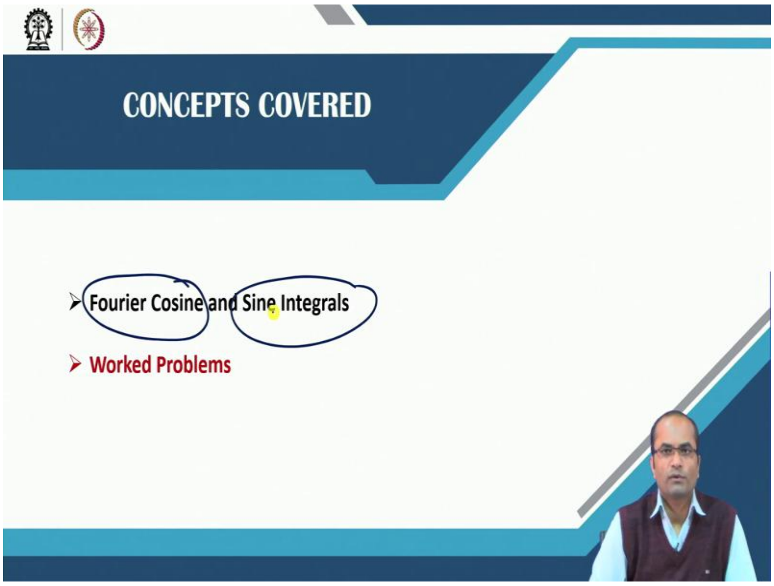


So, today we will discuss what are Fourier cosine integrals and Fourier sin integrals and then some worked problems based on these concepts.

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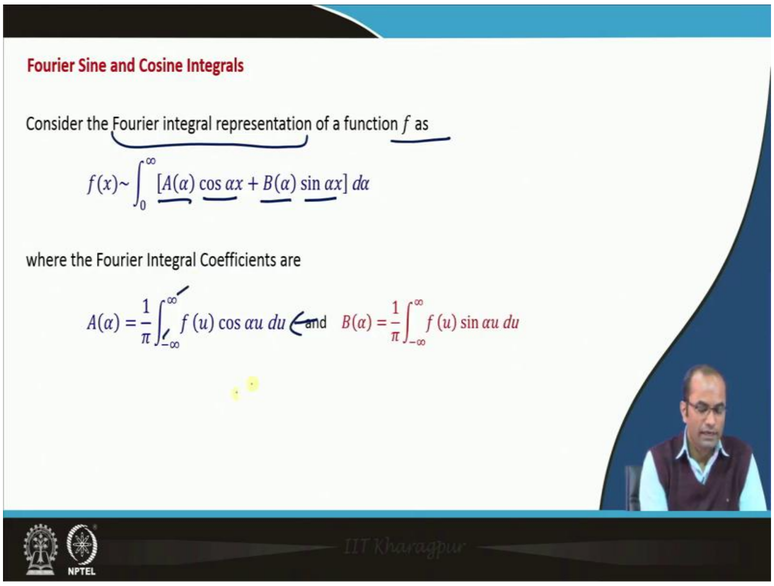


So, coming to the Fourier sin and cosine integrals, so in the last lecture we have seen that the Fourier integral representation of a function f is given as this a alpha cos alpha x, b alpha sin alpha x and d alpha this integral, where these Fourier integral coefficients a alpha and b alpha were given by these infinite integrals minus infinity to plus infinity.

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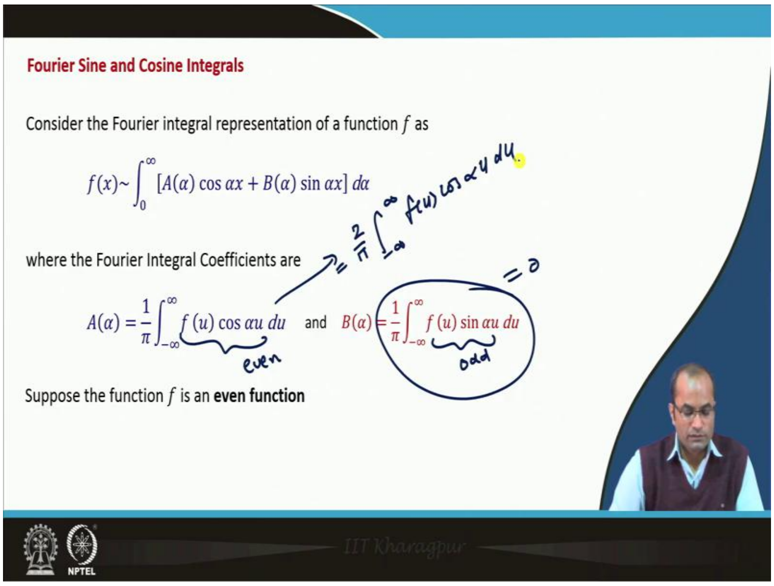


And b alpha was given again with this improper integral minus infinity to plus infinity fu and sin alpha u du. Well, now the point is if we consider that this function f is an even function or suppose the function f is an even function then what will happen whether there will be some simplification to the representation to the Fourier integral representation.

So, naturally because if this f is an even function, then what will happen for instance to this a, f is an even, cos is also even, so here the integrant will be the even function and in this case since the sin is sitting there with fu, fu is even but sin is odd, so this will be a odd integrant to this function.

So, in this case this b will go to 00 because the integrant is an odd function and because this integrant is an even function, so we can write instead that 2 over pi this integral minus infinity to plus infinity and fu and then cos alpha u and du.

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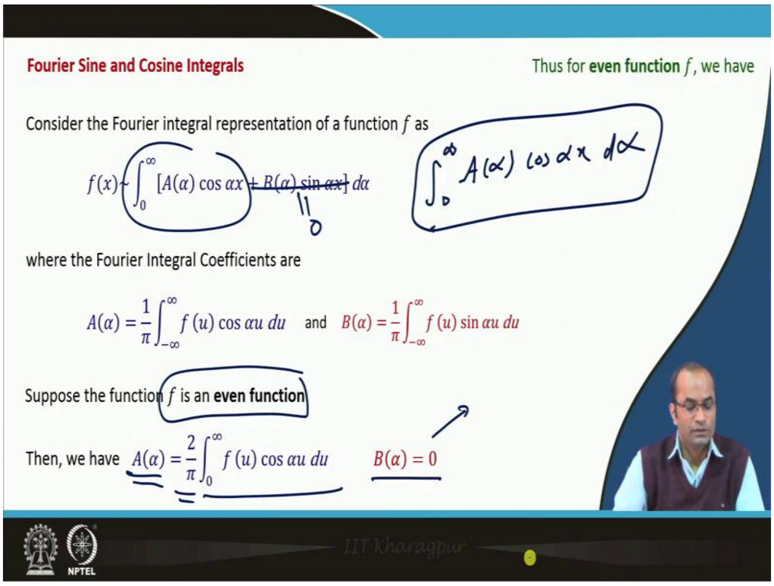


So, with these simplifications we now want to go, so if this f is an even function, then we have this a alpha 2 times 2 over pi integral instead of this minus Infinity to plus infinity this has become 0 to infinity fu cos alpha u and then b alpha will become 0, so this Fourier integral representation, this term will vanish and then we have rather simple integral representation which is 0 to infinity and then we have a alpha and cos alpha x d alpha.

This is again similar to what we have done in case of the Fourier series as well and similarly there also bn's were 0 for the function f which is an even function and for the odd function, the other way around.

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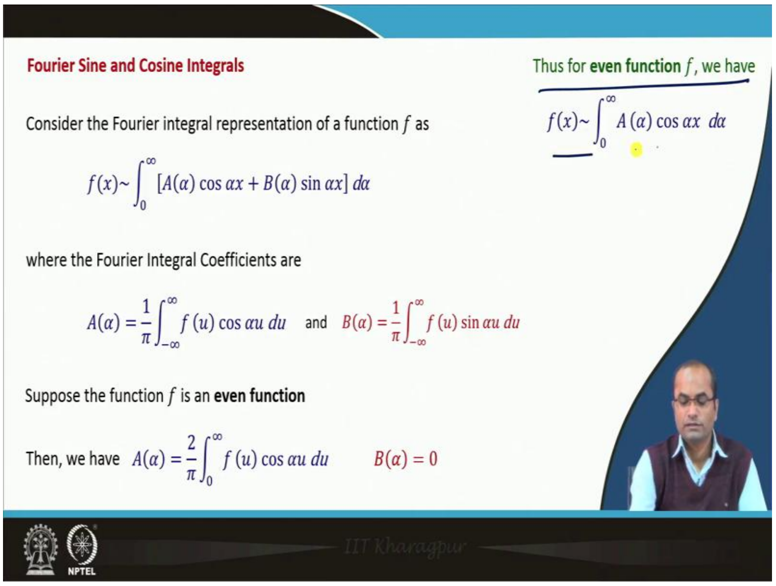


So, thus, for an even function we have this Fourier integral representation.

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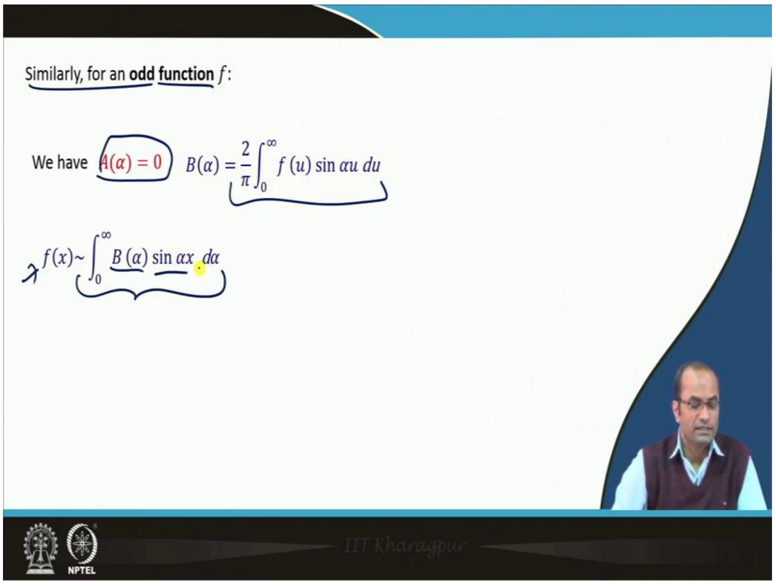


And now similarly when we have an odd function f, then the a alpha will becomes 0 and the b alpha will be computed by this 2 over pi 0 to infinity fu and sin alpha u du. And its Fourier representation will take this simplified form as b alpha and sin alpha x d alpha.

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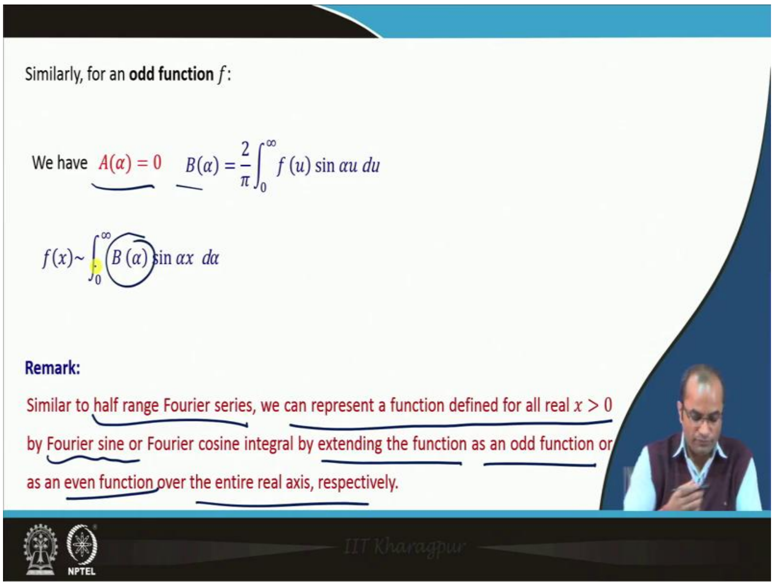


Just a remark here which is very for the discussion now similar to the half range Fourier series. So, if you recall the half range Fourier series, the idea was entirely different there where we were extending the function which is defined in the 0 to l interval and the way we extend the function in minus l to 0, the corresponding either the sin series or the cosine series will appear because if we make the entire function defined in minus l to l as an odd function, then we will end up with the sin series and we extend the function, in terms of, so that the function became even function and then we will end up with the Fourier sin cosine series.

Though we can have infinitely many extents of this function, but we have chosen those two extents to make the overall function minus l to l as an even function or an odd function so that we get some simple representation either as a sin series or cosine series and both of them will be valid for the or represent the function given in 0 to l.

A similar structure we have here as well that we can represent a function which is defined for instance for all real x, so a function is given which is defined for x positive and now defining or by extending that function as an odd function or an even function over the entire real axis, so for instance we extend this from minus infinity to 0, so that the function in the whole real axis become an even function then we have that simplified representation where a and a alpha will survive and b alpha will become 0 we have the rather simple representation with that a alpha.

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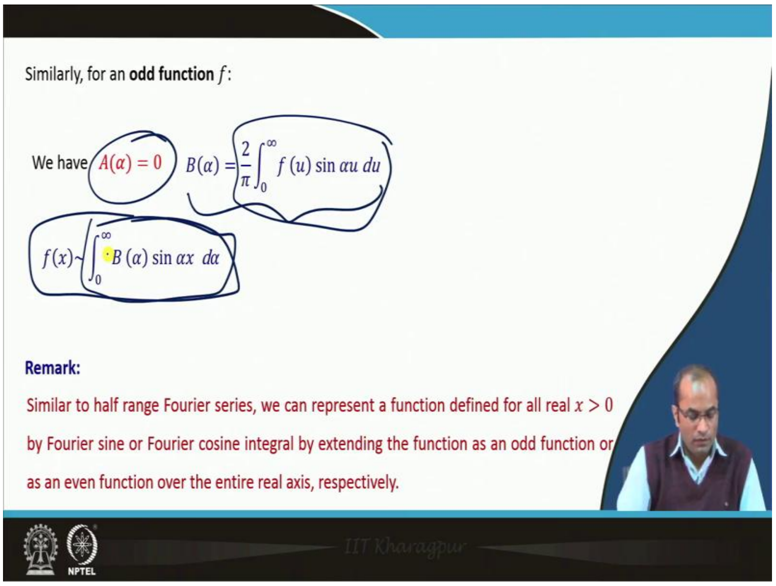


Or if we extend this function from minus infinity to 0 so that the function defined in the whole range now becomes an odd function in that case this a alpha will be 0 and b alpha can be easily computed with the help of this integral and the representation will be simpler now in terms of only this b alpha and sin alpha x.

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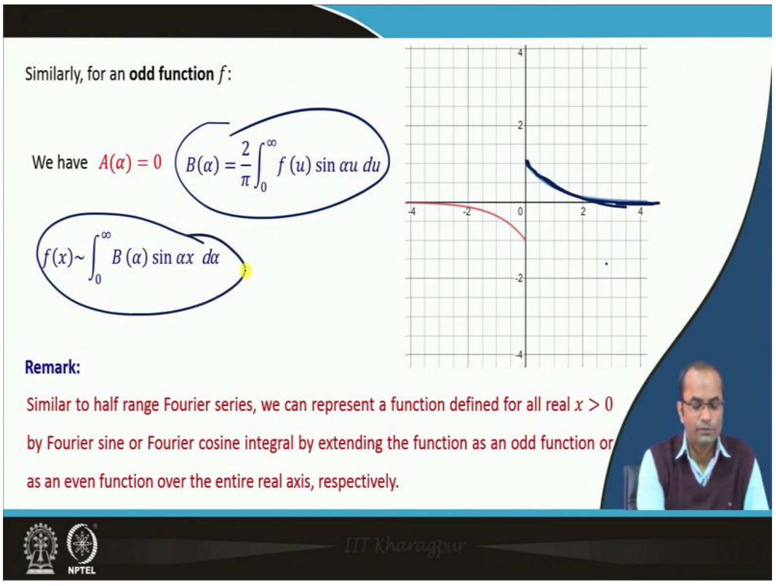


So, with this idea now let us just consider for instance a function is defined here in 0 to infinity by this curve, so we have the possibility of this extending as an odd extension or an even extension, so if we extend this so that in the whole real axis the function has become now an odd function, then its representation will have only this b alpha terms and then we have this simple representation.

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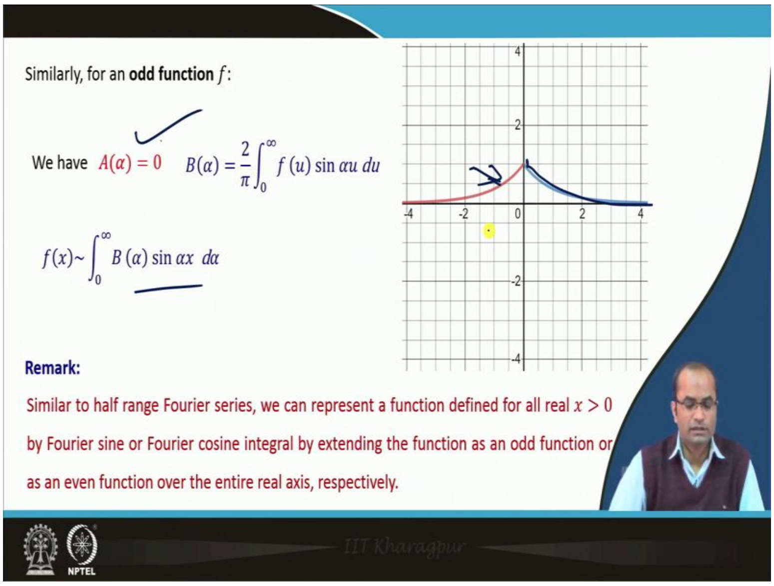


On the other hand, if we extend this in this way, then this will become an even function and if we write its integral representation that will have only the cosine term, only the cos term in the coefficient and as well as in this integral representation. So, these are the two possibilities where we can make use of the function defined in the interval 0 to infinity.

So, only in one half of this axis, the function is defined and in the other half, we can define according to our wish to have either a Fourier sin integral representation or to have Fourier cosine integral representation. So, when a function is defined in minus infinity to plus infinity, we will have a Fourier integral representation which will have in general both the terms sin and cosine.

But when the function is defined in 0 to infinity, we have the possibility to have either only sin functions in the representation or only cosine function in the representation. So, that will be the discussion of this chapter now.

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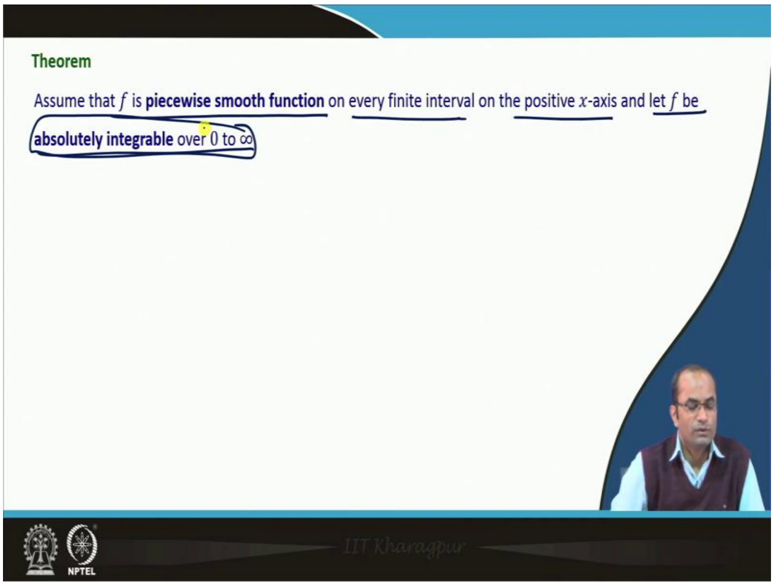


So, we have this result which we can summarise now. Suppose f is piecewise smooth function, the conditions are same as discussed earlier on every finite interval on the positive axis and let f be absolutely integrable, I mean, that is important here for that transition from Fourier series to this Fourier integral representation, we have additional condition on f to be this absolutely integrable.

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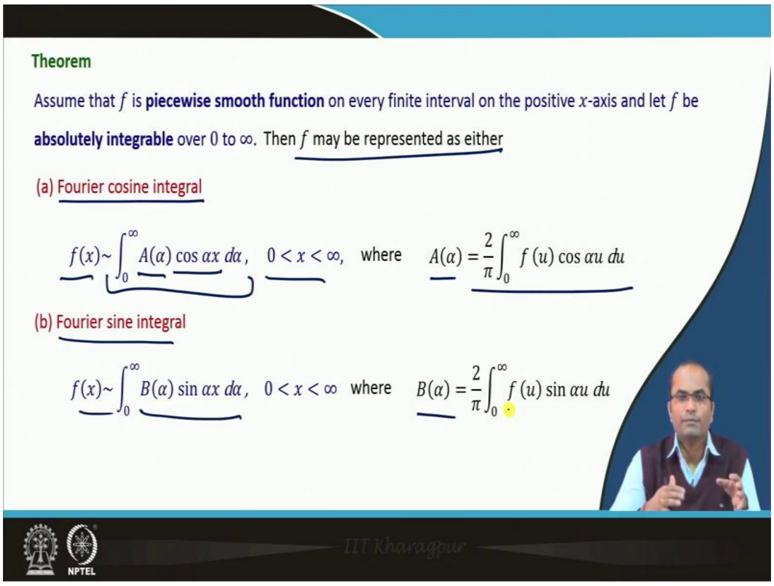


Then f maybe represented either by the Fourier cosine integral that is the first possibility we have discussed and the Fourier cosine representation is just 0 to infinity a alpha cos alpha x where a alpha is given by this integral. The second possibility would be that we can have the

Fourier sin integral which can be written here b alpha sin alpha x and b alpha will be given by this coefficient 2 over pi 0 to infinity fu sin alpha u du.

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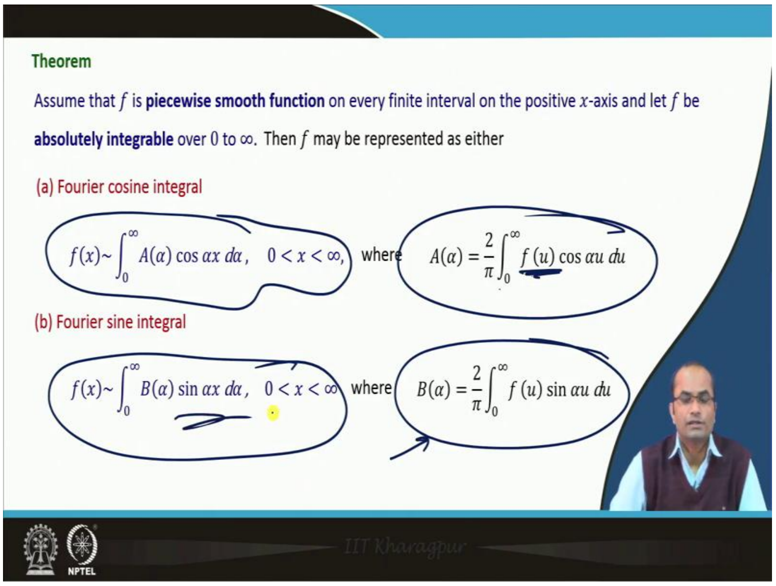


So, without this formally extending the function from minus infinity to infinity that is the idea behind, but we can directly use these results if we want to have Fourier cosine integral representation, we will compute a alpha by this formula 0 to infinity and fu is given in 0 to infinity, so there is no problem. Similarly, if you want to have, if we wish to have the Fourier sin integral representation we will compute this b alpha and then we can represent this f using this sin representation.

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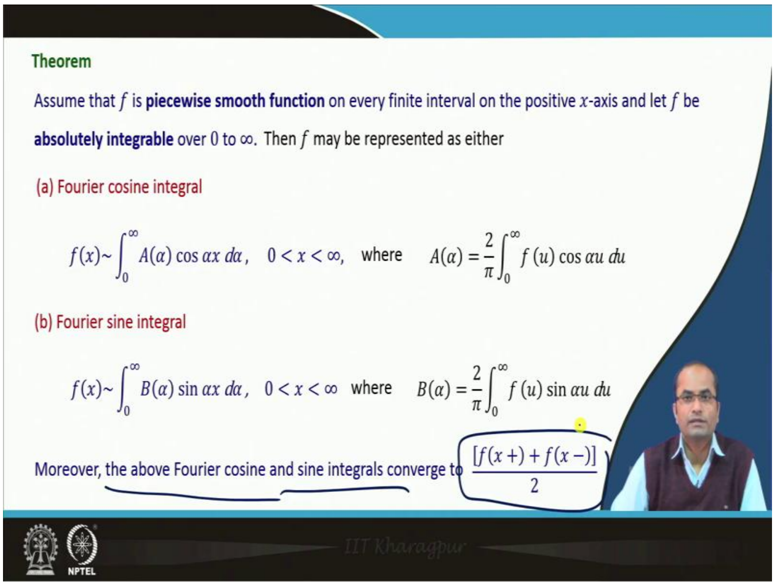

Well, so moreover the above Fourier cosine or sin integral converges the convergence is exactly what we have already discussed in previous lecture for the integral representation, so that these integrals will converge to the average value and at the point of continuity the convergence will be to the function value itself.

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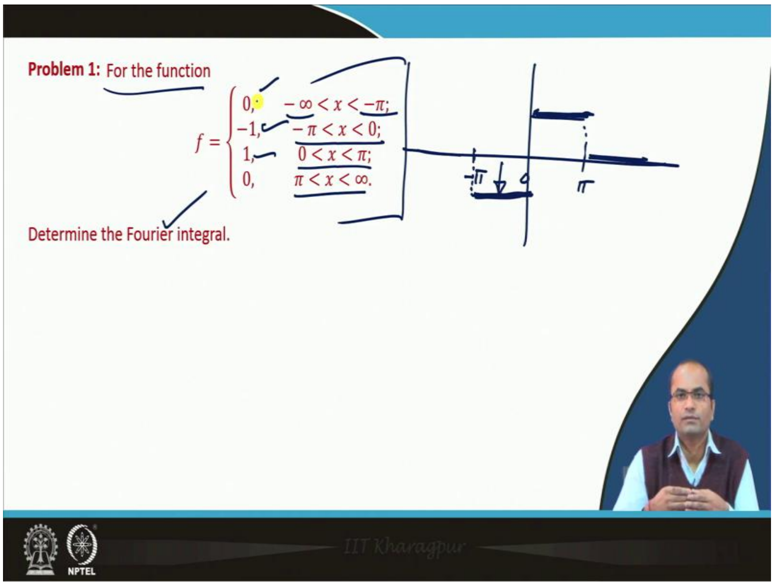


Well, so we consider this problem now that this function which is defined 0 as in the range minus infinity to minus pi, then minus pi to 0 it is minus 1 0 to pi 1 and then pi to infinity 0. So, we have the function here which is 0 up to this minus pi and then pi to 0 and then pi. So, minus pi to 0 the function value is minus 1 and then we have 1 there in 0 to pi and then again 0.

So, we have this piecewise continuous function and we want to find the Fourier integral representation of this function. So, what is to be noted in this case that this function is an odd function. So, here whatever values we have for the positive one, it is just minus of that in this negative range.

So, this function is an odd function and naturally this a alphas will become 0 and we have to only compute this b alpha but the point here is the function is defined in minus infinity to plus infinity, so we are writing its integral representation which is automatically coming as the sin integral representation but so, in this case we are not having the possibility of choosing whether sin integral representation or cosine integral representation because the function is defined from minus infinity to plus infinity.

Whereas in the next examples we will see that when the function is defines as 0 to infinity then we have both the possibilities whether we want to have a Fourier cosine representation or Fourier sin representation.

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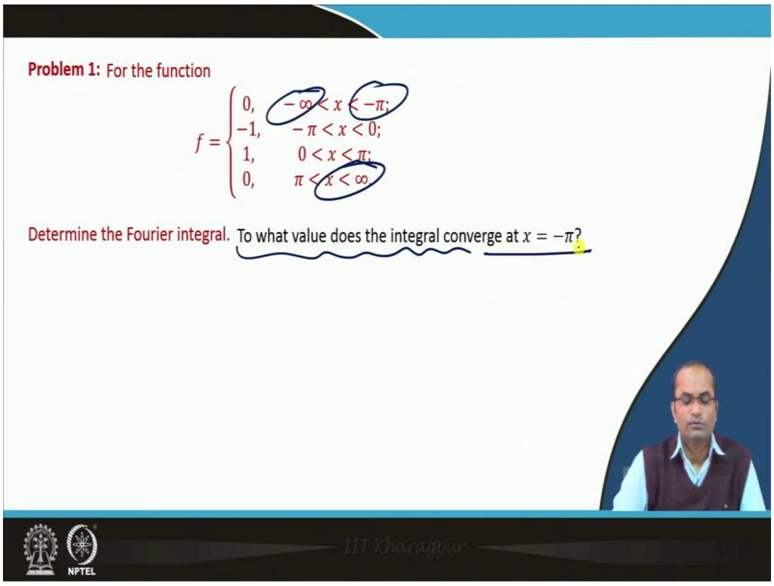


So, in this case, since the function was defined from minus infinity to plus infinity, we were simply write its Fourier integral representation and finally the question is that to what value does the integral converge at x is equal to minus pi, so that we will also discuss.

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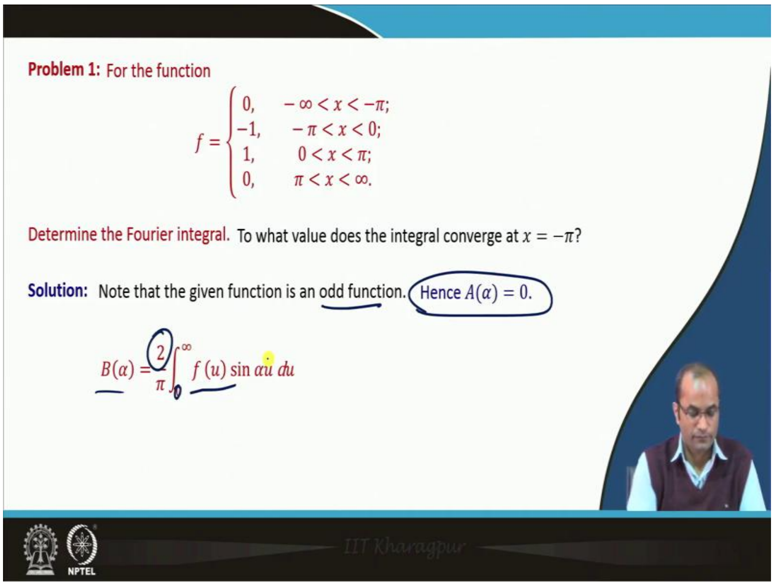


The given function is an odd function that we have already noticed, hence the a alpha will be 0, that is the automatic calculation here because of this odd function the a alpha will be 0 and the b alpha we can compute with this help of this integral which have now 2 there and 0 to infinity, so instead of minus infinity to plus infinity since the function is odd, we can just compute in the 0 to infinity fu and sin alpha u.

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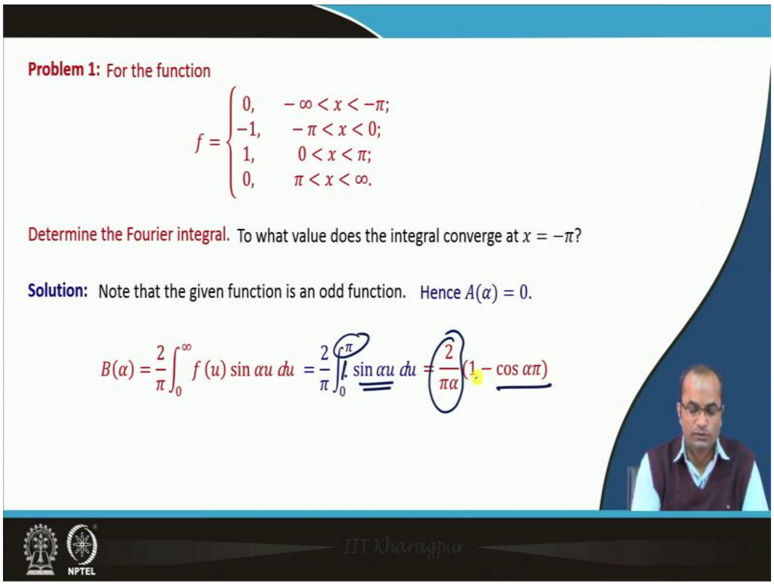


So, the function is given already that 0 to pi is 1, so 1 into this sin alpha u and this can be integrated now to have cos alpha u over alpha so we have 2 over pi alpha and then cos alpha, so we can put this with minus sin so minus cos alpha will becoming so we have the with this pi, we have cos alpha pi and then 0 we have 1.

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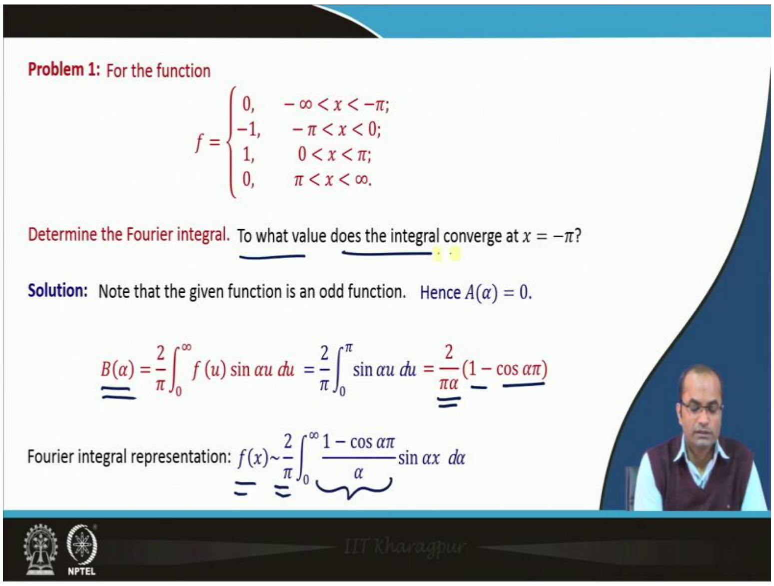


So the value of this b alpha coefficient is 2 over pi alpha and 1 minus this cos pi alpha. The Fourier integral representation because a alpha is 0 so in this case we have a rather simple representation fx is given by 2 over pi, we have this 1 minus cos alpha pi over sin alpha x dx. And now the question is that to what value does this integral converge?

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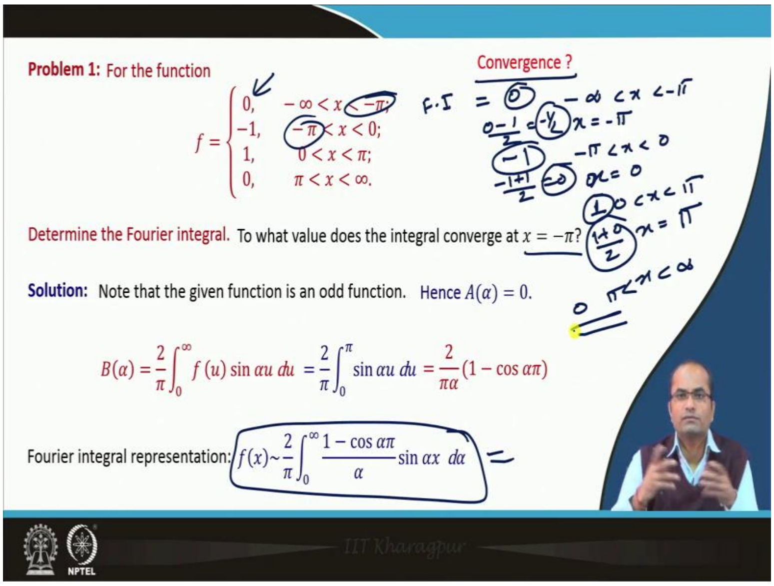


So, we have the Fourier integral representation of the function now the question is to what value does the integral converge at x is equal to minus 5. We know the convergence theorem already, so wherever the function is continuous, it will converge to the function value and if the function is not continuous it is discontinuous, then we will take the average value of the function at that point. So, for instance at x is equal to minus pi, indeed this function is not defined at x is equal to minus pi but it does not matter, we can take the average value of the limits at that point.

So, about the convergence at the point where it is continuous, it will directly go to the average value, so for instance in the range minus infinity to this minus pi, the Fourier integral will converge to this 0 and then at pi, when you have x is equal to minus pi, it will converge to the average value 0 minus 1 by 2 and then in the minus pi 2 x in to 0, it will converge to minus 1 and at x equal to 0, it will converge to again average value minus 1 plus 1 by 2 which is 0, here it was minus half and then when we have 0 to pi, it will converge to 1 and then at pi x is equal to pi, it will converge to 1 plus 0 by 2, so the half here its 1, here its 0, minus 1, minus half and 0 and then when x is greater than pi, then it will converge to 0.

So, we have convergence at all points here wherever the function is continuous, it will converge to the value itself and wherever it is discontinuous or the function is not defined at those points we will take just the average value.

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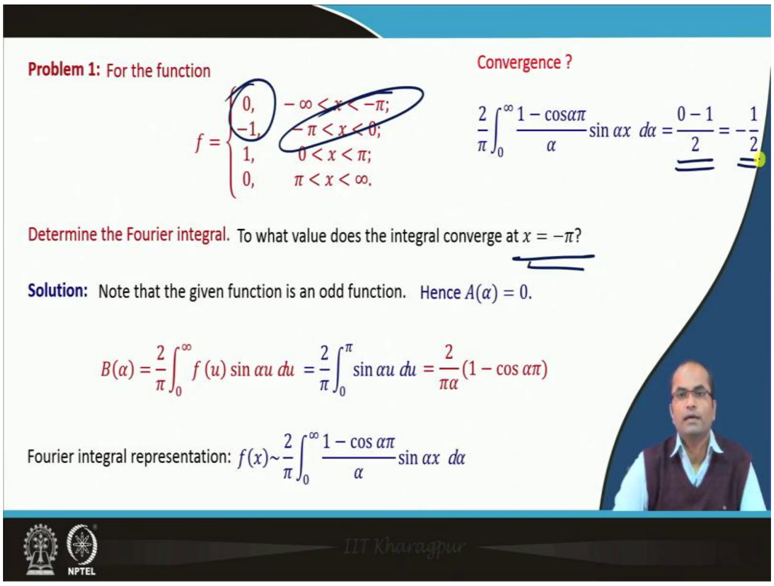


So, the question was at x is equal to minus pi we have already discussed at x is equal to minus pi, it will go with this average value which is 0 minus 1 by 2 and then we have minus half.

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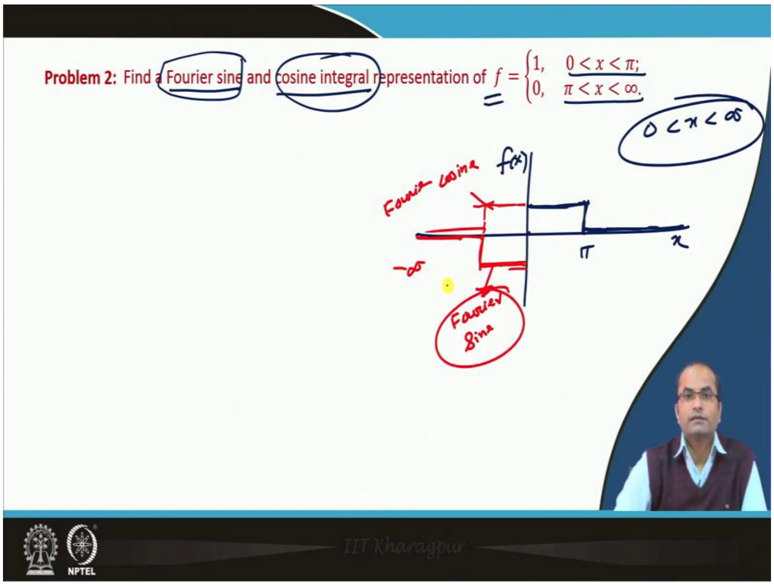


Well, so we will move to the next problem where we have to find Fourier sin and cosine integral representation. So, now in this question, we need to find Fourier sin as well as Fourier cosine integral representation of this function f which is defined 1 in this range 0 to pi and then we have here pi to infinity as 0.

So, the given function is in the 0 to pi is given as 1 and then it is 0, so after that it is 0, so the function is defined by this curve and now in this case it is different from the earlier one where the function was defining in the entire interval minus, entire real axis minus infinity to plus infinity but now the function is defined in 0 to infinity.

And now we have the possibility of extending this in this minus infinity to plus infinity if we extend it as in this way, then this will lead to a Fourier sin representation or if we extend this in this way, then it will extend to a Fourier cosine integral representation if we extend to have this odd function then we have the Fourier sin representation of the given function.

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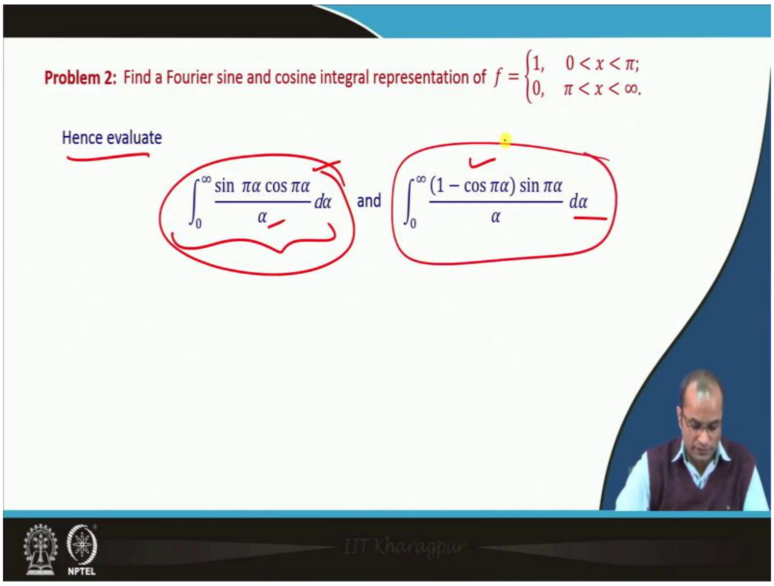


So, we will see now and the question further is that evaluate this integral 0 to pi sin pi alpha and this cos pi alpha over alpha and also this 1 minus cos pi alpha sin pi alpha over alpha d alpha. So, after evaluating this Fourier sin and the cosine representation you will see that we can easily evaluate these integrals there.

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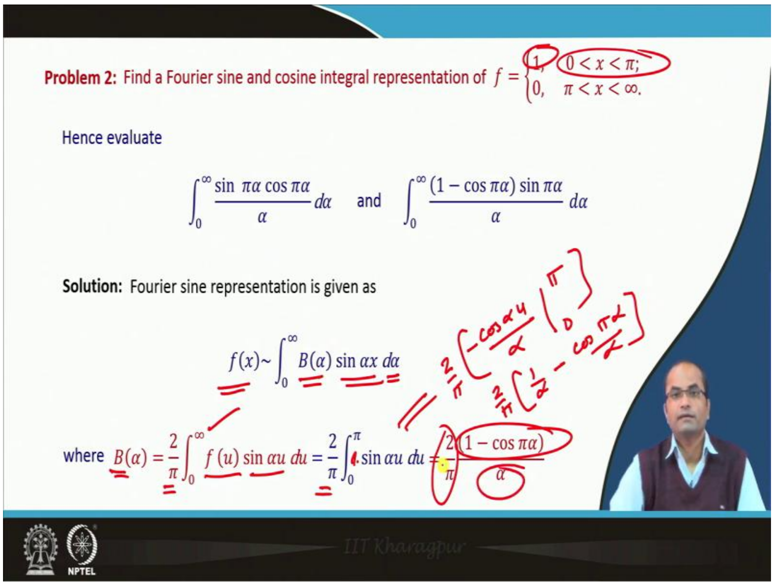


So, coming to the Fourier sin representation where this function is basically extended to have an odd function over the whole real axis minus infinity to plus infinity, we can write down its Fourier sin representation, so this fx as this b alpha sin alpha x d alpha where this b alpha is given by 2 over pi fu and sin alpha u du.

The f is defines 0 to pi as 1, so we can compute this 0 to, 2 over pi, we have 0 to pi because it is only defines 0 to pi rest it is 0 and then we have 1 in to this sin alpha u, so which can be now integrated, so we have this cos alpha u over this alpha and then 0 to pi with a minus sin because of this integral and then we have this 2 over pi naturally sitting outside.

So, here two over pi since it is a minus, so first we will put this 0, so we have 1 over alpha and then minus we have cos pi alpha over this alpha, so which is written here 2 over pi 1 minus this cos pi alpha divided by this alpha term.

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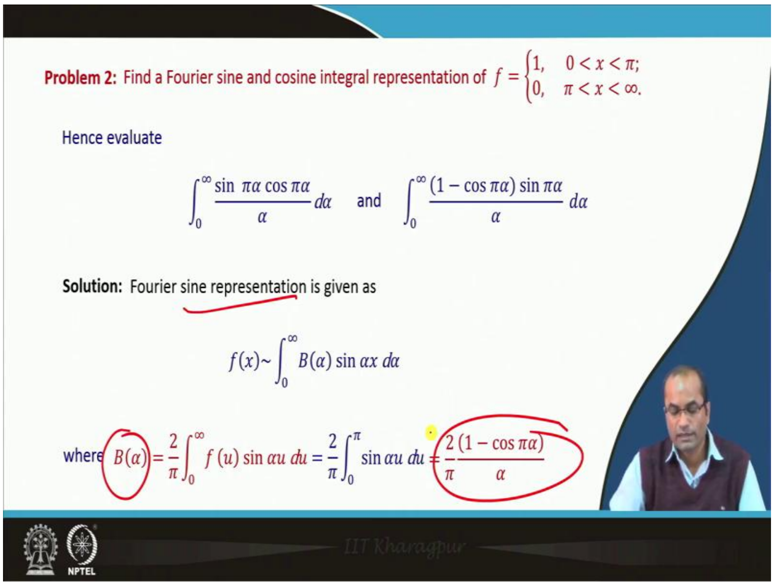


So, in this Fourier sin representation we have computed this b alpha as two over pi 1 minus cos pi alpha over alpha.

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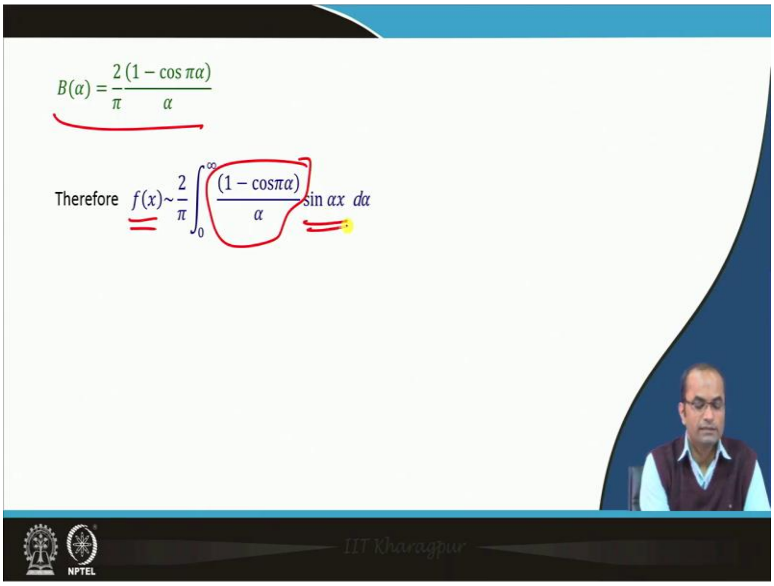


This is the Fourier coefficient and therefore we can write down this representation, the Fourier sin representation with this coefficient here 1 minus cos pi alpha over alpha and sin alpha x d alpha.

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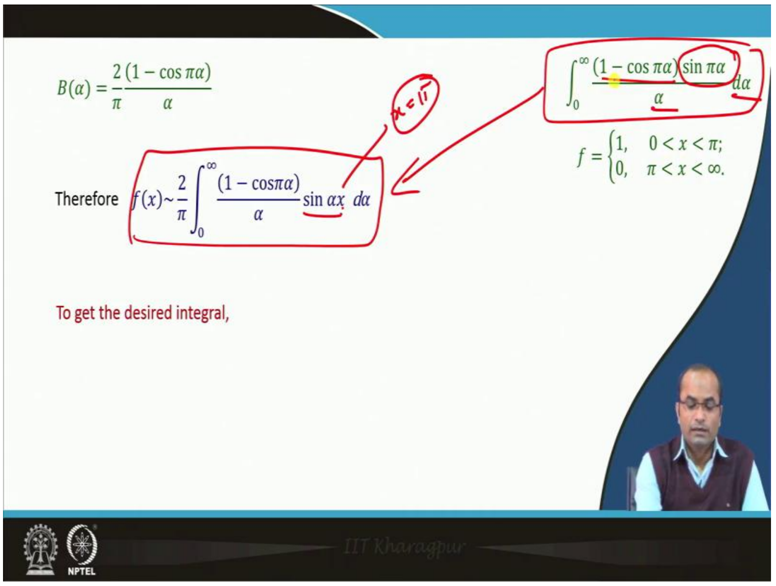


So, this is the Fourier sin integral representation of the given function. Now, if we look at the desired integral, which will be coming from this representation itself so this was the question that what is the value of this integral here, so if we compare with this Fourier sin integral representation, what we notice that this 1 minus cos pi alpha, cos pi alpha that is sitting exactly like this, we have sin alpha x but here we have sin pi alpha and then over alpha d alpha.

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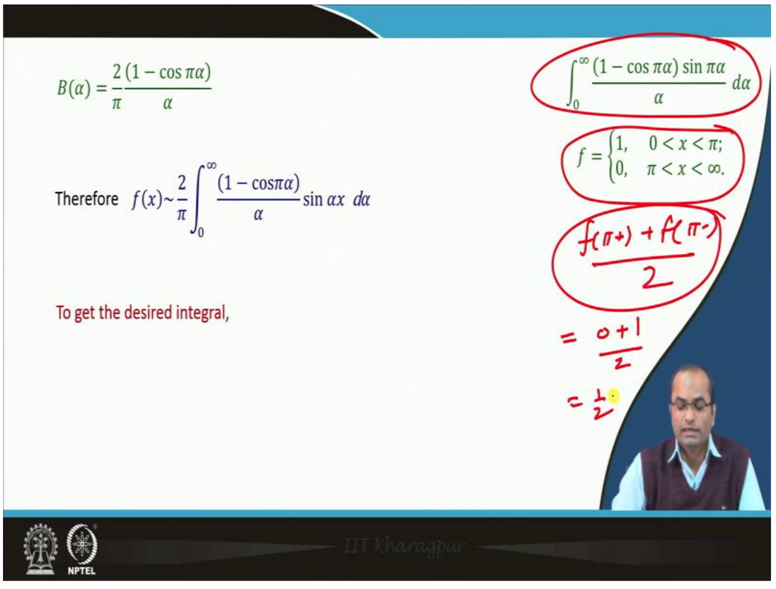


So, if we take here x is equal to pi, then we can get this desired integral which was asked in the question, so at x is equal to pi, then we have to also look at what is the convergence, so x is equal to pi again this function is not defined and therefore we have to take the limit, so pi plus and then f this pi minus divide by 2 this average value that Fourier integral at x is equal to pi will converge to. So, here fx pi plus is 0 and then f pi minus is 1, so divide by 2 so it is half, so that Fourier integral at x is equal to pi will converge to this value half.

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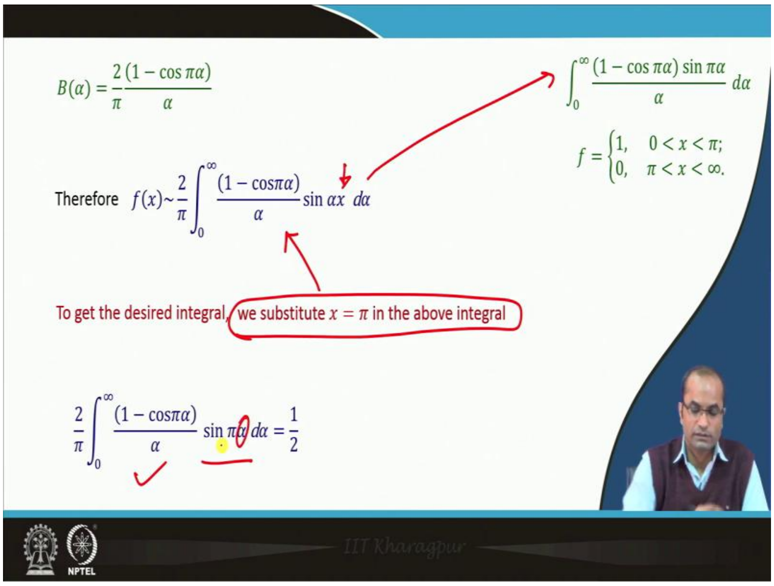


So, if we substitute this x is equal to pi in this above integral we will get exactly that integral which is the point of question now. So, here we have 2 over pi, 1 minus cos pi alpha over alpha and sin this x is replaced now with this pi, so x is replaced by this pi here.

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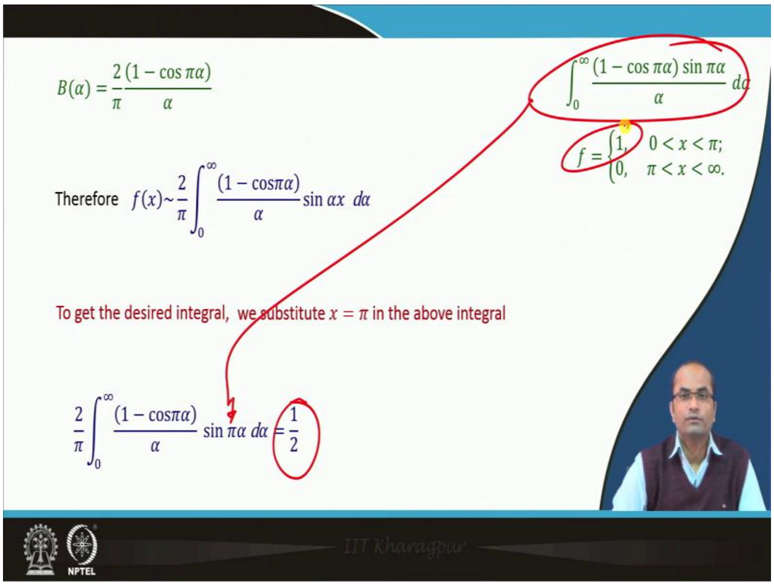


So, we have the desired integral coming and then this value is equal to half because that is exactly the average value of this function at x is equal to pi.

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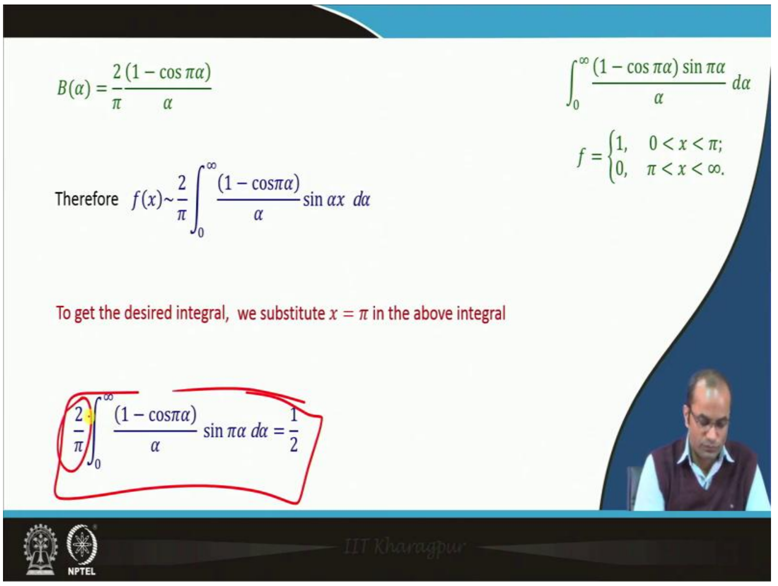


So, the value of this integral is 1 by 2 or this pi by 2, so this can go to the other side.

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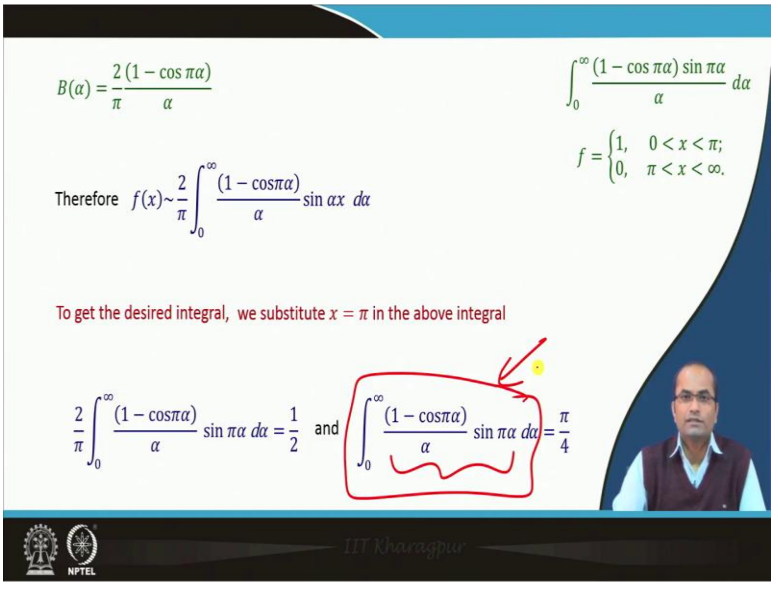


The value of this integral here is pi by 4. So, as we can notice that this integral again it is not easy, it is not simple to compute directly, but with the help of this convergence theorem and the Fourier integral, we can see that such complicated integrals can be evaluated easily and that's one of the applications now we do see in the computation of these complicated integrals.

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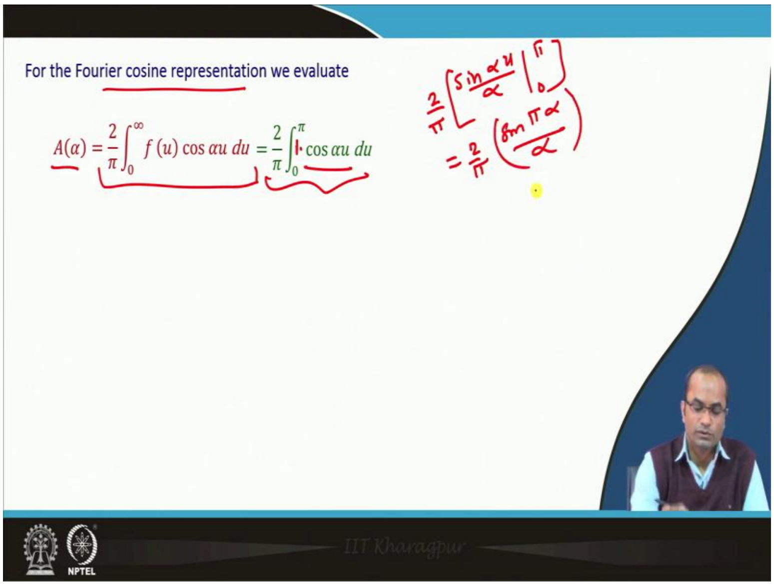


So, coming to the Fourier cosine representation where we have to extend the given function to make it as an even function and then its Fourier representation will be having a alpha computed using this 2 over pi 0 to alpha fu cos alpha u du and we can compute this as 2 over pi 0 to pi and the value is 1 and 0 to pi otherwise it is 0, then we have this cos alpha u du.

So, this will be, the integral will be sin alpha u over alpha and then we have 0 to pi and 2 over pi is sitting outside the integral. So, this value will be 2 over pi and when we put sin there, so we have sin pi alpha over alpha and then minus that will be 0, So this is the value of the integral sin pi alpha over alpha with two over pi factor.

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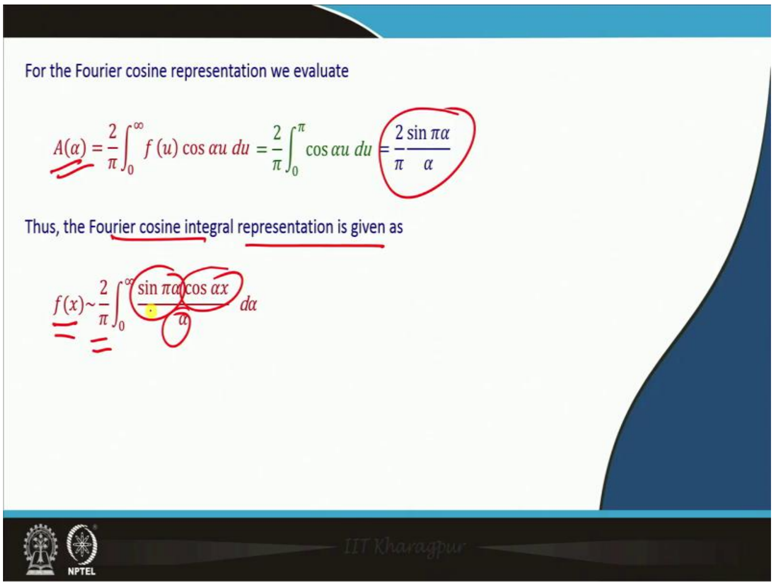

The Fourier cosine representation now because we have already a alpha which we can substitute in this representation, so the f alpha will be represented by this integral here 2 over pi. We have sin pi alpha over alpha and then this cos alpha x from the Fourier this integral representation.

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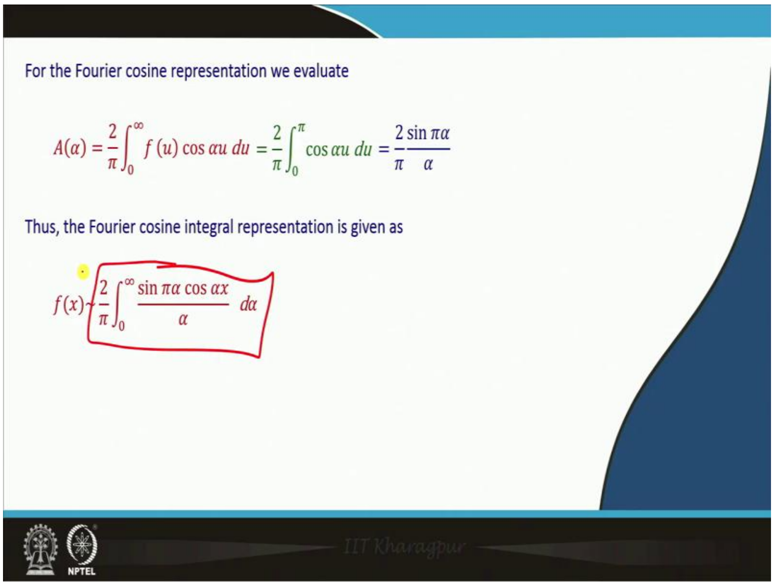


So, this is the Fourier integral representation of the cosine integral representation indeed for the given function.

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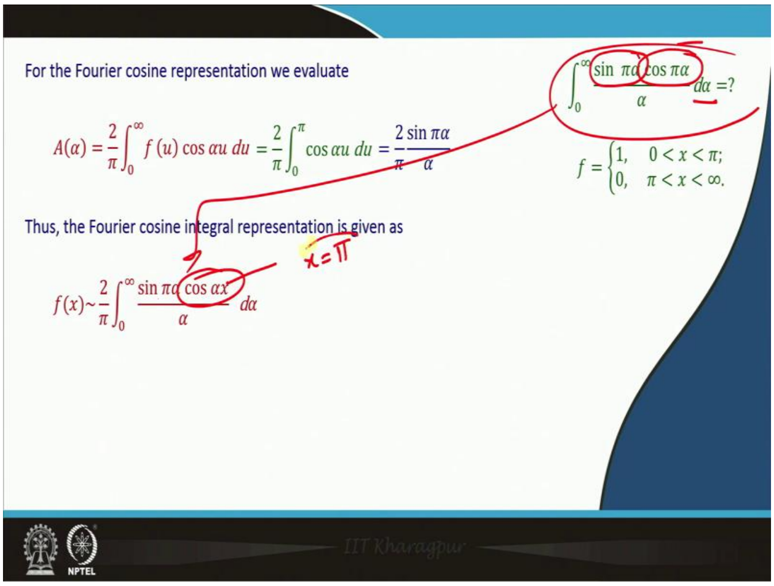


And then the question is that what is the value of this integral which is sin pi alpha cos pi alpha over alpha d alpha. So, if we compare with this now, we have sin pi alpha, here also we have sin pi alpha, we have cos pi x, we have cos pi alpha, so again if this x is replaced if its substituted by pi, then we are getting exactly this integral.

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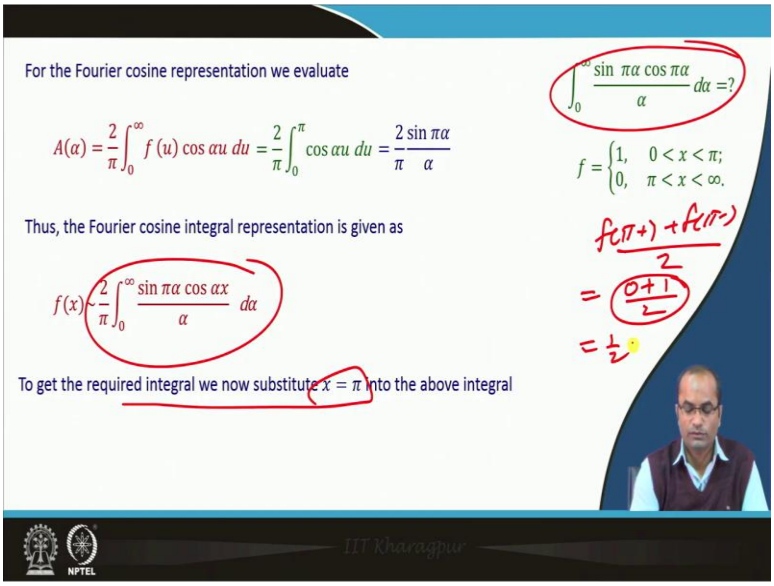


So, again to get this desired integral, we have to use x is equal to pi. So, if we substitute this x s equal to pi in to the above integral, then we can exactly get the desired value there. Now, the question is again that x is equal to pi this integral will converge again to half because we have this average value f pi plus and f pi minus divided by 2, so f pi plus we have 0 and then we have 1 divided by 2, so the value is 1 by 2.

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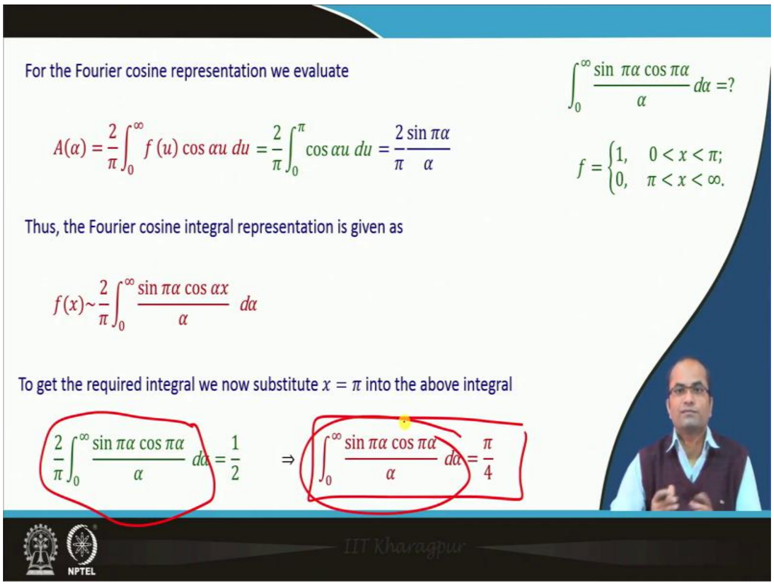

So, this we have value 1 by 2 and then this integral is said to be equal to half and again we have this value pi by 4 for this integral. So, again such complicated integrals can easily be computed with the help of this convergence theorem and this Fourier integral representation.

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Well, so here we have these references which are used for preparing this lecture.

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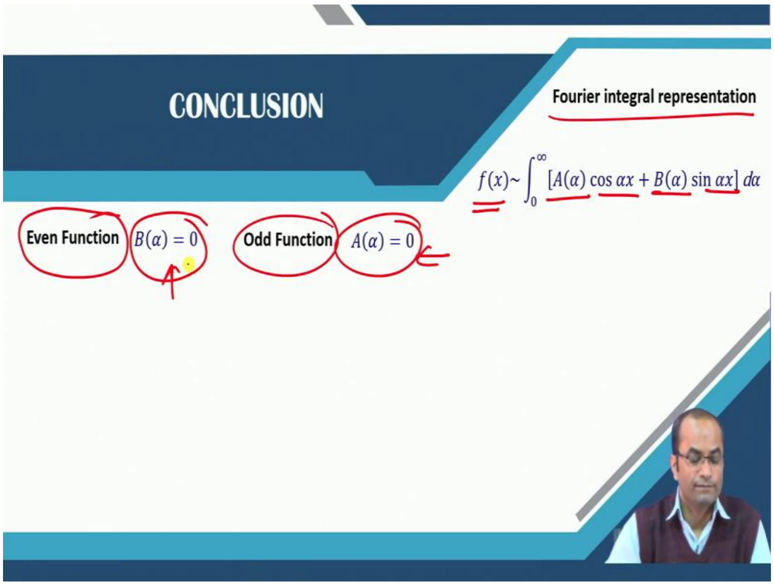


And now just to conclude that this Fourier integral representation which was given as a alpha and this b alpha term with cos alpha x and sin alpha which usually or in general it has cos terms as well as the sin terms but what we have seen in this lecture that for even functions, so this if f is an even function, then the b alpha will be simply 0 and for the odd function, the a alpha will become 0 and we can just compute b alpha there and in this case we need to just compute a alpha that to much more simplified integral.

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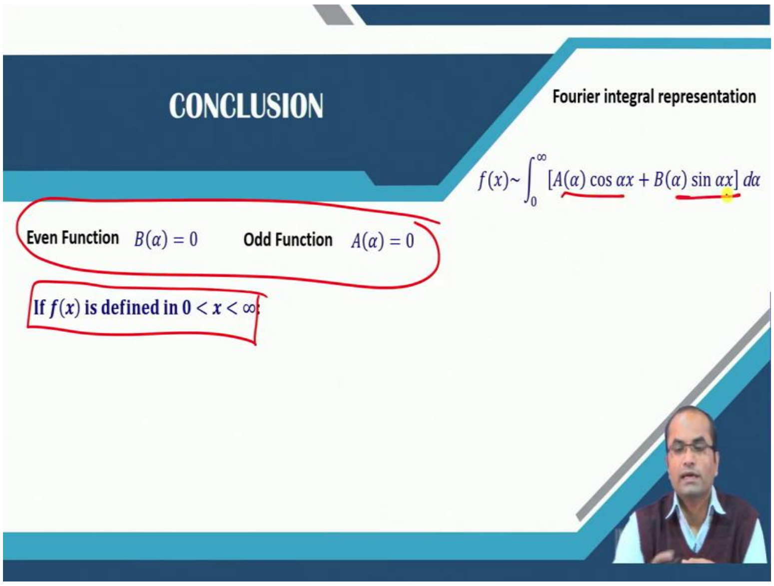


What was interesting here we have seen that if f is defined only in 0 to infinity, so earlier case this was when the function is defined in minus infinity to plus infinity and it is given that it is an odd function or we notice that it is an even function, then the calculations can be simplified to have representations which will carry only cosine term or only sin term.

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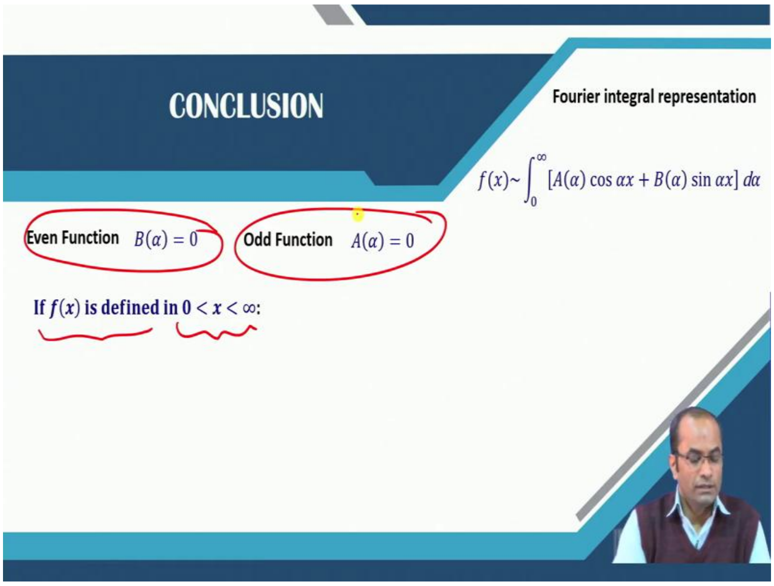


But we have discussed then or we have extended the idea that suppose the function is defined in the interval 0 to infinity, then we have both the possibility of extending this function to make it as an even function or we can extend it to make it as an odd function in the whole real axis. So, in either case, the idea which is discussed here will be applicable, will be used and we have a simplified formulation and the desired formulation.

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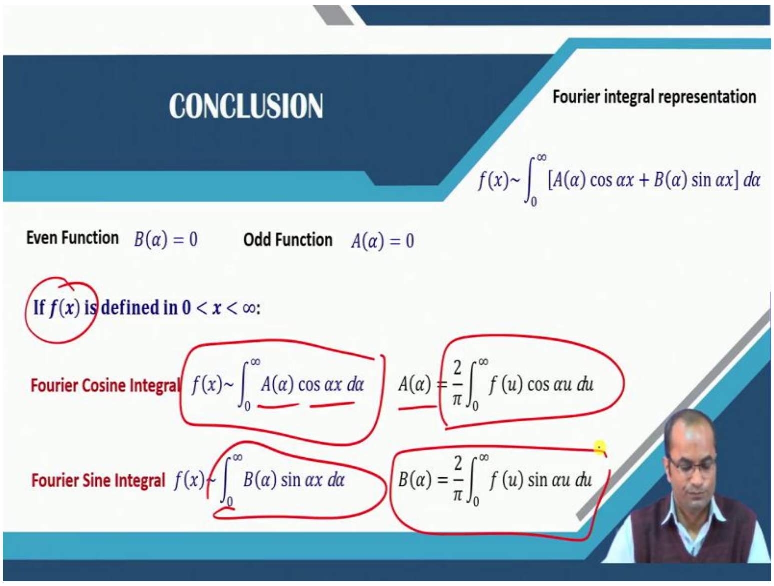


So, if we want to write this fx in terms of only the sin integral, we will extend it as an odd extension and if we want to have the cosine integral we will think as an even extension. So, for the Fourier cosine integral, the formula finally becomes a alpha cos alpha x where a alpha can be computed with this rather simple integral and further if you want to have Fourier sin integral representation, we can do with this help of this a alpha and this b alpha sin alpha and the b alpha can be computed with the simple integral 2 over pi 0 to infinity and we have fu sin alpha u du. So, that is all for this lecture and I thank you for your attention.