Engineering Mathematics 2
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Lecture 43
Fourier Cosine and Sine Transforms

Welcome back to lectures on Engineering Mathematics 2. So this is lecture number 43 on Fourier Cosine and Sine Transform.

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So today we will discuss what are these transforms? So one will be the sine transform and the other one would be the Fourier cosine transform and then we will discuss the properties of these transforms.

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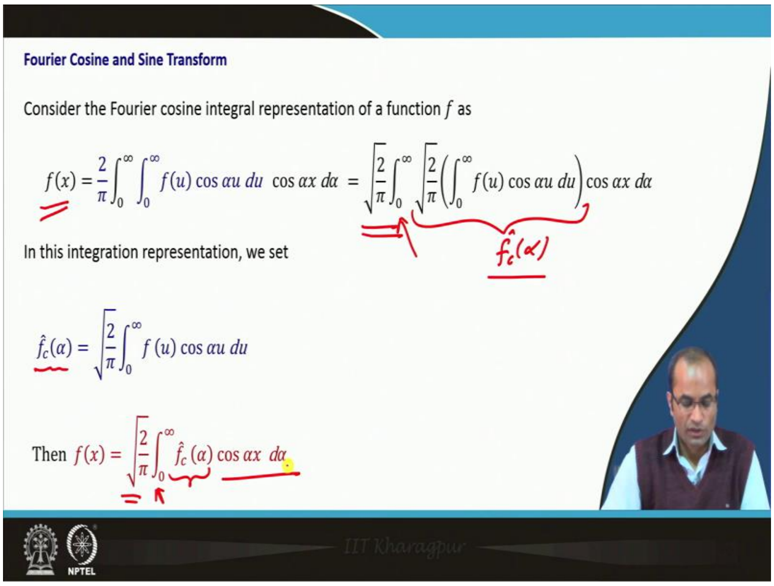

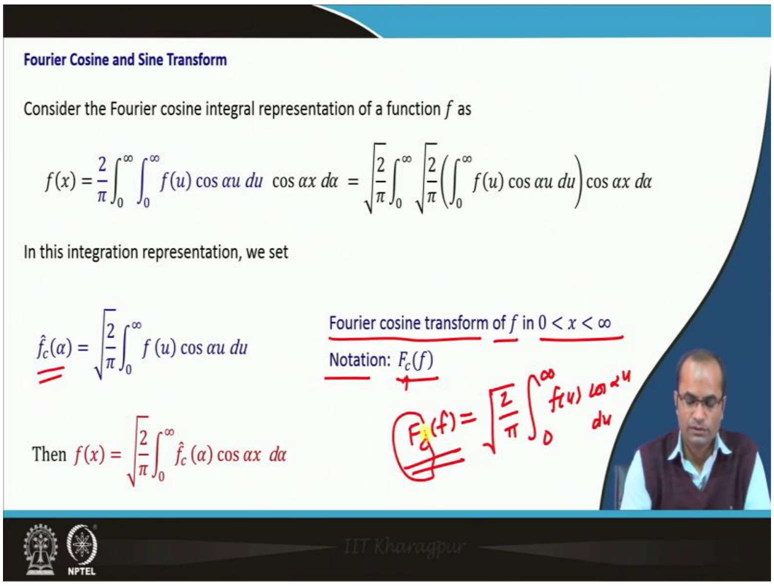

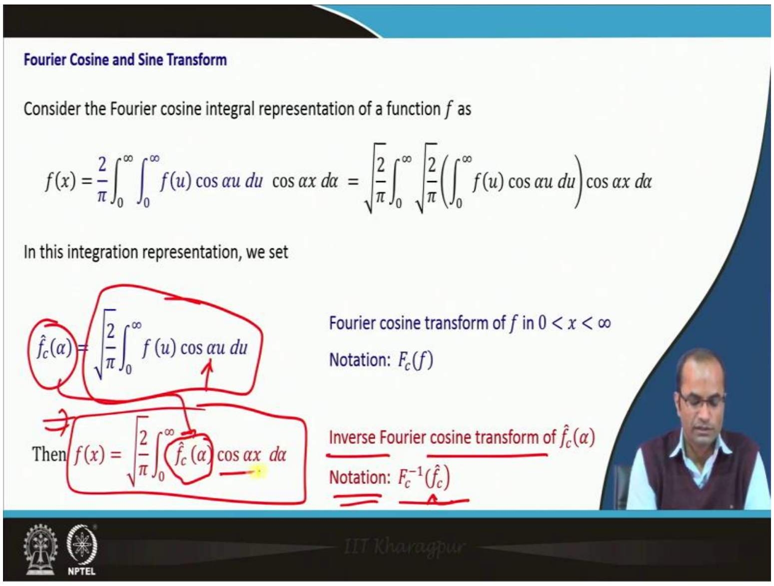

Well, so consider the Fourier cosine integral, which we have already discussed in the previous lecture and that was given as fx is equal to this 0 to infinity, 0 to infinity, fu cos alpha u du and cos alpha x dx. So this was the coefficients which we were denoting by a alpha in previous lecture and then fx was integral 0 to infinity a alpha, cos alpha x dx. So, and now we are, we have just substituted that a alpha in this representation, the Fourier cosine representation.

So finally we have this integral which we have assumed all convergence so that this is equal to fx, so that was already discussed in the previous lecture. Now, what we are doing, we are just breaking this 2 over pi into the product of square root 2 over pi and again square root 2 over pi and then this part, which is with this square root 2 over pi and this portion of this a alpha, we will denote that fc hat alpha as this is square root 2 over pi fu cos alpha u du.

So this portion we have given a new name, this f hat c for the cosine, so c stands for cosine and since this is a function of alpha, so we are calling it as f hat c alpha and then once we have denoted this f c hat alpha then fx would be square root 2 over pi and then integral here 0 to infinity and then fc hat alpha and then we have cos alpha x d alpha.

So this Fourier cosine integral we have with this new notation now fc hat alpha, it is defined as fx is equal to this square root 2 over pi and this integral for this, via this function fc hat alpha a serving as its integral. And now we can introduce, so this integral here because its transforming the function f to a new function, which is a function of alpha, so no more the u is, there u has been already integrated and we have a new function which is a function of alpha so this is called Fourier cosine transform of f.

And since we are talking about the 0 to infinity, so Fourier cosine transform in 0 to infinity and the notation we use usually Fourier and c for cosine of f, so whenever we are writing that Fourier cosine transform of f as per this definition, this will be 2 over pi and then integral 0 to infinity fu and then cos alpha u and du, so that is what we use this kind of operator or the notation we can consider for getting the Fourier cosine transform.

Similarly, this term because having this Fourier cosine transform, so this function of alpha and then we want to get back to the original function, so this formula maybe used and therefore this is called the inverse Fourier cosine transform of this fc hat alpha and notation we use here that is fc inverse of fc hat.

So this hat is used just to denote that this integral, now since it is a function of alpha so we are just naming it fc hat alpha. And then if this such a function of alpha is used here in this second integral and it is integrated over alpha after this multiplication of cos alpha x then we will get this fx the function and therefore this is called inverse Fourier cosine transform.

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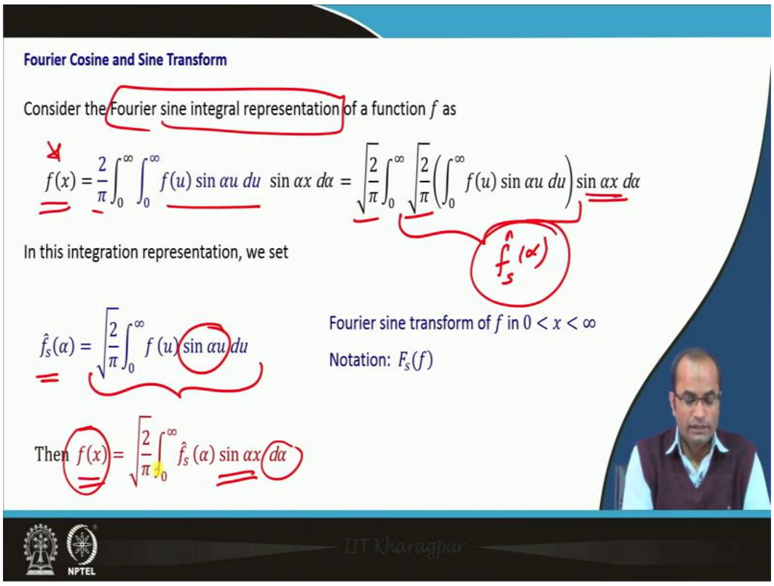

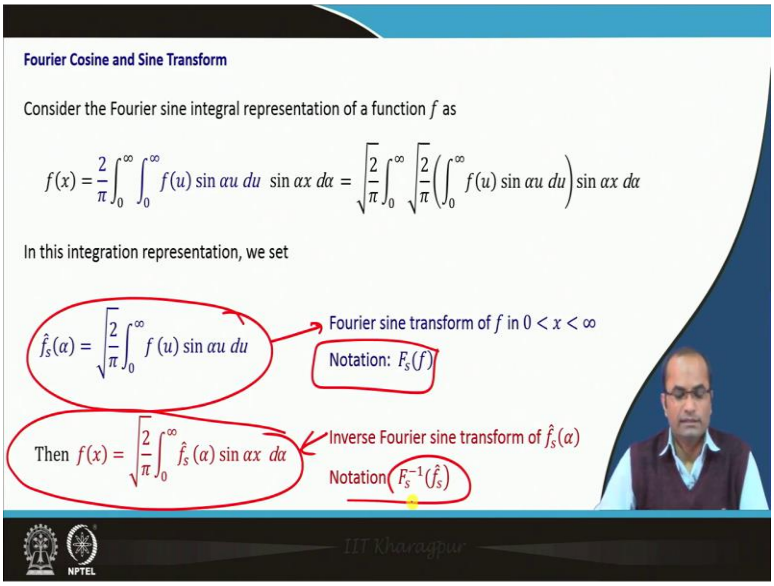

So similarly we can also define the Fourier sine transform and that exactly will be coming from the Fourier sine integral representation, so this is for instance define the Fourier sine integral representation. Again we have a similar split and then this portion, we will call at f the sine hat alpha and then we can define the Fourier sine transform as well as inverse Fourier sine transform.

So Fourier sine transform is given by this integral and now we have the sine alpha u instead of cos alpha u and fx now, getting back to fx we will use again this Fourier sine integral representation, so having this fs hat alpha if we integrate after multiplication with the sine alpha over this d alpha we will get back to the function fx and therefore this is called Fourier inverse transform.

So the first one we have here and this is called Fourier sine transform, the notation for sine transform is fs f and then for the inverse Fourier sine transform, which is given by this integral and here the notation we will be using that is fs inverse f hat s.

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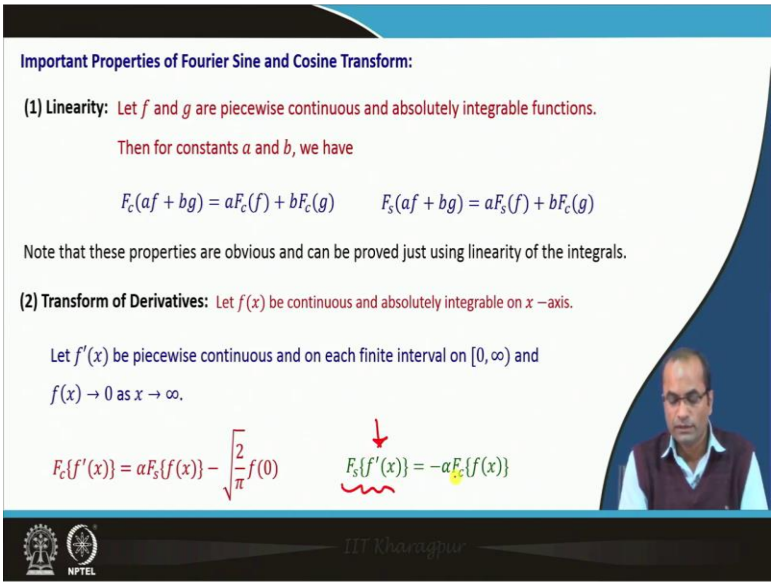

So we will now discuss some important properties of Fourier sine and cosine transform. So the first one is the linearity property, which is a very common property of all integral transforms including the Fourier transform, later on we will see and then the Laplace transforms, et cetera. So let this f and g piecewise continuous absolutely integrable, so all those conditions for the existence of such transforms.

Then for constants a and b we have this result. So if we apply the Fourier cosine transform on this linear combination af plus bg, so that will be equal to a the Fourier cosine transform of f, b into Fourier cosine transform of g and similarly for the Fourier sine transform, if we apply on af plus bg, the result will be equal to a times the Fourier sine transform plus b times the Fourier cosine transform.

And this result can easily be obtained just plugging this af plus bg in to the definition and as we know now it is a integral and integral enjoy such linearity property, therefore all these integral transforms have this linearity property. The another one which is most important from the point for view that we will be using this derivative theorem for solving partial differential equations or other kind of differential equations and these derivatives, the Fourier transform or Fourier sine or cosine transform in this case of derivatives will be used.

So again if f is continuous and absolutely integrable on this and we assume that f prime is piecewise continuous on each finite interval on this 0 to infinity and we let that fx goes to 0 as extending to infinity, so we have this additional property of f, which says that if must go to

0 as x goes to infinity, then we have this nice result that Fourier cosine transform of the derivative f prime will be equal to alpha Fourier sine transform of fx.

And then this is another factor with minus sign square root 2 over pi and then we have this f0. So the nice property of this derivative which we enjoy for solving the differential equation, so left hand side we have derivative and when we are taking the Fourier cosine transform of derivative, the right hand side is free from derivative and this is exactly the point which helps us to solve the differential equation.

Because when we take the Fourier cosine transform for instance then in the differential equation so we get rid of the derivative term and this is how we use Fourier cosine or Fourier sine transform for solving differential equation. So similarly, for Fourier sine transform when we apply on derivative of f, then we get minus alpha times the Fourier cosine transform of the function f.

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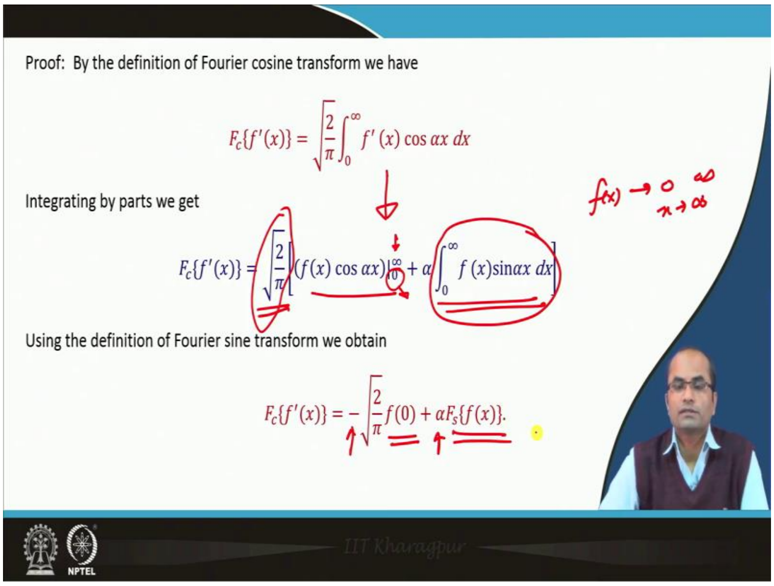

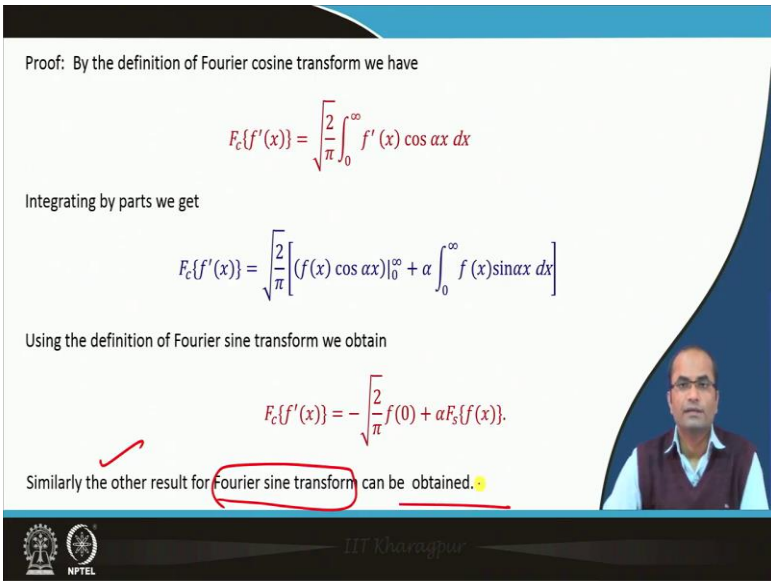

This we can easily prove this by the definition itself so the Fourier cosine transform for instance of f prime as per the definition we can write down f prime x cos alpha x dx and then if we integrate this by parts we will get the desired result. So integrating this by parts what we will have? We have here fx now and then the cos alpha x as it is, then we have the limits. then this cos alpha x will be after differentiation sine alpha x with the minus sign and there was a minus sign already in the formula so we have the plus there.

And one alpha will also come as a result of this differentiation and then we have fx which is the derivative of this integral of this f prime. So this is just integrating this by parts and then we can use here, we can substitute the upper limit in f cos alpha x and also the lower limit. So we have already discussed that or we have already assumed that this fx goes to 0 as x goes to infinity, so the upper part here when we put this x to infinity this will go to 0 and for the lower one we will get an f0.

So using this definition of Fourier sine transform in the second one what we will get, the first one says when we put this 0 there so we will get this f0 with the negative sign and then the second one alpha and as per the definition of the sine transform with this factor square root 2 over pi this will become Fourier sine transform of fx. Similarly, the other results on this Fourier sine transform can be obtained just by integrating by parts. So, we are leaving that portion here, so these ae the steps basically which can be used for Fourier sine transform.

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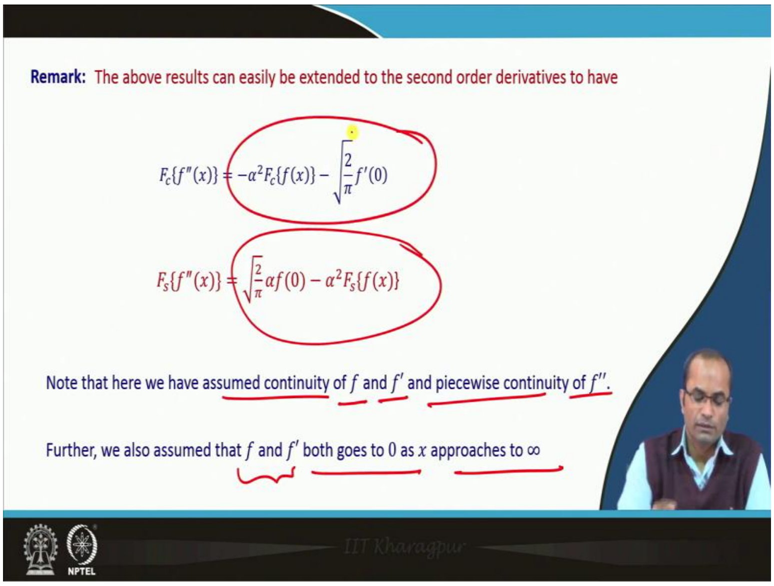

So these above result can easily be extended to the second order derivatives that means if we apply the Fourier cosine transform on the second derivative we will get minus alpha square with Fourier cosine transform of fx and then there will be additional term here square root 2 over pi f prime 0. So again this enjoys the same property that when we apply the Fourier cosine transform on this f double prime, then we get the right hand side which is free from the derivative.

So we have only the Fourier cosine transform here but there is no derivative, only the derivative here at a 1 point x is equal to 0. Similarly, for Fourier sine transform we will get this formula so square root 2 over pi alpha f0 and minus alpha square the Fourier sine transform of fx. Well, so and again the assumptions what we have used for proving this Fourier cosine or Fourier sine transform or f prime.

They are also extended here that means we assume the continuity of f and now also for f prime and piecewise continuity of f double prime. And then we have also assumed that f and f prime goes to 0 as x approaches to infinity to get these results. But such assumptions are very obvious which we will also observe during the applications to solving differential equations.

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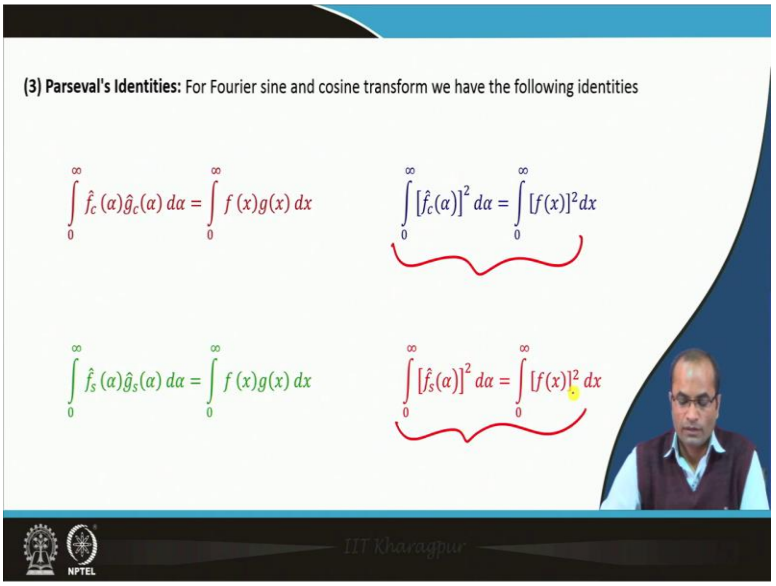

Well, so we have also the so called Parseval's Identity in this connection. So, if we have the Fourier sine and cosine transform, we have the following identities. So Fourier cosine transform and Fourier cosine transform of the function g, so we are considering 2 functions here f and g having all properties for the existence of such transforms.

So the product of the 2 transforms here when we integrate over this alpha from 0 to infinity that result will be equal to integrating fx gx over this x from 0 to infinity that is what we call here the Parseval's identity. And of we take here f equals to g in this left formula, basically we will get another identity which is fc hat alpha whole square because that will be product and the right-hand side will also be the product of same function so fx whole square.

So this is basically the same identity. For the sine transform also we have similar result so when this product of the sine transform is integrated over alpha, we will get this result equal to the product of this f and g integrated again over this x from 0 to infinity. And similar to this result we can also write down for Fourier sine transform.

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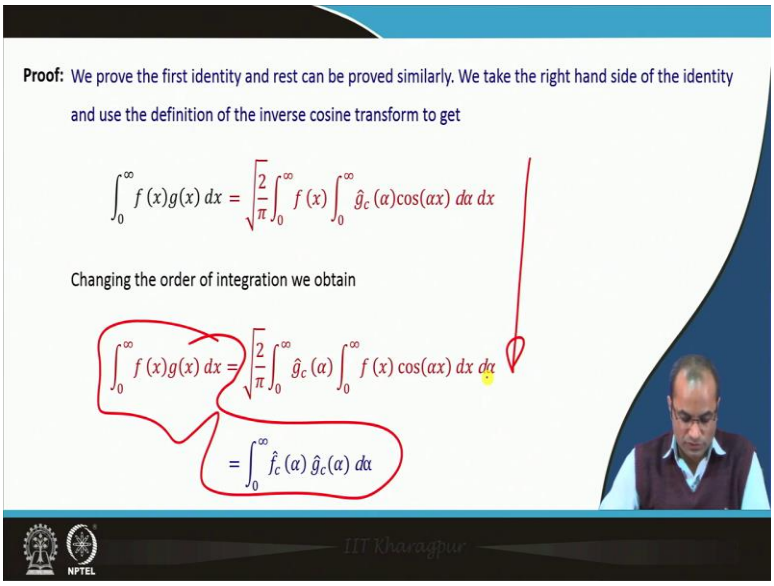

The proof of this these Parseval's identities can easily be done, so we will take right hand side of the identity fx gx integrated over 0 to infinity and the we will use the definition of inverse Fourier transform of g so which is square root 2 over pi and then we have g hat alpha cos alpha x d alpha. So this is what we are doing for the cosine transform and later on for the sine and other 2 inequalities can easily be extended.

So here we have used the inverse Fourier cosine transform of this g which is given here with the square root 2 over pi and then we can interchange this order of integration so this changing this order of integration in this above double integration what we will get, now we have here g hat alpha, so it was d alpha dx, now we have here dx d alpha, so that is the change so the inner integral has now become over the x.

So we have to have the x there fx and cos alpha x g hat c alpha was free from x, so we have written out of this integral and now if we take a closer look, there is a fx cos alpha x dx, this is exactly the Fourier cosine transform of this f, which we can denote by the fc hat and then gc hat is already there and then we have this d alpha. So this is how we have proved this identity which is cosine transform whether similar steps we can use for proving the identity or the Parseval's identity for the sine transform.

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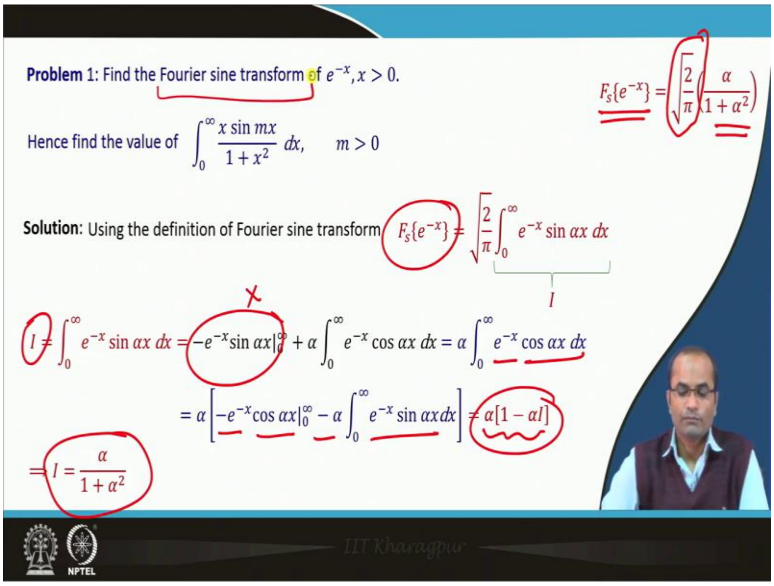

Well, now we will do some example problems. So the first problem here is to find the Fourier sine transform of this function e power minus x where this x is greater than 0 and then later on, once we have this Fourier sine transform we can find the value of this integral 0 to infinity x sine mx over 1 plus x square dx. So let us first go with the Fourier sine transform, so using the definition of the Fourier sine transform, which we know that e power minus x.

And then sine alpha x dx as per the definition and lets denote this integral here by I so that we can solve first for this I and later on we can multiply by this factor 2 over square root square root 2 over pi to get the Fourier sine transform of e power minus x. So taking this integral as 0 to infinity power minus x sine alpha x, we can integrate this by parts, so this e power minus x is being integrated here with minus sine e power minus x.

And then sine alpha x as it is and then we are putting this limit it 0 to infinity plus again the integration of this e power minus x, so that minus minus will make it plus here, then we have this alpha which is coming because of the sine alpha x was differentiated here, so we have the cos alpha x with this alpha term. So now considering the first part, when x goes to infinity, this e power minus x will make this term 0 and when x goes to 0 then the sine 0 will be 0.

So, the first term will disappear and then we have from the second term alpha e power minus x cos alpha x dx. So again we can integrate this by parts, so we have this minus e power minus x and then cos alpha x and again the same argument to get this result. So finally we get here alpha 1 minus alpha I and this was the result of the alpha itself, I itself, so we can bring the I to the other side and finally we conclude that this I is alpha over 1 plus alpha square.

Having this integral done then we can write down the Fourier sine transform of power minus x just by multiplying this factor 2 over square root pi and then we have this alpha over 1 plus alpha square. So that was the first part where we need to find the Fourier sine transform of this e power minus x function. The second is we want to evaluate this integral so we proceed further.

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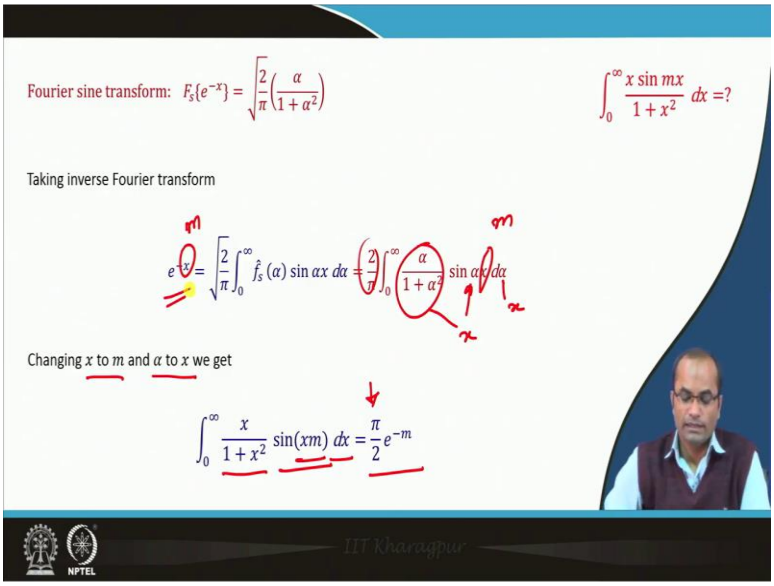

So we have the Fourier sine transform which we have just evaluated and now we take the inverse Fourier transform so this is always the trick for the evaluation of the integrals and the inverse transforms says that e power minus x because this was the function which was used to get this Fourier sine transform and now we are applying this inverse formula, so here Fourier sine transform of alpha which is already given there, and then sine alpha x and then we have a d alpha term.

So this 2 over pi and then 0 to infinity we have this Four Square, f hat alpha which is alpha over 1 plus alpha square with this factor square root 2 over pi but there was already 2 square root so that has become now 2 over pi and for fs it help now this term now this alpha plus 1 plus alpha square and then sine alpha x and we have this d alpha term. So now we want to evaluate this integral x sine mx over 1 plus x square dx.

And if we take a closer look here, this is matching with this integral itself, having this integral over x, 1 plus x square dx here we have alpha 1 plus alpha square d alpha and there is a sine term also where x must be there so here we have the x. So somehow for this x we are using here this alpha and for this x we are using m there. So if we interchange in this integral now, this alpha if we replace by x, but then this x must be replaced by some other number m.

So we can replace this x here and this x of course by m and then this alpha which is appearing here and here, then we can replace by this x, so replacing this x to this m and then alpha to x what we will get the right hand side we will have x plus 1 over x square, we have sine and then alpha will become x and x will become m, so mx.

And then we have dx because this alpha has become now x and the right hand side, this 2 over pi can go to the other side to make it pi by 2 and we have e power minus m there. So the value of this integral x over 1 plus x square sine mx dx is equal to pi by 2 e power minus m.

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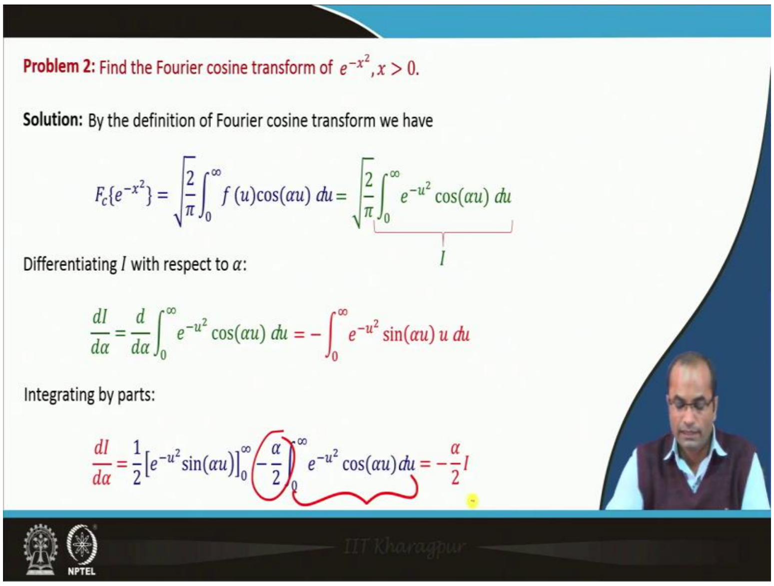

So find the Fourier transform this is the another problem, find the Fourier cosine transform of e power minus x square. Now, the function is e power minus x square. So by the definition of the Fourier cosine transform w know that it is a fu cos alpha you, so this fu is now e power minus u square and this cos alpha u and then we have du. So here there is a difficulty now to compute this integral by parts.

Because e power minus u square is sitting which is not easy to differentiate or to integrate this and having the similar steps which we have followed in the earlier examples where e power minus u was there. Now we have e power minus u square, so there is another trick which we can use now, so denoting this integral as pi, we can differentiate this I with respect to alpha and then see.

So when we differentiate with respect to alpha this integral what will happen you will get because alpha is here in cos alpha us so that will become minus sine alpha u and then one u will appear there so as du and now in this new integral we can easily do because this u will be clubbed with e power minus u square so that we can integrate this e power minus u square easily and then integrating by parts what we have dI over d alpha is equal to we have minus half there because of this e power minus u square in to u when we integrate.

So basically we need 2 u there and then 1 by 2 we can have outside, so that will be e power minus u square the integral of e power minus u square in to 2 u and then this sine alpha u as it is and in the second case here the sine will become cos alpha u and then we have this alpha from here so that is already there now and e power minus u square.

So that is the result of this integration by parts. And now looking at the first term when this u is 0 so this sine will become 0 and when u is infinity, the e power minus u square will make it 0. So what we have then minus alpha by 2 and then this is exactly the integral which we have started with that is I itself.

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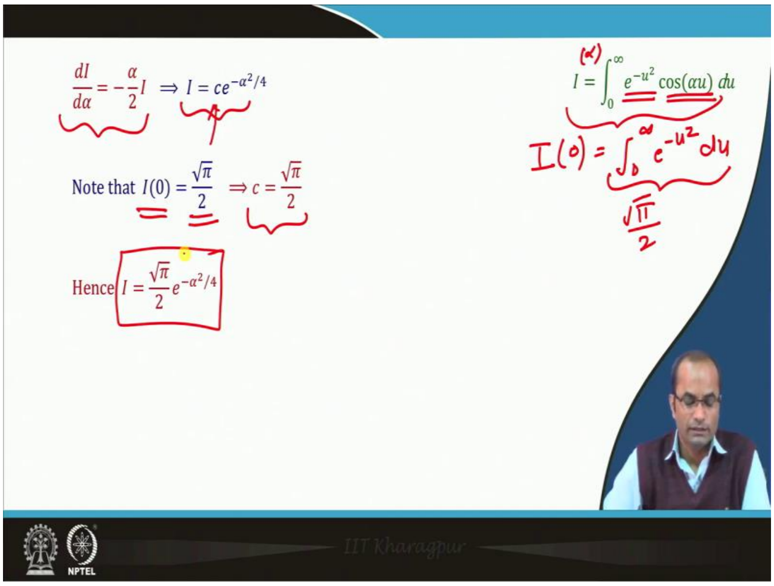

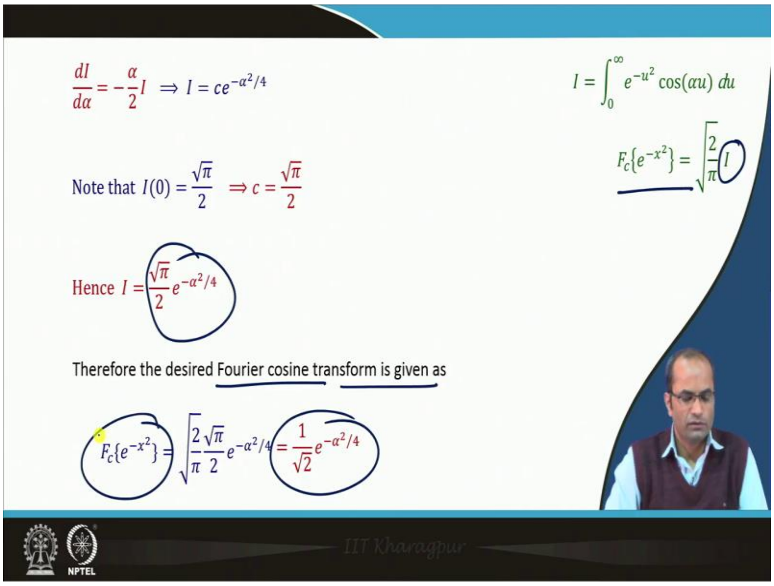

So, we have a differential equation here dI over d alpha is equal to minus alpha by 2 I and which can be solved now by I is equal to ce power minus alpha square by 4. Now the question is that what is this arbitrary constant of integration? But if we recall that I was e power minus u square cos alpha u du and I at 0, so this I is a function of alpha, so if we take I at 0, then 0 to infinity we have e power minus u square cos 0 is 1.

And then we have du and then this integral we are familiar with this, the value of this is square root pi by two. So we can use this I naught as square root pi by 2 which can give us the c gained square root pi by 2 and putting this in the integral back we have I square root pi by 2 e power minus, so therefore the desired Fourier transform is given by this equation here. So we need to just substitute I there, so the I is given by square root pi by 2 e power minus alpha square so just by substituting, we got the Fourier coefficient of e power minus x square.

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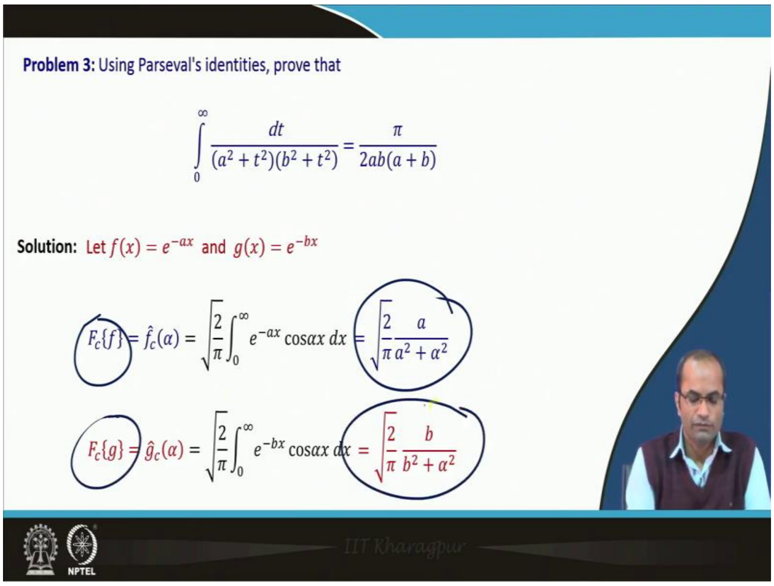

So we have another example and this is the last example of this lecture. Using the Parseval's Identity, we will prove that this integral 0 to infinity dt over a square plus t square in to b square plus t square is equal to pi over 2 ab and a plus b. So here we have to consider these 2 functions fx e power minus x and then gx e power minus bx.

A similar example we have done for computing Fourier cosine transform of e power minus x. And this will follow the similar steps, so the cosine transform of f will be given by this formula e power minus ex and then cos alpha x dx and then integrating by parts, we can end up with e square root 2 over pi a over a square and plus alpha square.

Similarly for gx because the only difference is instead of a, we have b, so the Fourier cosine transform of g can be similarly obtained as a square root 2 over pi and b over b square plus alpha square. So having these 2 Fourier cosine transforms for f and g we can now apply the Parseval's identity.

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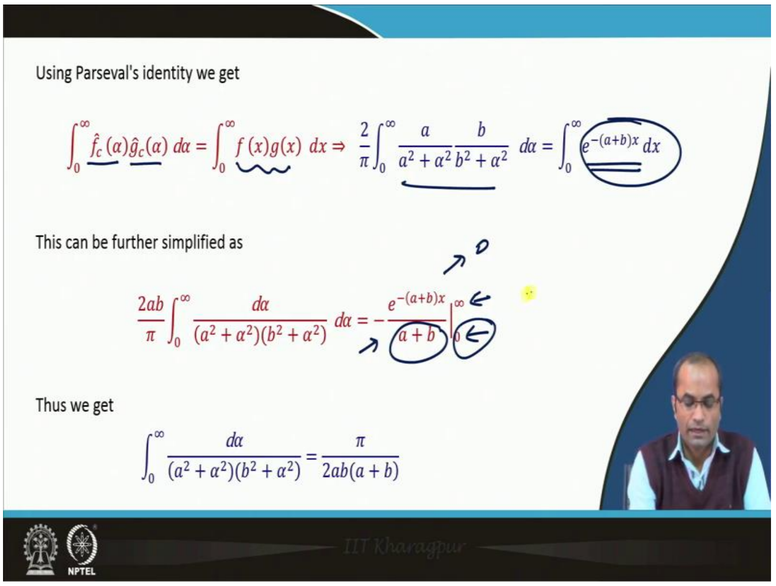

So just to recall so the Parseval’s identity was the multiplication of the Fourier cosine transform when integrated over 0 to infinity, the value is equal to the integral when the 2 functions are multiplied and integrated over 0 to infinity. So we have this information now the Fourier cosine transform of alpha and Fourier cosine transform of this f and g, so we can substitute there and the right hand side because the function was the exponential minus ax and exponential minus bx, so that can be clubbed now and it can easily be integrated.

So now e power minus a plus bx that the integration will give e power minus a plus bx over a plus b and then the limits we have 0 and infinity. So by substituting this e power infinity, this e power a plus bx will go to 0 and then we have for 0, 1 over a plus 1 over a plus b when we substitute this value 0 and then there is a factor 2 ab over pi sitting to the left hand side, which will contribute here 2 pi over 2 ab and then we have 1 over a plus b coming from this limit there. So in this way we have easily computed this integral using this Parseval’s identity.

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And here these are the references used for preparing this lecture.

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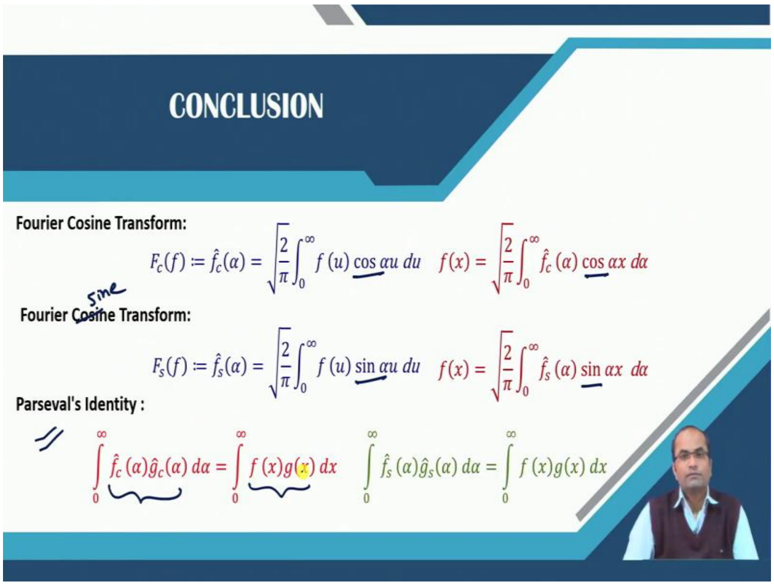

And just to conclude that we have discussed todays the Fourier cosine transform of a function f, which is given by this formula having this fu and cos alpha u in its integral and its inverse Fourier cosine transform can be obtained, so if we substitute this fc alpha there in the formula and multiplied by cos alpha x after this integration, we can get back to fx again and this is called inverse Fourier cosine transform.

Similarly, we have Fourier sine transform and Fourier sine transform instead of this cos we will have sine there and here also this cos will be taken as sine alpha x d alpha. We have also discussed the Parseval's identity and its application was also demonstrated for evaluating integral, so here this product of this Fourier cosine transform when integrated over 0 to alpha, the value is equal to the product of f and g integrated over 0 to infinity and the same result we have for Fourier sine transform.

So that is all for this lecture and I thank you for your attention.