Engineering Mathematics 2
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Lecture 45
Properties of Fourier Transform

So welcome back to lectures on Engineering Mathematics 2. So this is lecture number 45 on properties of Fourier transform.

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So today we will go through various properties of the Fourier transform mainly the change of the scale property, then we have shifting properties and then dual properties, derivative theorems, convolution etc. and also we will look in to the Parseval's identity for Fourier transform.

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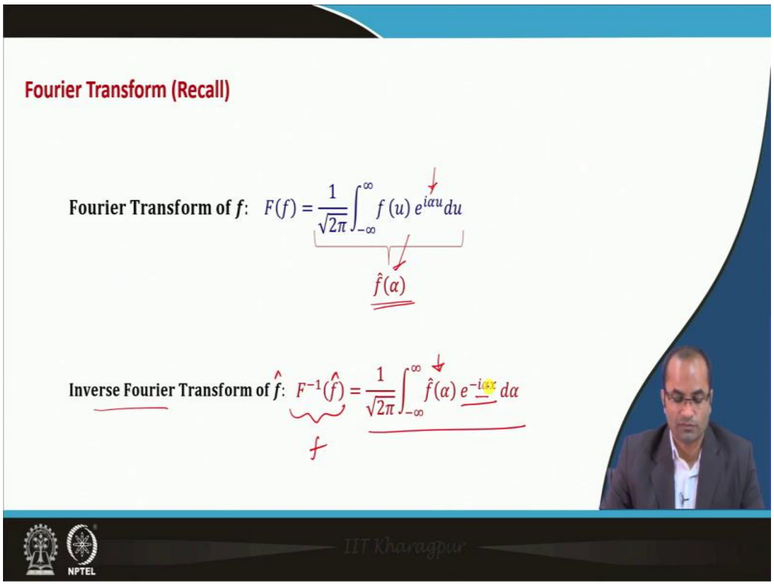

So just to recall in the last lecture we have introduced already the Fourier transform and then the Fourier transform of a function f was given by this integral, the factor 1 over square root 2 pi and then we have the function f multiplied by exponential i alpha u du and for the sake of simplicity we have denoted this by a function alpha because this is the transformed function, now so it is a function of alpha because alpha is a parameter here.

And then if you want to find the inverse Fourier transform of this, for instance this f hat alpha, then we can go back to the fx, so this is f inverse f we will have basically this f and then this is nothing but the function this f hat multiplied by e power minus i alpha, so here it was i alpha and then we have minus i alpha in the inverse transform of this f hat.

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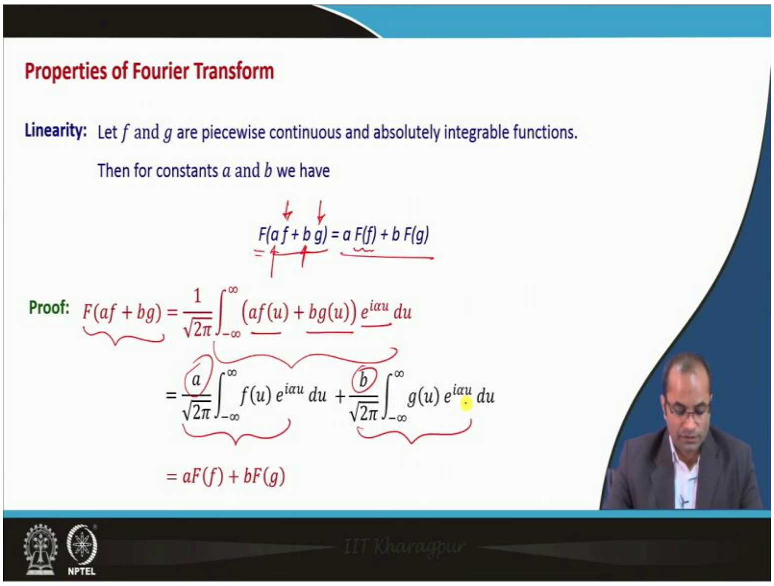

So coming to the properties of the Fourier transform, first property is the linearity which we have already discussed in, while discussing the Fourier cosine and sine transform and this is a common property of a all integral transforms so because of the integral this property that the Fourier transform of this a f plus b g, so it is a linear combination of these two functions f and g, a and b are some constants.

So that is given by a, the Fourier transform of f plus this b the Fourier transform of g, which is again trivial because of the integral, so if you want to get the Fourier of f, af plus bg that means af plus bg e power i alpha u and since it is an integral, we can break into two parts and this constant can be taken out, so here this b and then this remaining portion is nothing but the Fourier transform of f and Fourier transform of g. So this is a common property of all integral transforms, later on we will see again that Laplace transform also enjoys this property.

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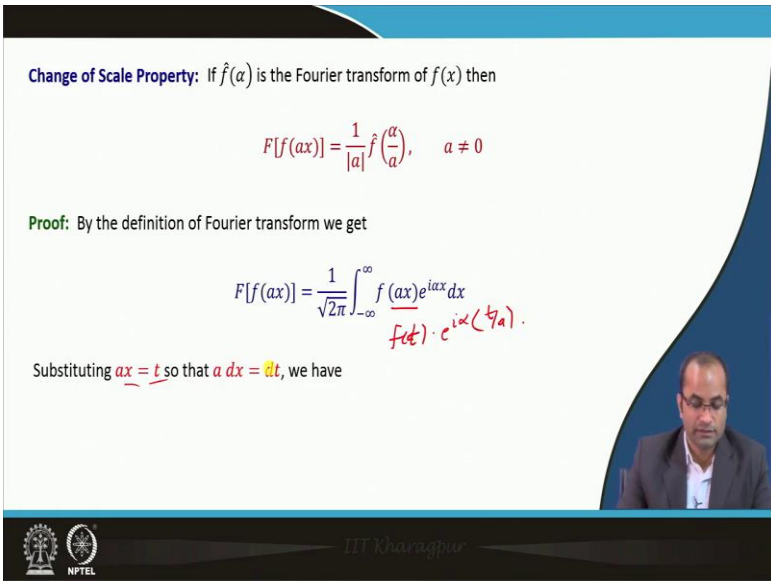

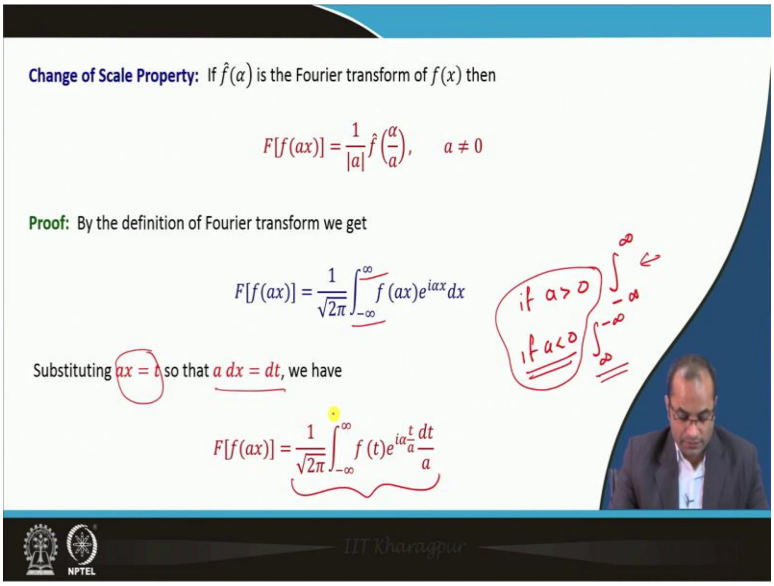

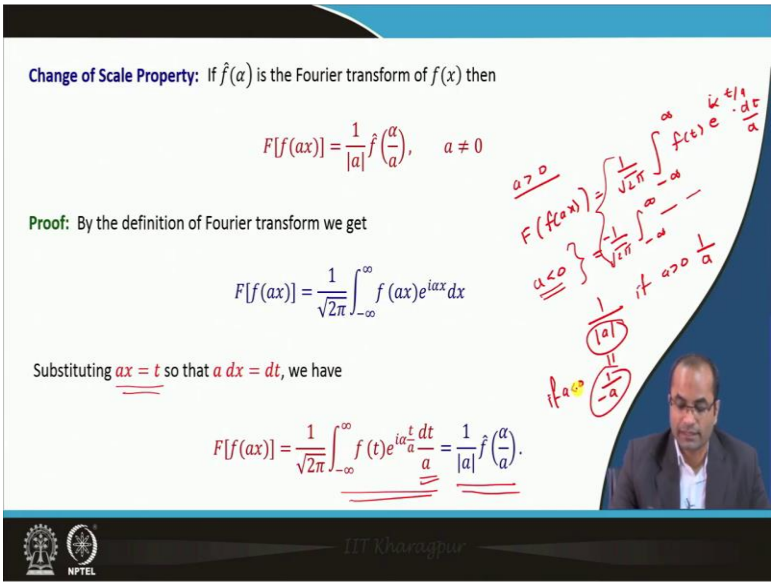

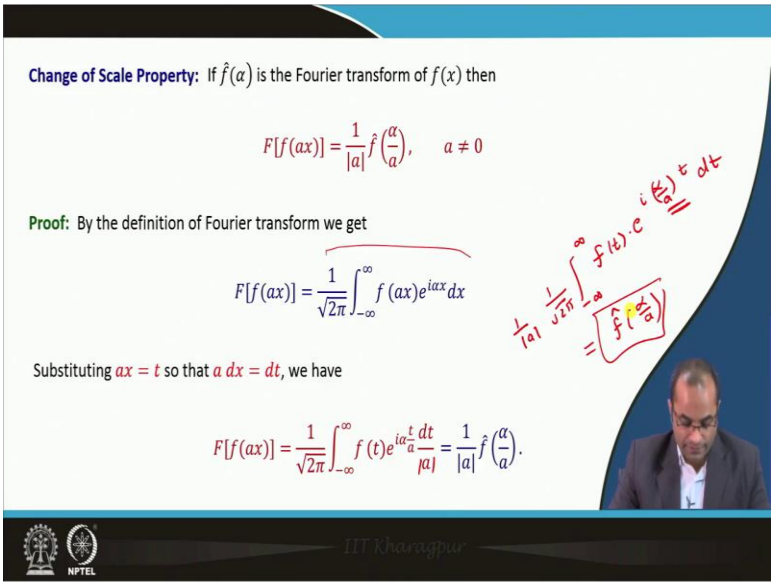

So the next one we have this change of scale property so if this f hat is the Fourier transform of this f as per our standard notation, then the Fourier transform of f ax, so that is what this is called the scaling property, we have instead of x now, we have ax, so what will be the Fourier transform of f ax in terms of the f hat which is the Fourier transform of this function f and here a is naturally a non-zero number.

So by the definition of the Fourier transform, we have the Fourier transform of this f ax given by this f ax exponential i alpha x integrated over x from minus infinity to plus infinity and then if we substitute ax is equal to t and so that we have a dx is dt and then we can have here this f at ax will be just t and then we have e power i alpha and then x will be t by a and similarly this tx will be dt by a.

So we can make this substitution in this above integral. However, the range of integration which was minus infinity to plus infinity, that will remain the same of this a is positive, so if this a is positive, the range of this integral after this substitution will remain the same, but if this a is negative because we have not fixed the sign of a in that case this will change from plus infinity to mins infinity or there will be a minus sign and then we can put minus infinity to plus infinity.

So depending on what is the sign of a if a is positive there will be no change in the limits, but if a is negative then because of this axt, the new limits of t will have minus infinity to plus infinity orientation. So then suppose this a is positive then we have this situation where dt over a is coming from this dx and then we have minus infinity to plus infinity.

But if we have a negative then what will happen this limit will change from minus infinity to plus infinity and then we have again a similar structure whatever is there, but now if we want to combine it in terms of this Fourier transforms, so what will happen that when a is negative, so we can instead of this a we can have the absolute value of a because, so suppose this a is positive then our integral, so the Fourier transform of this f ax will be given by this formula.

So minus infinity to plus infinity and then ft e power this i alpha and we have the t over a and similarly we have dt over a there, but if this a is negative then what will happen to this integral again this factor here and minus infinity to plus infinity so we will take minus sign there again minus infinity to plus infinity and then everything will be as it is.

So in the second case when this a is negative we have somehow this minus and a is appearing, so irrespective of this a, if we change this to absolute value of a, 1 divided by this absolute value of a, so that will take care of the sign, because when a is positive so if a is positive then this 1 over modulus a will be 1 over a and if this a is negative then this 1 over a will become 1 over this minus a.

So in both the cases we have incorporated in this integral instead of this a that is we have here the absolute value of a and then we have this 1 over absolute value of a common or out of these integral and the remaining part is just the ft e power i alpha and instead of the t we have now here t over a dt, so that is the only difference, so just rewriting this again, so we have 1 over square root 2 pi and 1 over, so 1 over square root 2 pi will be anyways taken care with the definition of the Fourier transform.

So we have 1 over absolute value of a and 1 over square root 2 pi minus infinity to plus infinity and we have then ft and ei, instead of alpha now we have alpha over a and then we have t dt, so as per the definition of the Fourier transform if we go, then the only change here is that this is exactly f hat, but instead of alpha we have alpha over a, so if this is only alpha then this is just the f hat alpha but now instead of alpha we have got there alpha by a, so this is nothing but the f hat alpha over a.

So this is a scaling property we have which says that f the Fourier transform of f ax then this 1 over absolute value of a will come out and then we have this Fourier transform of this alpha over a. So this f hat was the Fourier transform of fx and now the alpha will be replaced by alpha over a.

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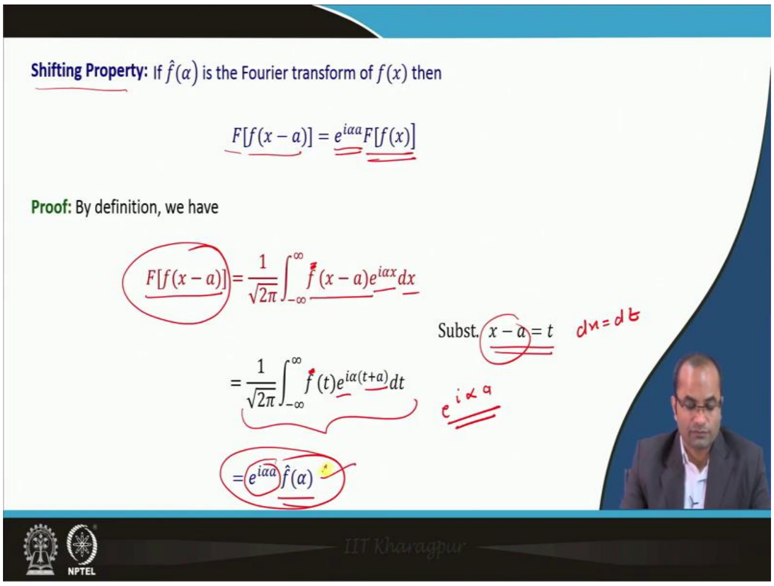

The another property we have the shifting property which says the Fourier transform of f x minus a, so there is a shift here in x, so x minus a this is equal to e power i alpha a and the Fourier transform of fx. So again if we go by the definition, we have the Fierier transform of f x minus a is equal to 1 over square root 2 pi and minus infinity to plus infinity, so f of x minus a e power i alpha x dx.

So if we substitute here x minus a to t that means dx is just simply dt and there will be no change in the limit, so we are getting 1 over square root 2 pi minus infinity to plus infinity and this f t exponential e power i alpha and x is t plus a dt and this is exactly what we have the f hat of alpha if we take this e power i alpha a out of this integral, then we have minus infinity to plus infinity ft e power i alpha t dt which is just the Fourier transform of fx.

So here again we have this result that the Fourier transform of this fx minus a is just e power i alpha a and the Fourier transform of fx.

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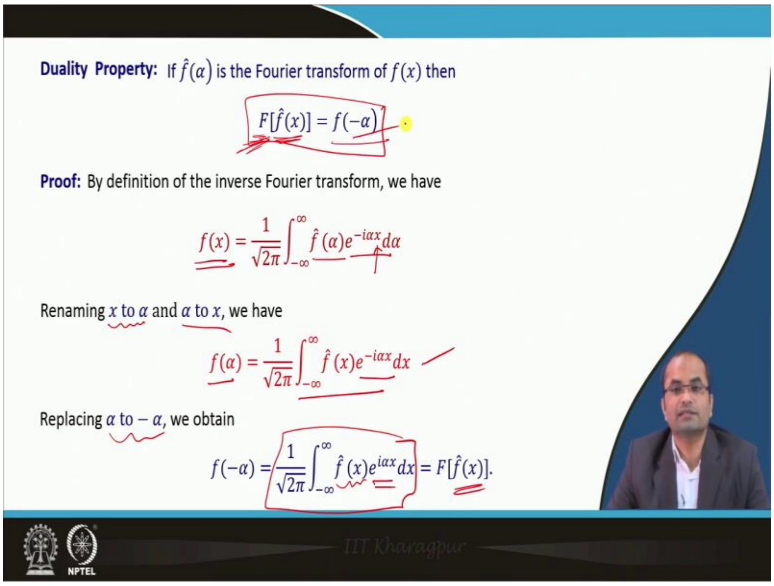

There is one more property which is called the duality property, which says that the Fourier transform, if we take the Fourier transform of f hat, so f hat is the Fourier transform of fx, so if we take again the Fourier transform of this f hat, then we will get back to f but with this minus alpha, so we can just check again with the help of this inverse Fourier transform, so we know that fx is one over square root 2 pi f hat alpha e power minus i alpha x d alpha.

And then we just need to rename that x here, we will rename 2 alpha and the alpha 2x we will just change their name so we have here f alpha is equal to 1 over square root 2 pi minus infinity to plus infinity f hat x e power minus i alpha x and then we have this dx there. And now we just replace this alpha by minus alpha, so we have now this definition of the Fourier transform because in the Fourier transform, we re taking the positive sign there i alpha x we have f hat at x, so this is the Fourier transform of this f hat x.

So the result is interesting which is called the dual property. So if we take the Fourier transform of the Fourier transform of x then we are getting back to the f the function itself evaluated at this minus alpha.

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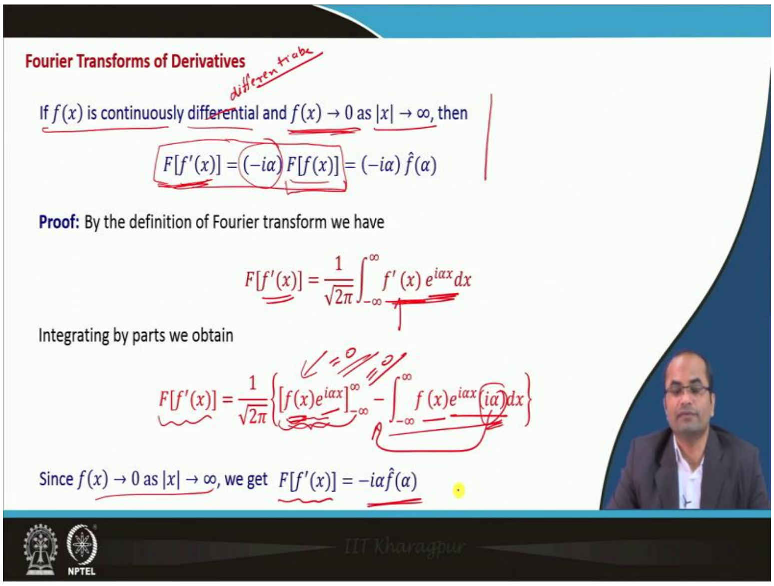

The Fourier transforms of derivative because this is going to be very useful for solving differential equations as discussed before because when we take the Fourier transform of the derivative the interesting property of these transforms are that we get the result which is free from the derivative and this is the property, which is useful for solving the differential equation.

So this derivative theorem says that if fx is continuously differential, differentiable and this fx, so this is differentiable, so if this is differentiable and this fx is equal to 0 as this mod x goes to 0 that means when x goes to plus infinity or x goes to minus infinity, this fx is going to 0, so that is the additional property we are taking here, but in most of the practical problems or later on we will also realise while solving PDEs - Partial Differential Equations that this property is quite natural.

Then we have this result that the Fourier transform of f prime is equal to minus i alpha so this factor will come out and then we have the Fourier transform of fx. So again by the definition of the Fourier transform we will apply to this f prime x which is f prime x e power i alpha x dx and then integrating by parts what we getting now here because we have two functions so we will integrate naturally this f prime that will get us to this fx.

And them e power i alpha x as it is then we have minus sign again fx and the differentiation of this e power i alpha x which is e power i alpha x into i alpha. So concerning the first part here when x goes to infinity or minus infinity we have discussed already that is our assumption that this f is going to be 0 and the e power i alpha x that is something which is finite, so we have this, the first part of this integration that becomes 0.

And then we have only the second term because this goes to 0 we have simple result that the Fourier transform of this f hat is equal to minus this i alpha go out here and then this is the Fourier transform of fx.

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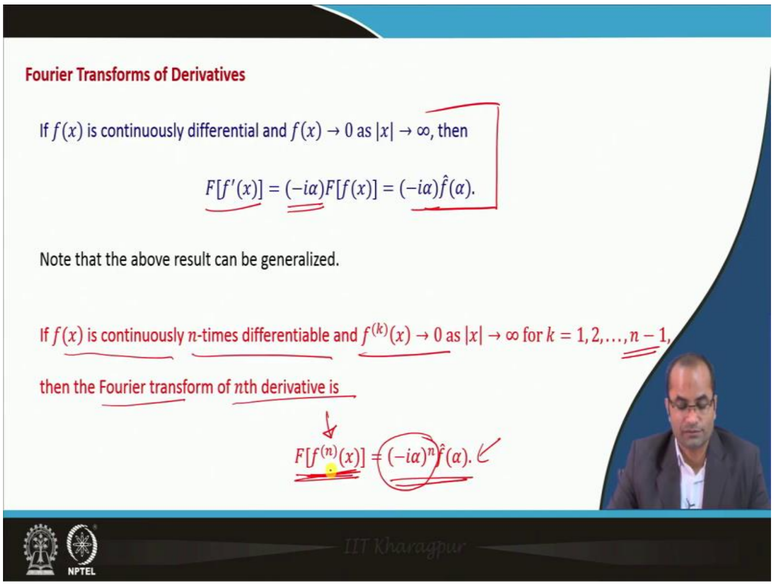

Well, so having this property of the Fourier transform which says that the Fourier transform of f prime is minus i alpha of Fourier transform of fx, we can generalise this result for the higher order derivatives that says that if fx is continuously n times differentiable and we have this property again that f kx, so the kth derivative of x goes to 0 up to we have k is equal to 1, 2, 3, n minus 1.

Then this Fourier transform of the nth derivative that means Fourier transform of this nth derivative will be simply minus i alpha every time will come out from this Fourier transform so every time minus i alpha that means i alpha powered this n and the Fourier transform of f, so that is a very simple property, which can remove or we can reduce the derivatives with the application of this Fourier transform to the derivative of the function.

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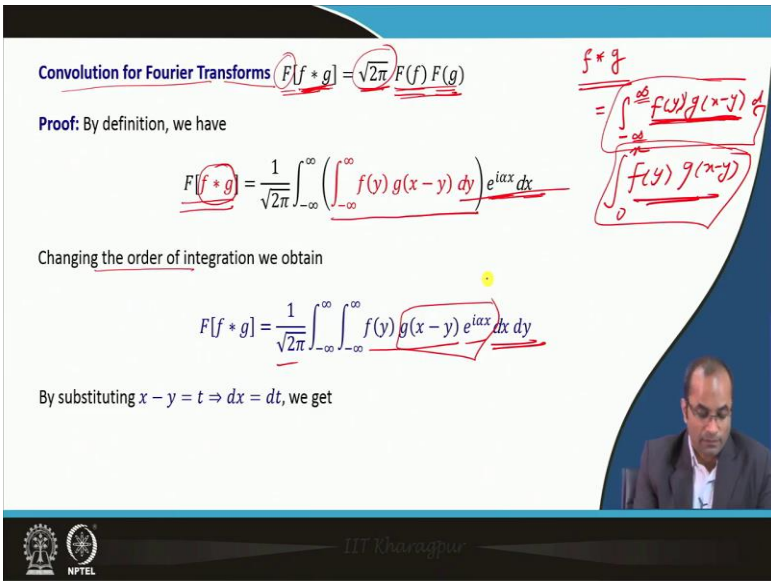

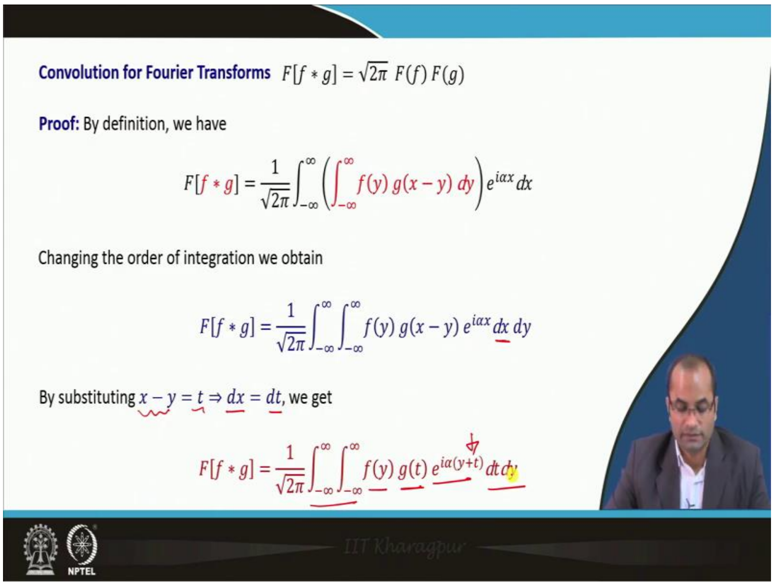

Well so another interesting property which is also very much used for getting the inverse Fourier transform and then finally to the applications of differential equations, we have the convolution of the Fourier transform, so f the convolution of f and g this is the symbol used for the convolution, so the convolution f and g that is equal to square root 2 pi and the product of the Fourier transform of f and g.

So the property says that the Fourier transform of this convolution here is the simple product of the Fourier transform multiplied by square root 2 pi. Again, by the definition, we will go so we will apply the Fourier transform to f star g and remember this f star g that is the symbol we have used for the convolution, so f star g means we have this integral here, so f star g is nothing but the integral minus infinity to plus infinity and fy and gx minus y and dy.

The important property of the convolution is this its integrant here, so one is this fy then another one is gx minus y. So the limits for instance here we are taking minus infinity to plus infinity, but sometimes this is also are known that we take the limit 0 to x for instance and while discussing the Laplace transform, we will take this definition of the Fourier transform for the Laplace transform.

But here this is more appropriate so we are going with minus infinity to plus infinity because this Fourier transform is defined for the entire real axis. So here we have minus infinity to plus infinity and this is the special structure of this convolution integral that we have fy and the other function is just shifted by this x minus y and then dy and then because of this Fourier transform we have e power i alpha x and then dx.

So here the trick is that if we change the order of integration that what will happen, so instead of this dx, dy dx there we have now dx dy 1 over square root 2 pi and then this product is as it is we have e power i alpha x and now we can segregate, so the inner integral is over x, so these functions will be there for the inner integral.

But before that we will just substitute here x minus y to t so that we have dx becomes dt the inner one so dx becomes dt and accordingly the limits because minus infinity to plus infinity so they will not change now. So we have the limits fy and then gt e power i alpha and this x is replaced with y plus t here, and then we have dt and dy.

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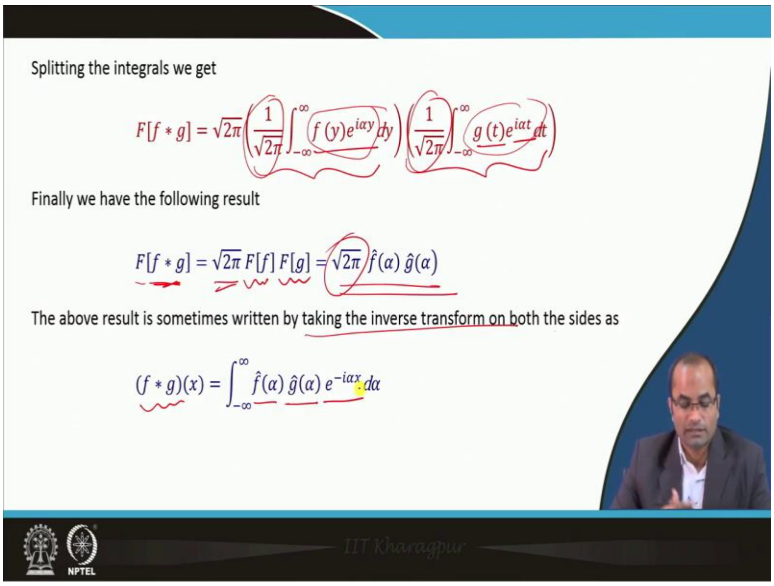

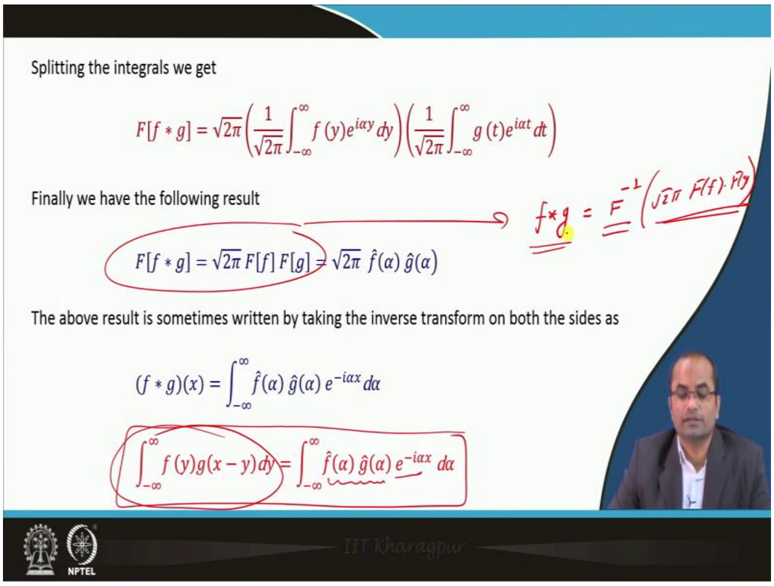

And then we can segregate into two integrals that fy e power i alpha y and then gt we have with e power i alpha t, so we have a splited into two integrals now because we have the separate functions which depends on y only, then we have a functions which depends on t only. So finally what we have that this Fourier transform of this convolution is square root 2 pi there and we have adjusted this 1 over square root 2 pi so that it becomes exactly the Fourier transform of f, the other one Fourier transform of g and we have the desired result.

And sometimes it is very useful and we will realise in may applications that if we take this inverse here on both the side then what will happen? We have this product the convolution product here f start g and then we are taking the inverse so inverse of this f star f hat and this g hat and this 1 over square root 2 pi will cancel out so we have e power minus i alpha x, the definition of the Fourier inverse transform.

And then so what result we have now which can be now summarised here that this convolution basically is equal to the product or the Fourier inverse, Fourier transform of the f star g star or from here itself we can just rewrite that the convolution f star g star is nothing but the Fourier inverse of this square root pi and the Fourier of f and product Fourier of g. So if we take the Fourier inverse of this product of the Fourier transforms, then we get back to the convolution of these two functions.

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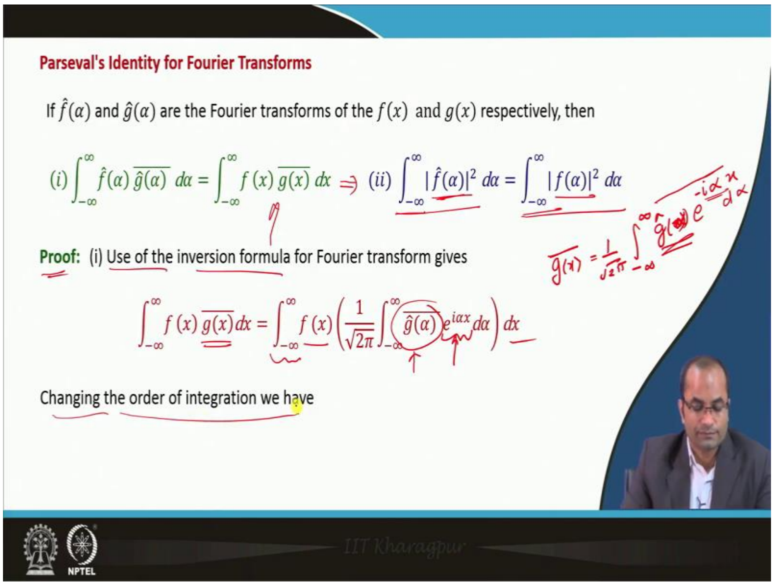

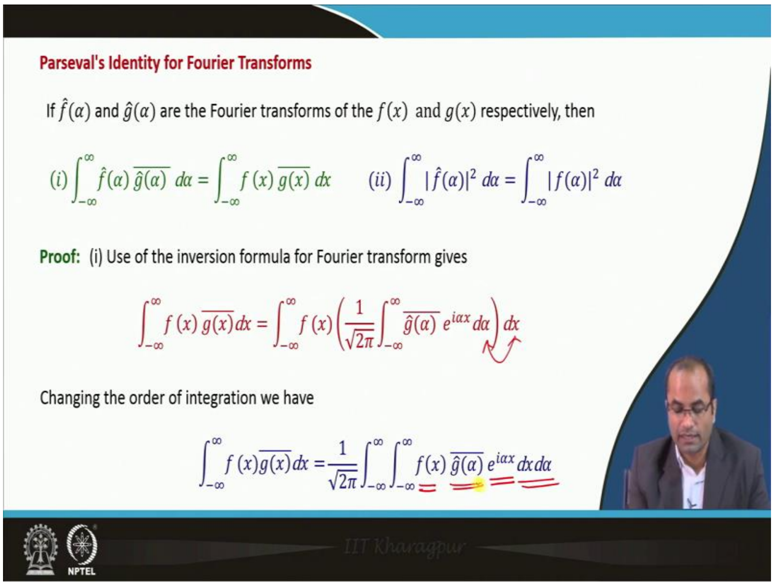

Well, now the Parseval's identity for the Fourier transform, so we will discuss and here suppose this f star n this g star are the Fourier transforms of this f and g respectively, then what we have? We have these results. So the integration over this product of this f hat and g hat and this is a complex conjugate here is equal to the integral, minus infinity to plus infinity the product of f and g and again this complex conjugate of this g so dx.

So that is the result of this Parseval's identity for the Fourier transform and as a special case we will see that we can also write this as f hat the absolute value square and here also the absolute value of this f square integral form minus infinity to plus infinity can their integral from minus infinity to plus infinity. So what we now use the inversion formula for the Fourier transform to get back to these identities.

That means if we take this fx and g the complex conjugate of this g dx, then what we getting now again minus infinity to plus infinity f as it is, for g we are writing the inversion formula so for gx the inverse Fourier transform says that this will be 1 over 2 pi and then minus infinity to plus infinity and we have this gx, we have this gx and exponential i alpha x with negative sign.

So gx that means g at, g hat alpha, so this is inverse Fourier transform g hat alpha e power minus i alpha x and d alpha, so this is the inverse Fourier transform, and then its conjugate when we take so the Conjugate will go to this integrant there. So here we have e power minus i alpha x that will become e power i alpha x and as a result we have here e power i alpha x and then g hat alpha the conjugate will go to g hat alpha and then we have this dx there.

So we change the order of integration now that's a trick again, so if we change the order of integration here instead of d alpha dx we will make dx d alpha so we dx d alpha there and e power i alpha x and we have this fx and this g hat alpha conjugate can go now out of this first inner integral.

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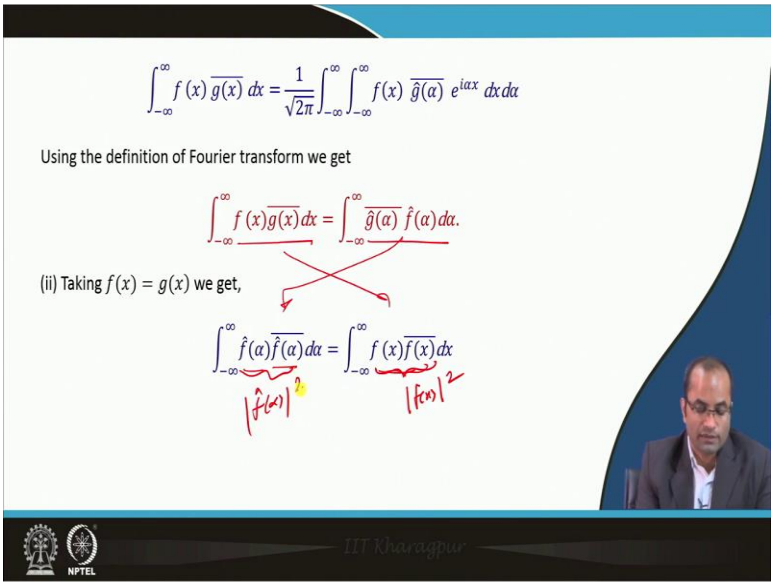

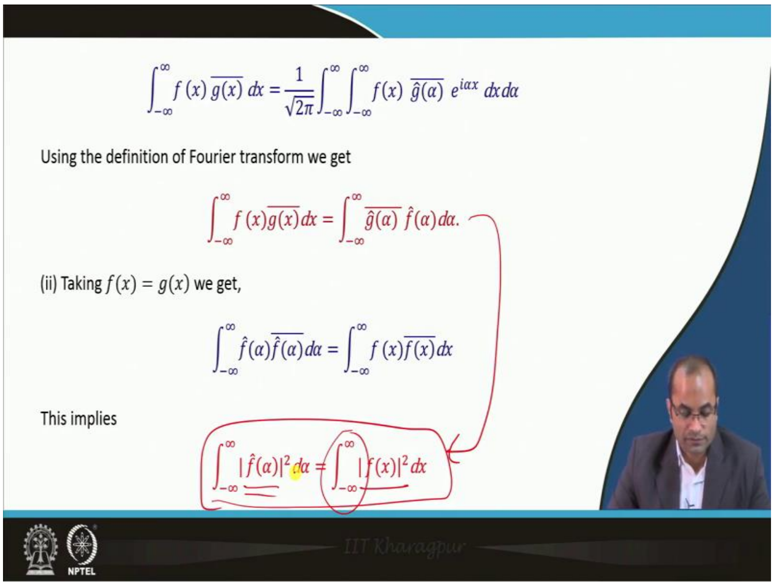

So what we have now? We can use the definition of the Fourier transform because having this outside this integral, the inner integral is over fx e power i alpha x dx with this 1 over square root 2 pi that is the definition of the Fourier transform, so what we have fx and gx, the conjugate equal to minus infinity to plus infinity and then g hat alpha that is there already with complex conjugate and this inner integral which was 1 over square root 2 pi and then we have minus infinity to plus infinity fx e i alpha x dx.

That has become now this f hat alpha the Fourier transform of the function f which is written here now and then this integral, the outer one is over alpha, which varies from minus infinity to plus infinity. For the second part, so if we take the two functions equal that means this fx equal to gx then what will happen, we have the fx and gx equal, so in the above formula we have taken now the same function.

So we have this right hand side we have f hat alpha and f hat alpha conjugate, the other side we have the integral of fx and fx conjugate dx and just recall from the complex analysis that this fx and its conjugate that is nothing but the fx absolute value square and here also we have this f hat alpha, the absolute value square.

So this is direct implication of the earlier result to get this just of the same functions, so we have the integral the Fourier transform of the f square with the absolute value is equal to the absolute value of fx square integral from minus infinity to plus infinity. So that was the Fourier Parseval’s identity for the Fourier transform.

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These are the references we have used for preparing this lecture.

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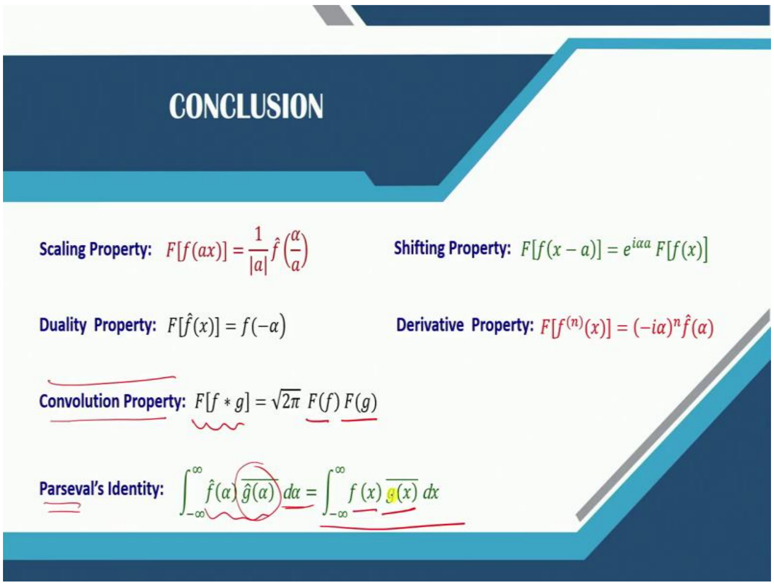

And just to conclude that we have seen the scaling property which says that the Fourier transform of f ax is equal to 1 over the absolute value of a and the Fourier transform of this x, which was the Fourier transform of this fx was denoted by the f hat alpha. So in this alpha now, the alpha over a will appear and then this factor will go out.

There was a shifting property which says the Fourier transform of fx minus a then there will be additional factor here e power i alpha a out and then we have the Fourier transform of fx.

The duality property which says here that this Fourier transform of this f hat x equal to f minus alpha.

Derivative property which is the most important now for the application point of view to solve a partial differential or the ordinary differential equation that the Fourier transform of the nth order derivative of the function is equal to the minus i alpha and this power n and the f hat alpha. And remember this property the most important result which is used to get this nice property was that f goes to 0 as x goes to plus infinity or minus infinity.

The convolution property, it is very much used for getting the inverse transform and later on we will see in many examples that the Fourier transform of the convolution of this f and g is equal to the product of the Fourier transform of f and g. And then at last, we have also seen this Parseval's identity, which says that the integral over this alpha of the product of this Fourier transforms of f and g having this complex conjugate over g, then this is equal to the integral of the product of the function which is fx and complex conjugate of this g.

So that is all for this lecture and I thank you for your attention.