Engineering Mathematics - II
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Lecture 46 - Evaluation of Fourier Transform (Part - 1)

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So welcome back to lectures on Engineering Mathematics 2. So this is lecture number 46 on Evaluation of Fourier Transform. So here we will consider several functions and evaluate their Fourier Transform.

So for example, the rectangular first function will be discussed and also the some forms of the exponential functions as well as we will consider the Fourier Transform of Dirac-Delta function and finally, some examples on inverse Fourier Transforms.

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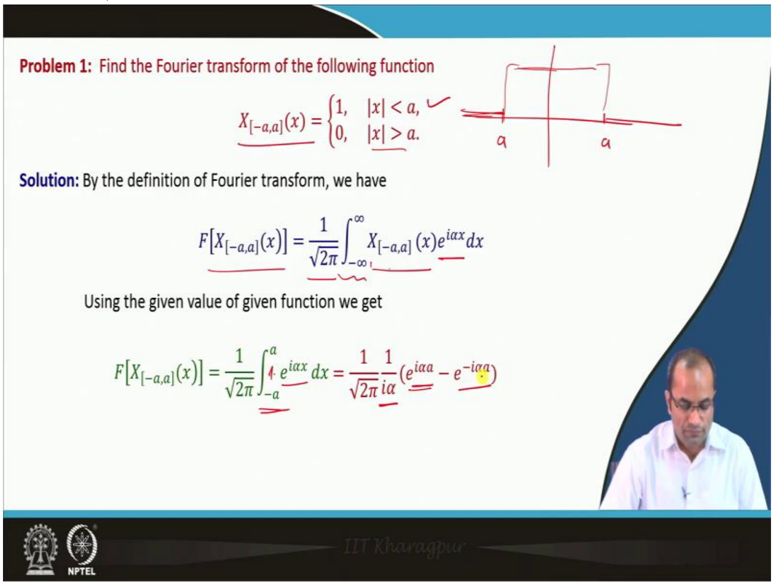

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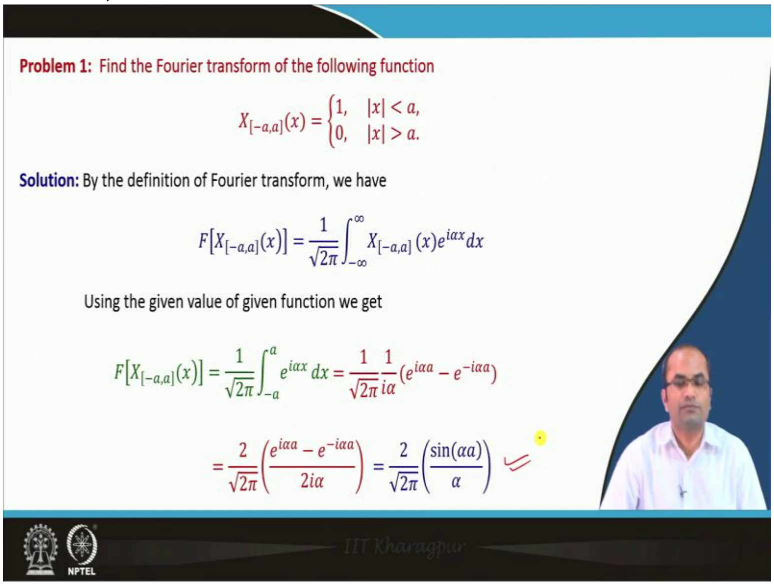

So considering this first problem where we need to find the Fourier Transform of this following function which is defined by this, so it is 1 in the region from minus a to a and then outside this, this is 0. So if we have a here some positive number then in this region it is defined as 1 and outside this it is defined as 0.

So this is the so-called rectangular function, first function and then with the definition of the Fourier transform we can apply to this function which says that 1 over square root 2 pi and this function e power i alpha x, and since this function is defined only between minus a and a so this integral will be simplified to minus a to a and the function value in the region, it is 1. So we have just exponential i alpha x which can easily be integrated to give again e power i alpha x and divide by i alpha and then we need to put the lower and the upper limits. So while substituting the upper limit there e power i alpha a and then the lower limit e power minus i alpha a, and then we can also rewrite this in terms of the sine function.

So by introducing here this 2 i then we have basically this is sin a alpha and divided by this alpha and then the outside factor we have this square root 2 over pi or 2 over square root 2 pi. So this was a very simple function because this function is defined only in this minus a to a, so this integral which is from minus infinity to infinity has become just from minus a to a and then because of the exponential function the integration was easier and we got this value as its Fourier transform.

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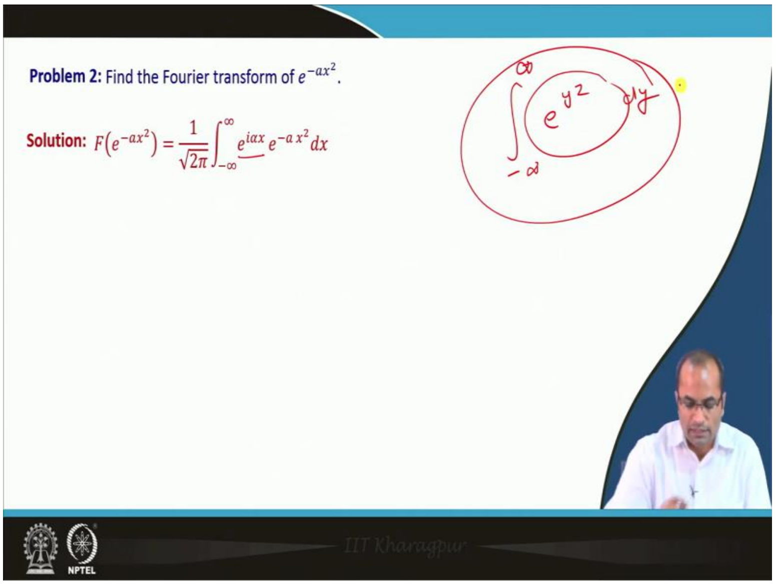

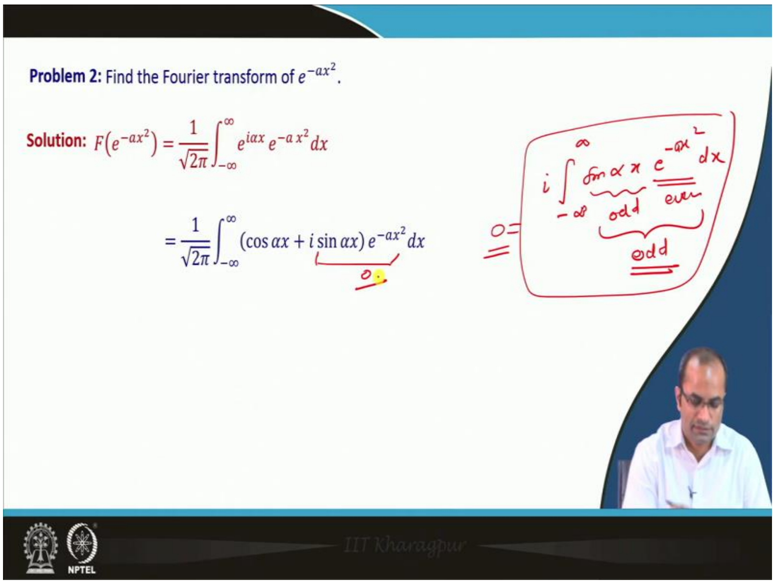

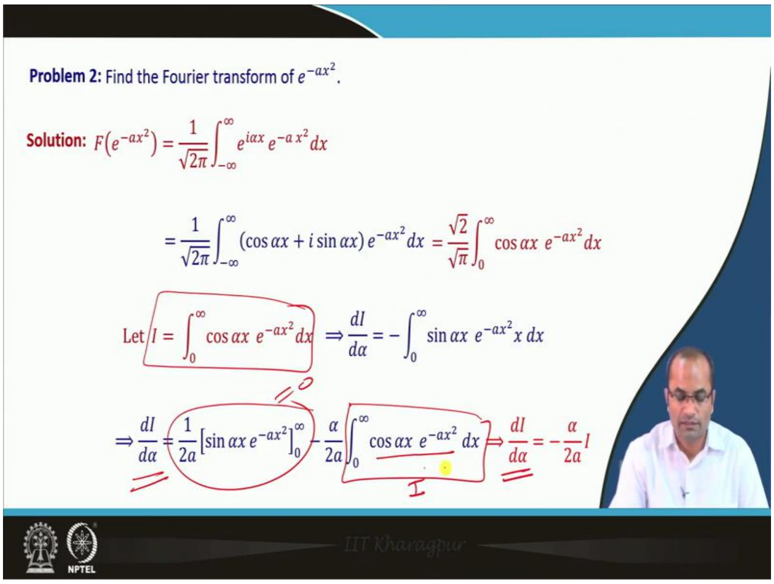

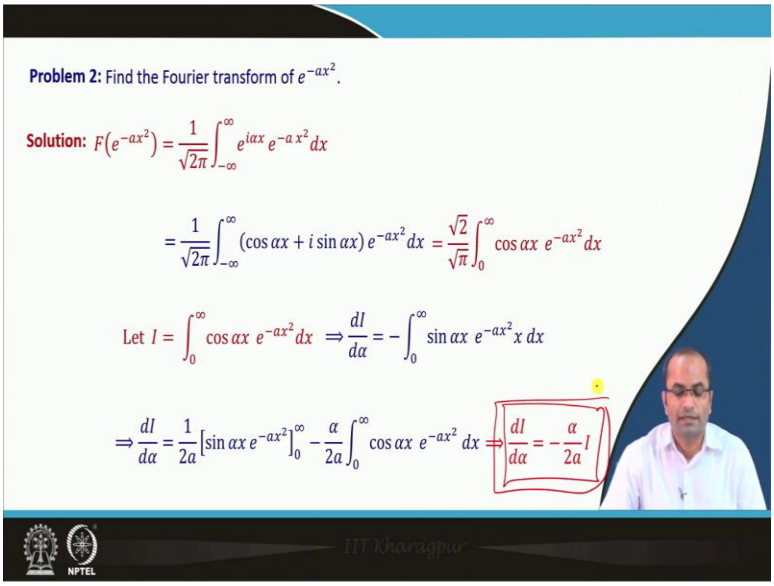

In another problem, we will consider here e power minus a x square, so a is again here something positive, some positive number and for this problem so we will again apply the definition of the Fourier transform, that means e power i alpha x and then we have e power minus a x square dx.

So in this case now things are bit complicated because of this minus a x square. So it is not a simple integration anymore but we have to think now that how to deal such a situation where we have in one exponential function e power minus a x square and then other one is already there because of the definition e power i alpha x.

So in this case, the trick is that we have to make power of e as a square function and then finally, if we are able to convert to something like e power y square and then this integral minus infinity to plus infinity, so we know the value of such integrals. So that is the objective here. If we can convert this integral into that type of integral then we can evaluate this in a closed form.

So now the trick here what we are using now, it is exponential alpha x, we can write down as cos alpha x plus i sin alpha x and then we have here e power minus a x square. So this is the other way of treating this function simply because now if we look at the second integral which is together with i and again this minus infinity to plus infinity, we have the sin alpha x and e minus a here x square and then we have dx.

So this is even function and then here the sin is an odd function, so as multiplication of the two we have this odd integrand of this integral which varies from minus infinity to plus infinity, and as a result this value will be just 0. So the second part of the integral with the sin alpha x will become 0 and then we have to deal only with the first integral.

So then we have this cos alpha x and e power minus a x square dx, where if we set this, the integral there except this constant term, we set this as I and then we know the trick that if we have this e power minus a x square and somehow we can manage to get one more x there then the integration of this e power minus a x square is possible. So we will exactly do this.

So we will differentiate this I with respect to alpha and then, so here we have this cos alpha that will become sin alpha with the negative sign and then one x will appear there, and then we can integrate this integral by parts easily, so integrating this by parts, so we have this exponential function together with x which can be handled now for the integration. So we have the sin alpha x as it is and the integral of this e power minus a x square x will be just, we have to adjust this factor 2a and then this is exactly e power minus a x square the limit 0 to infinity and similarly, here the sin will now become the cos alpha x and one alpha will also appear and there is a 2a already there.

So we have this cos alpha x and e power a x square. And then regarding this first term when x goes to infinity, so if we take a look here with the exponential function e power minus a x square and when x approaches to infinity, this will go to 0 and this is a finite quantity sitting, so there is no problem. This everything will go to 0 and when we take the limit as x goes to 0 because of this sin, again this term will become 0.

So this first term will disappear now and regarding the second term so we can take a close look at this I there, the integral I which we are evaluating and here we have exactly the integral I there with cos alpha x e power minus a x square. So we have a very simple differential equation now. This d I over d alpha is equal to minus alpha by 2a into I. So we need to now differentiate, or integrate this differential equation or solve this differential equation to get this I, the desired integral.

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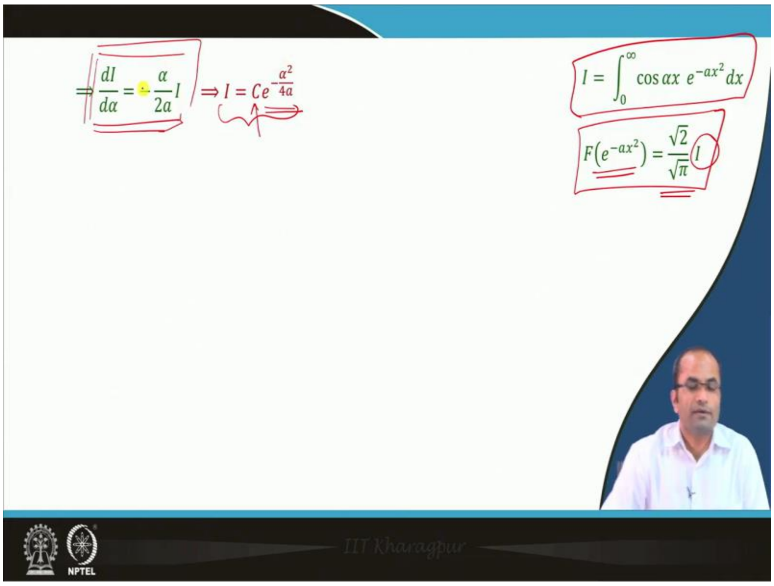

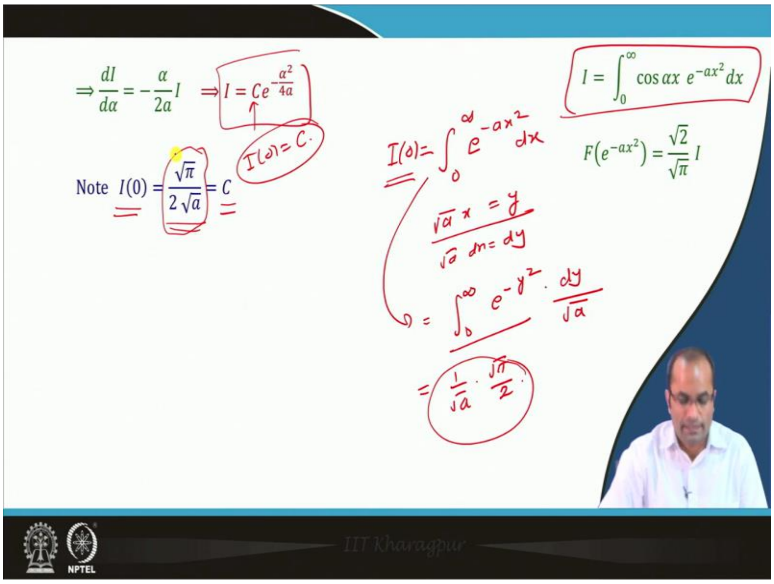

$$\Rightarrow \frac{dI}{d\alpha} = -\frac{\alpha}{2a} I \quad \Rightarrow I = Ce^{-\frac{\alpha^2}{4a}}$$

$$I = \int_0^\infty \cos \alpha x \, e^{-ax^2} dx$$

$$F\left(e^{-ax^2}\right) = \frac{\sqrt{2}}{\sqrt{\pi}} I$$

Note $I(0) = \frac{\sqrt{\pi}}{2\sqrt{a}} = C$

$$\Rightarrow F\left(e^{-ax^2}\right) = \frac{\sqrt{2}}{\sqrt{\pi}} \frac{\sqrt{\pi}}{2\sqrt{a}} e^{-\frac{\alpha^2}{4a}} \quad \Rightarrow F\left(e^{-ax^2}\right) = \frac{1}{\sqrt{2a}} e^{-\frac{\alpha^2}{4a}}$$

Remark: If $a = 1/2$ then $F\left[e^{-\frac{1}{2}x^2}\right] = e^{-\frac{\alpha^2}{2}}$.

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Remark: If $a = 1/2$ then $F\left[e^{-\frac{1}{2}x^2}\right] = e^{-\frac{\alpha^2}{2}}$. This shows $F[f(x)] = f(\alpha)$

Such a function is said to be self-reciprocal under the Fourier transformation.

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So what is the situation we have? This I which we have defined there and the Fourier Transform of e power minus a x we have written as square root 2 over pi and this integral I. So now for solving this integral I we turned up to this differential equation which is d I over d alpha is equal to minus alpha over 2a I which can easily be solved to give this I some constant term and exponential minus this alpha square over 4 a, and now the question is that how to get his arbitrary constant of integration, but just to recall that we have this I and the I at 0 is a much more simplified function now, 0 to infinity and we have e power minus a x square dx.

So if we take a substitution here, square root a is equal to y that means square root a dx is dy and then this integral will become 0 to infinity and we have e power minus y square and this dx will be dy and there will be factor square root a. So this value will be 1 over square root a and e power y square with dy that is square root pi by 2. We have also used this in previous lectures.

So this I naught, the value of this I naught, the value of this I naught which we have just evaluated here, that is square root pi by 2 and this is square root a and that is exactly the value of the constant because if we put here this I 0, this is, so alpha is 0 so we have only the constant term there. So this constant is nothing but this value of I at 0.

So we got this constant term that means we have this I and once we have this integral I we can use this to get this Fourier Transform of e power minus a x square which is nothing but 1 over square root 2a because this will cancel out and then here we have also square root 2, so 1 over square root 2a and exponential minus alpha square over 4a.

Well, so just a short remark here, that if we set this a is equal to half so in this Fourier Transform, Fourier Transform of this e power minus a x square is 1 over square root 2a e power minus alpha square over 4a. So if we set here a is equal to half then what is happening here, the Fourier Transform of minus half x square and now we will set this a half there too, so this is e power minus alpha square by 2.

So what is interesting here that the Fourier Transform of minus x square by 2 is e power again, minus alpha square by 2, only this x is replaced with, x is replaced with alpha. That is the only difference but we have the same function.

So the Fourier Transform of this function is exactly the same function. And such a function is said to be self reciprocal under the Fourier Transformation because the Fourier Transform of f x is just f alpha and this is one of the examples of such functions.

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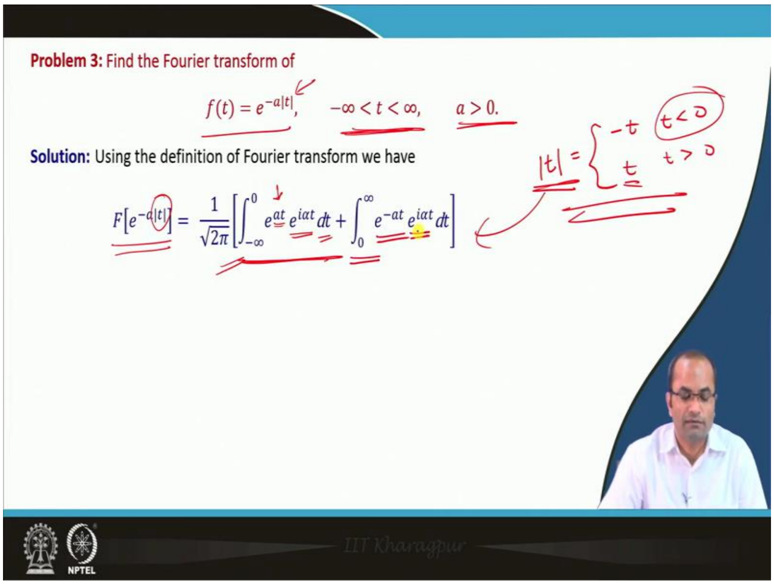

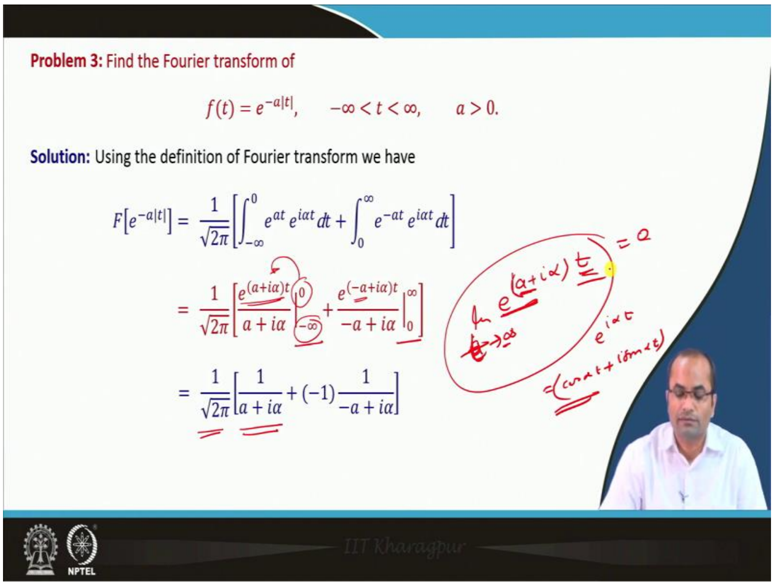

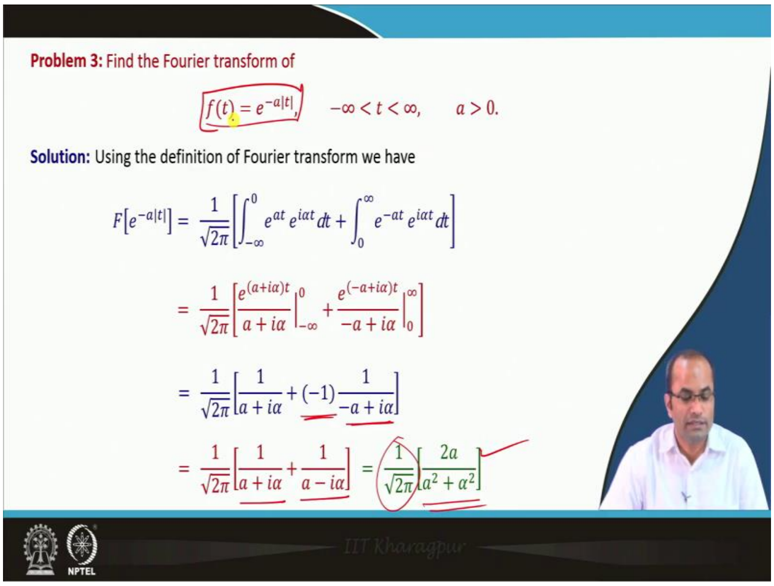

Well, so we have another problem where we will find the Fourier Transform of this function e power minus a, the absolute value of this t, the t is from minus infinity to plus infinity and a is a positive number. So using the definition of Fourier Transform again we can apply here, so this t is the absolute value. So we can break this into two portion because when we are talking about the absolute value so if t is negative this is like minus t and when t is positive this is equal to t.

So this property we can use here. So in this region minus infinity to 0 when t is negative so we will be using this absolute value of t as minus t, so therefore this has become a t now, and then we have e power i alpha t coming because of the Fourier Transform, and then this integral dt, and in the region 0 to infinity we have this t is equal to just t, so we have minus a t and e power minus i alpha t, the same factor again.

So now since we have the exponential at both the places, we can merge them and then we can integrate easily as done previously. So we have here a plus i alpha, a plus i alpha t and then after this integration a plus i alpha will come in the denominator. Here we have minus a plus i alpha t and after integration minus a plus i alpha will appear in the denominator and then we have these limits, minus infinity to 0 and 0 to infinity. So 1 over square root 2 pi we have 1 over a plus i alpha and because when we put this 0 then only we will get here 1.

When we put this infinity, so we have e power a plus i alpha t and we are looking at the limit when this a goes to, so this t, sorry t goes to infinity, and this a is positive and t goes to minus infinity in the first case. In the second case, t goes to infinity but then we have minus a. So in both the cases, this quantity will go to 0 because this e power i alpha t we can write as cos alpha t plus i sin alpha t which is a finite number and then we have e power a t, then we will have this as 0. So then because of this we have, at the 0 this will survive the negative sign of course because this is the lower limit and then we have 1 over minus a plus i alpha.

So the finally, when we incorporate this there, so we have a minus i alpha and a plus i alpha which can be written as this 2a over a square plus alpha square and there will be a factor this 1 over square root 2 pi. So that is the Fourier Transform of this exponential function e power minus a and the absolute value of t.

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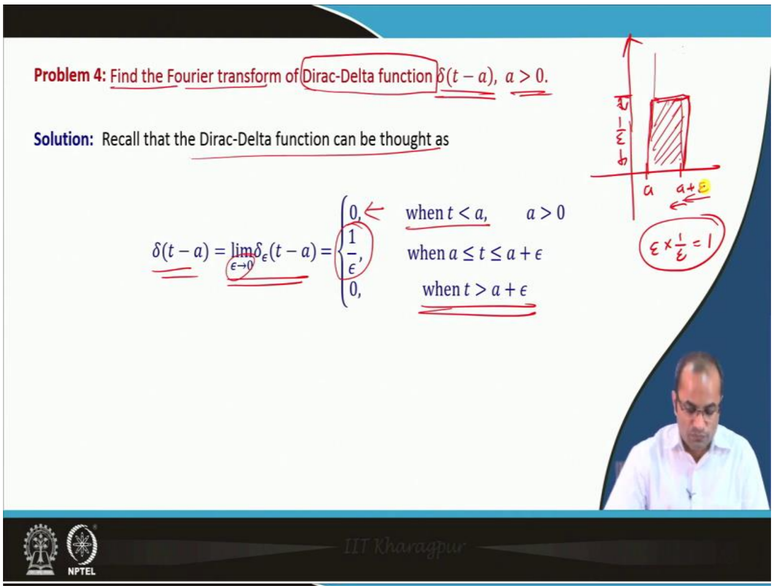

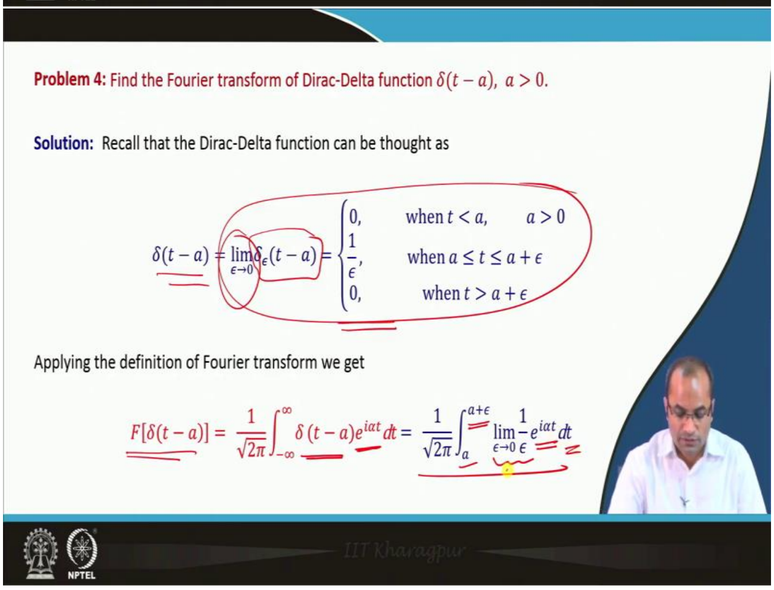

Now we will compute the Fourier Transform of Dirac-Delta function delta t minus a where we take this a again positive. So recall that the Dirac-Delta function can be thought as, there are various ways to define this Dirac-Delta function and the simplest way of imagining this,

this Dirac-Delta function as this rectangular pulse in the limiting situation. That means if we consider that when t less than a the value is 0, t greater than a plus epsilon it is again 0 and it is giving value 1 over epsilon. So in the region here a to, for instance this a plus epsilon, the value is 1 over epsilon of this. So this is 1 over epsilon.

So as this epsilon goes to 0, if we take this epsilon goes to 0, this height of this rectangle will increase. And our interest is, and I have this Dirac-Delta is defined that what will happen when this epsilon goes to 0?

So in that case, this peak will go to actually infinity but what is interesting in this Dirac-Delta case that if we compute this area which is epsilon over 1 over epsilon this is 1, so the area is always 1 and when, even the limiting situation also the area would be 1. So this is how we imagine this Dirac-Delta function as the limiting case of this delta epsilon. And with this definition, we can easily compute the Fourier Transform now of this Dirac-Delta function.

So if we take this Fourier Transform of this delta t minus a then we have here delta t minus a e power i alpha t as per the definition and then we have used this limiting definition of the Dirac-Delta. So a to a plus epsilon we have the value 1 over epsilon and later on we have to take this epsilon goes to 0 to get this delta. So we have this epsilon goes to 0, 1 over epsilon and e power i alpha t and then we have this dt.

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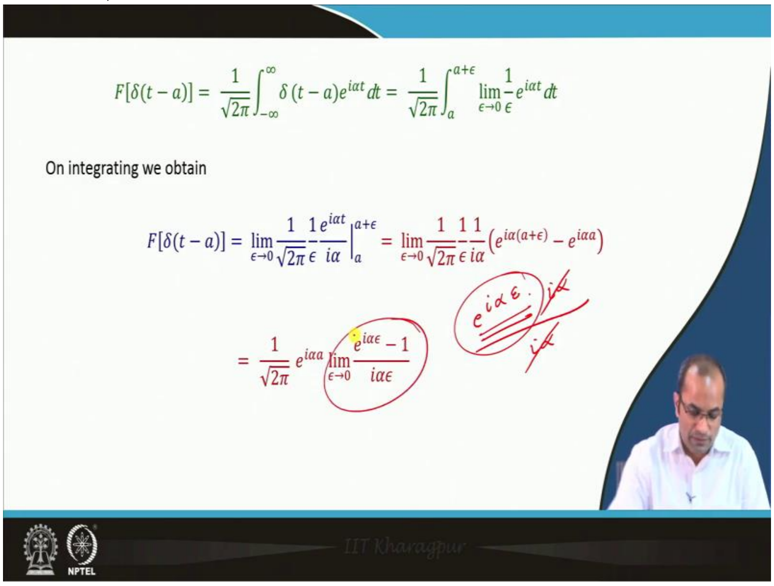

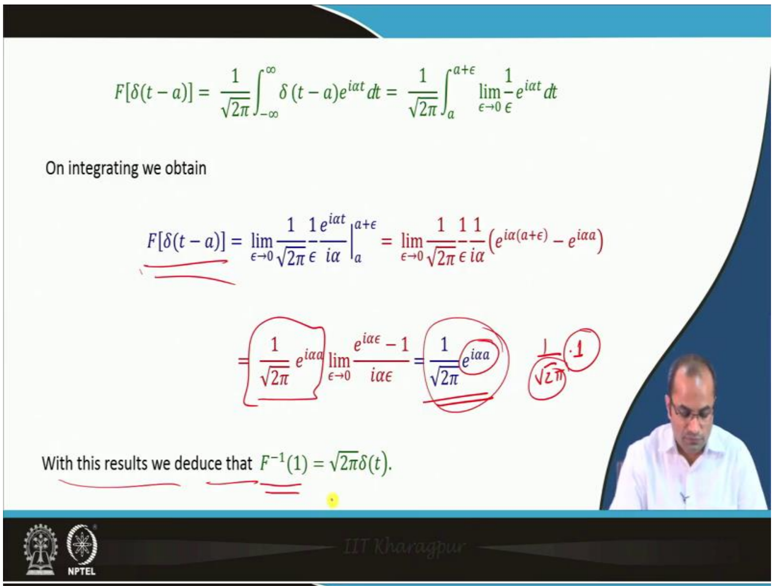

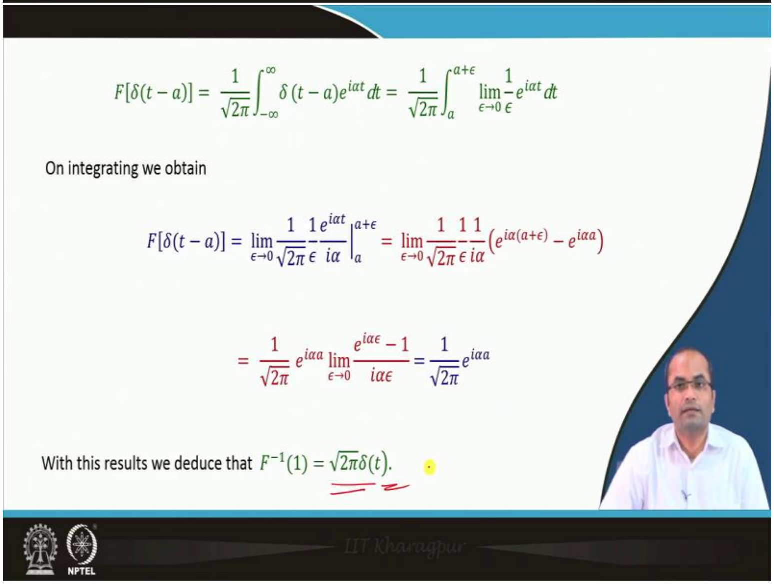

So again, we have this integral to evaluate. So this limit we can pass later on. So first, this 1 over epsilon will go out of the integral and then e power i alpha t dt will be integrated to give e power i alpha t over i alpha and then we have a to a plus epsilon.

So 1 over square root 2 pi and then we have 1 over epsilon and 1 over i alpha and exponential i alpha a plus epsilon and minus e power i alpha a. When we substitute a plus epsilon for this t, we are getting this number here. When we put a, we are getting the second one.

So now e power i alpha a is common at both the places, so we can bring this out, e power i alpha a and then we need to compute this limit as epsilon goes to 0, e power i alpha epsilon minus 1 divided by i alpha epsilon. And now here in this case, so this is going to 1 minus 1 and then so that is 0, so we have this 0 over 0 form where the L' Hospital's rule we can apply and once we apply this L' Hospital's rule there, so e power i alpha epsilon and then we will get this i alpha and in the denominator, we will have i alpha so this will get cancelled and then limit epsilon goes to 0, this will go to 1.

So this limit will be simply 1 and we have this answer 1 over square root 2 pi e power i alpha a. So with this result we can also deduce that F inverse of 1 will be, so if we put this a to 0, so here a to 0 that means we have 1 over square root 2 pi and then 1, the right hand side. So if we take this inverse now, so we have the delta function there, delta t, this square root 2 pi can go there and the Fourier Inverse of 1. So this is 1 there, the right hand side function so the Fourier Inverse of 1 is exactly square root 2 pi and delta t.

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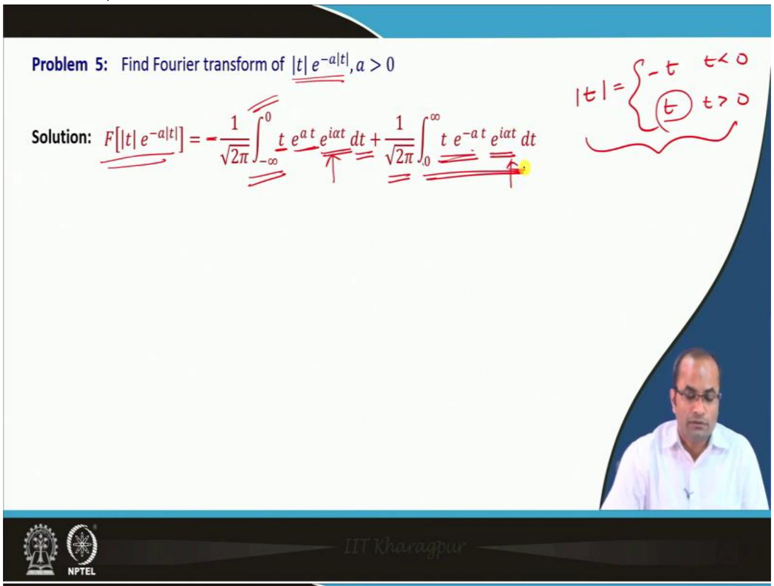

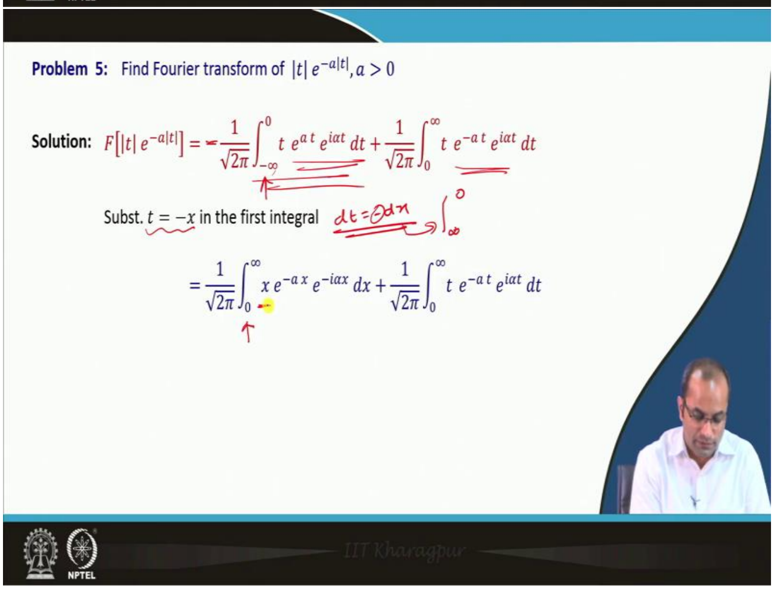

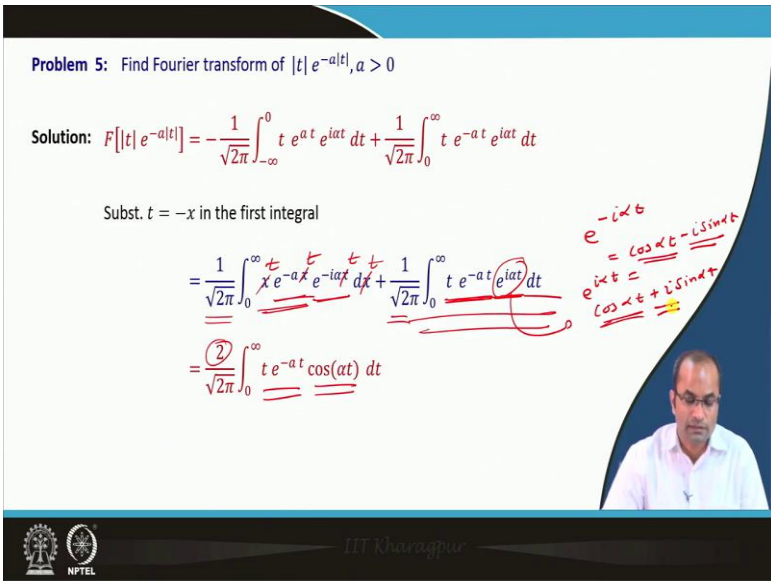

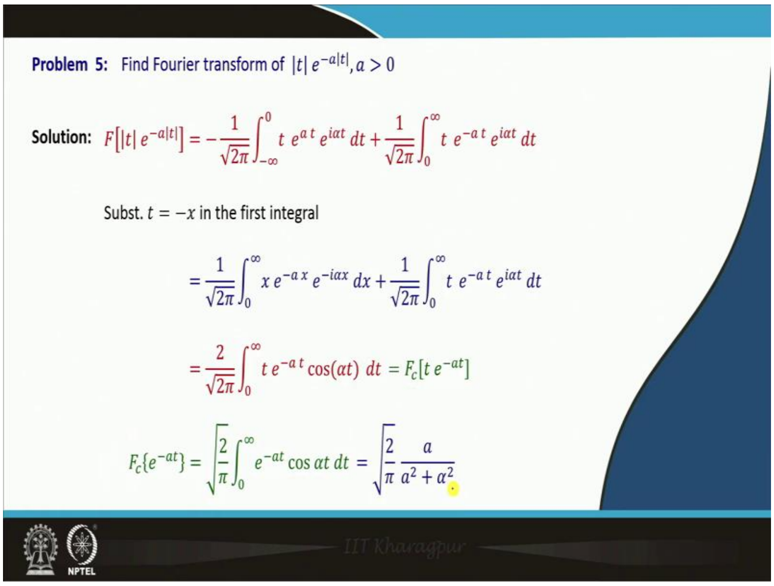

So another problem where we will look for the Fourier Transform of this function. Now we have this absolute value of t and e power minus a, absolute value of t. So this e power minus a absolute value of t we have already evaluated but this is different because here absolute value of t is also sitting with e power minus a absolute value of t.

So in this case, again, we will apply the definition of the Fourier Transform over this given function and what we observe because again this here we have this absolute value of t which says that it is minus t when this t is negative and t when t is positive. So with this consideration, what we have in the minus infinity to 0, the mod t will become minus t so we have minus sign there and t, and e power again this minus, minus plus, so e power a t and e power i alpha t and we have this dt there.

So 1 over square root 2 pi and then 0 to infinity we have t e power minus a t because here in this region 0 to infinity this will be t only. So we have this t e power minus a t and e power i alpha t that is the part of the definition of the Fourier transform.

So having this now, we will just substitute t equal to minus x in the first one to make similar integral again. So when t minus x, the limits will become infinity to 0 in this first case and then the some more changes will happen to the integrand and also the dt will be minus dx. So there will be another minus there but then here we have the limits are now infinity to 0 but with this minus we can replace again 0 to infinity.

So we have the limit 0 to infinity. t is replaced by minus x so there is minus here now which will be compensated with this minus which was already at front there. So e power minus a, so because now t is minus x so a power minus x and e power minus i alpha x dx, and the second integral as it is. So again, this t may be replaced by, x may be replaced by t so we have basically the same integral with dt, e power, here we have the plus sign, here we have minus sign but this is e power minus a t and here also we have this t e power minus a t.

So what is now? So in the first one we have minus i alpha t. So e power minus i alpha t which can be written as the cos alpha t minus i sin alpha t. And in the second case, we have this e power i alpha t which is cos alpha t and plus i sin alpha t. This is from the second integral where we have e power i alpha t. So the rest everything matches.

So we have the factor, the same factor there, we have t e power minus a t, t e power minus a t. The first term of both the integrals will be coming, this cos alpha t so that we have considered here with the two times and then the second, in one case we have the minus sign, in the other one we have the plus sign. So that integral will cancel out.

So here the sin integral from both, because they will have the opposite sign, it will cancel out and we will get only with the cos term and that is the factor 2 will be appearing there. So we have here the Fourier cosine transform. If we take a look here, the t e power minus a t and cos alpha t dt so that was the definition of the Fourier cosine transform.

So this and this is here square root, this 2 over pi. So that is exactly the factor we have considered also in Fourier cosine transform. So this is precisely the Fourier cosine transform of t e power minus a t. However, we do not know what is the Fourier transform of this t e power minus a t, so we can compute this easily either directly by solving this equation or there is another trick here which we are using.

So just notice that the Fourier cosine transform of e power minus a t is e power minus a t and then cos alpha t dt with this factor which can be easily computed and we have done this in earlier problems. That will be a over, a over a square plus alpha square. So this Fourier cosine transform of e power minus a t is given as square root 2 over pi a over a square plus alpha square.

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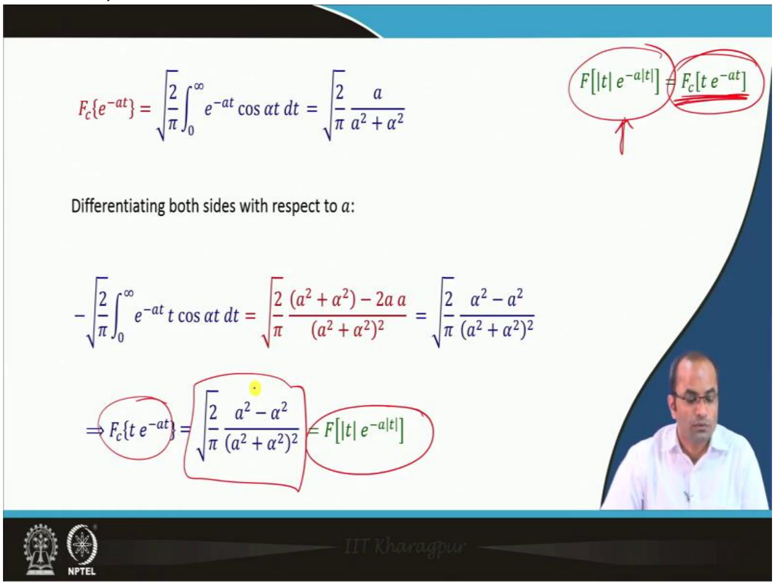

So having this in hand, so we want to get the Fourier transform of the absolute value of t into e power minus a t, that is Fourier cosine transform of t e power minus a t. So here the Fourier cosine transform of e power minus a t we have here, a over a square plus alpha t. So from this we will deduce that how to get the Fourier cosine transform of t e power minus a t.

If we differentiate this relation, obviously both the sides with respect to a then what will happen? So we differentiate this with respect to a. So here e power minus a t and then t factor with minus sign will come extra, and the right hand side again we can differentiate, so the whole square term and then this quotient rule will be applicable, so which can give us this alpha square minus a square over a square plus alpha square whole square. That means the Fourier cosine transform of this, because if we take a look at this one here, now this is a Fourier cosine transform of t e power minus a t with this factor 2 over square root 2 pi, this minus sign we can bring to the right hand side.

So the Fourier cosine transform of t e power minus a t with this minus adjusted here, a square minus alpha square now over alpha square plus a square whole square, this is exactly the Fourier cosine transform of t e power minus a t.

So the Fourier cosine transform of this t e power minus a t is exactly the desired Fourier transform which we want to evaluate this as absolute value t e power minus a absolute value t. So this is the desired Fourier transform we have evaluated using or with the help of this Fourier cosine transform.

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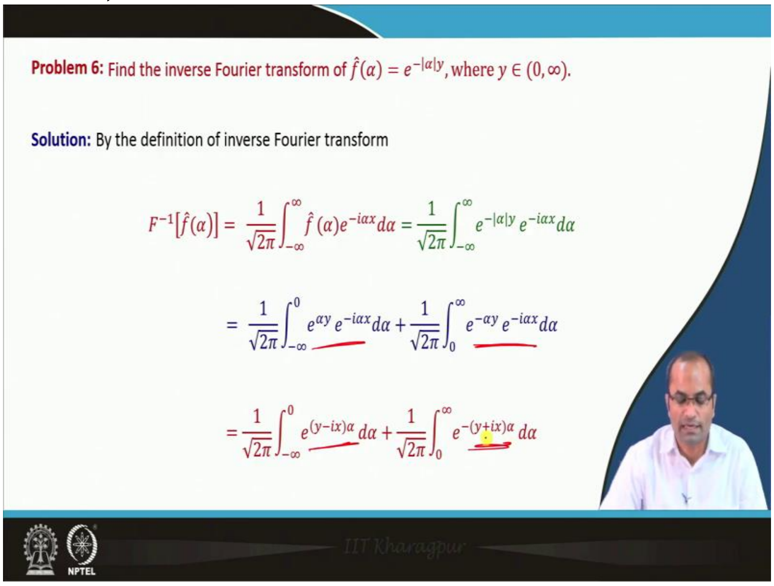

In this example, we will find, so there are two examples based on the inverse Fourier transform but they are simple. So here we want to get the Fourier inverse transform of e power minus alpha y where y is a positive number.

So by the definition of the inverse transform we have e power minus i alpha x there. So we can substitute this e power minus, so the function itself and then so from minus infinity to plus infinity where this alpha is negative so, because again this absolute value of alpha is there, so that will be minus alpha in this range here. So that is already incorporated. And in the positive range this absolute value of alpha will be just alpha.

So with this definition now we can club the two because these are the exponential functions. We have clubbed y minus i x and then y plus i x and each of them can be integrated with respect to alpha easily.

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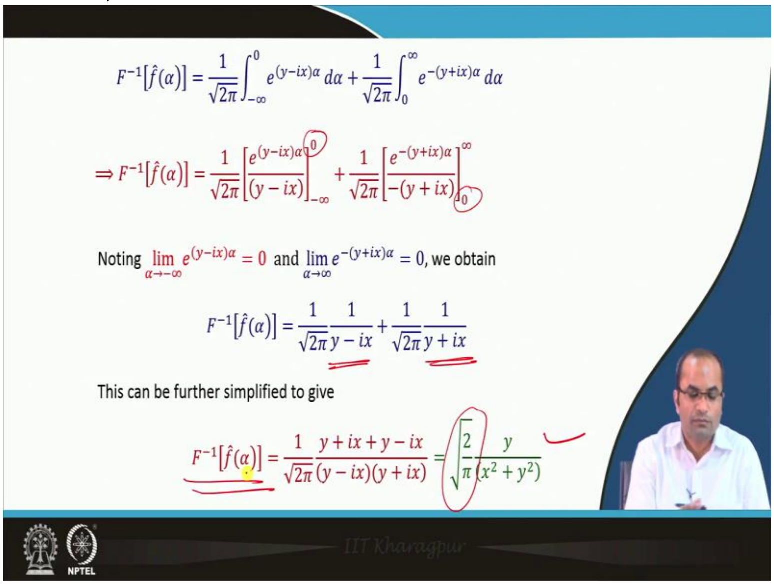

So we have this exponential then in the denominator y minus i x will come. Here y plus i x will appear. And we should also note that when alpha goes to 0, this e power y minus i x alpha will go to 0 because this y is positive and in this case again, because the minus y is there and this alpha goes to infinity, this will again go to 0 and then we have rather simplified form which will just come because of this 0 there.

So we have y minus i x and here we have this y plus i x which can be further simplified to give y over x square plus y square with the factor here 2 over square root pi. So this was the example where we have used the definition of inverse Fourier transform to get the inverse transform of such a function.

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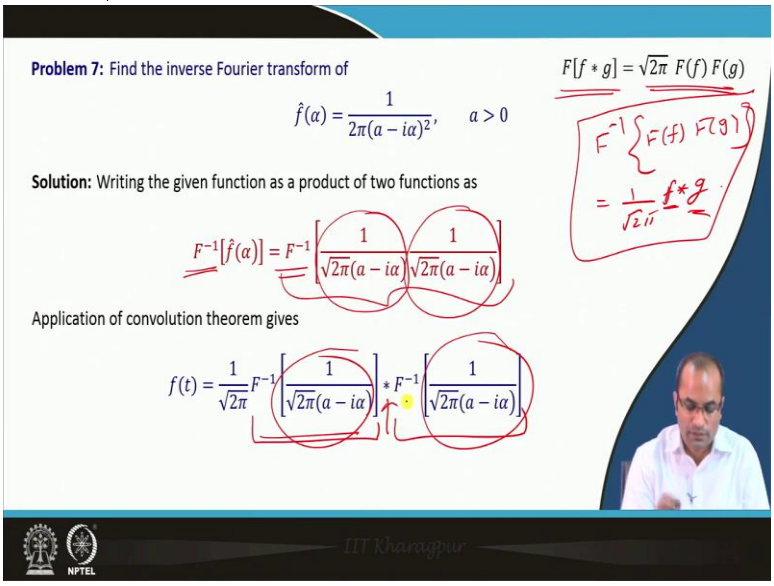

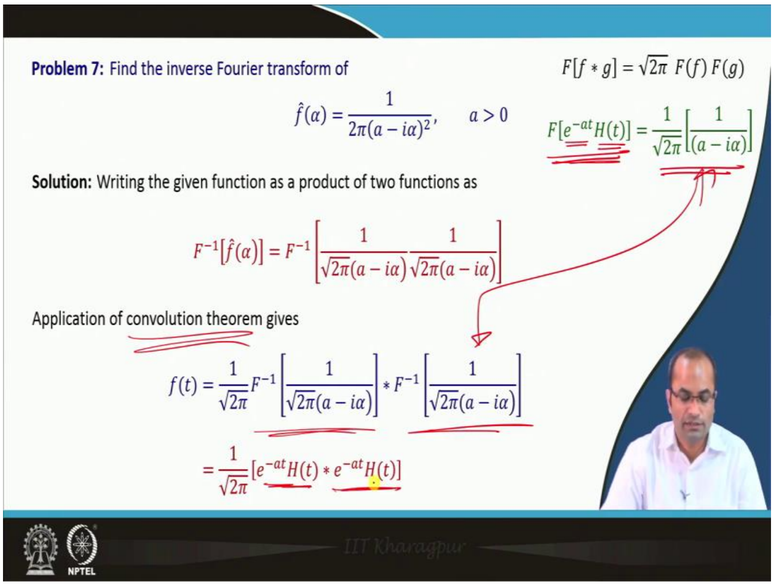

So this is the last example where will be talking about the inverse Fourier Transform of this 1 over 2 pi a minus i alpha whole square and here again we assume that a is positive. So we write this as a product of two functions, that means 1 over square root 2 pi and a minus i alpha and here again this 2 pi, so is made into square root 2 pi and square root 2 pi, and then again a minus i alpha, and we take the Fourier inverse both the sides now, so we want to compute this what is the inverse of this, and we know already that there is a Convolution Theorem which says that the F inverse of this Fourier Transform of f and Fourier Transform of g is equal to 1 over square root 2 pi and the product, so this convolution of f and g.

So we will apply this result here now. So because this is the product of two Fourier transforms, so we can apply this and we will get the Fourier inverse of this, so the f is the Fourier inverse of this and then again the second one g is the Fourier inverse of this function and then this convolution has to be done.

So with this Convolution Theorem we got this result. Now the question is what is this F inverse of this 1 over square root 2 pi a minus i alpha, and if you recall from the previous lecture, we have done this Fourier transform of e power a t, the Heaviside function of t and was exactly this one. So we know now the inverse of this is e power minus a t H t. So here we have then e power minus a t H t and e power minus a t H t and the convolution of the two.

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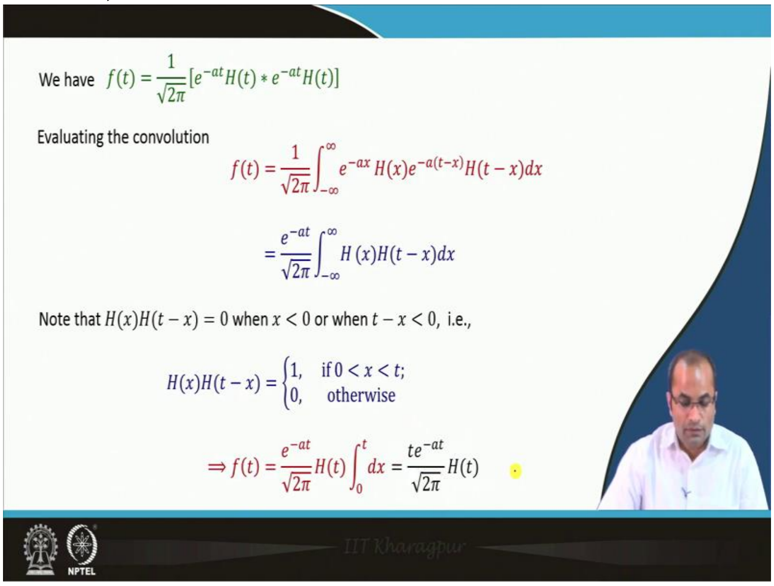

So having this convolution now we can evaluate this. So e power minus a x H x and this shift will be there in the second case minus infinity to plus infinity. So this simplification because we have e power minus a x here, we have e power a x that will get cancelled and we have e power minus a t outside because this integral is over x.

So the integrand is just H x and H t minus x which we know that when x is positive, when x is negative this will be 0 because of this and if t minus x is negative so this argument is negative, so again this product will become 0. That means we have the situation, between 0 and t, if x is between 0 and t the value will be 1. Otherwise, the value will be 0 of this integrand.

Therefore, we can write down here e power minus a t over 2 pi and H t is important because this integral will survive only when this t is positive, and then we have here 0 to t, the value this is 1 over dx which we can evaluate this as just t. So we have t e power minus a t divided by square root 2 pi and this Heaviside function of t.

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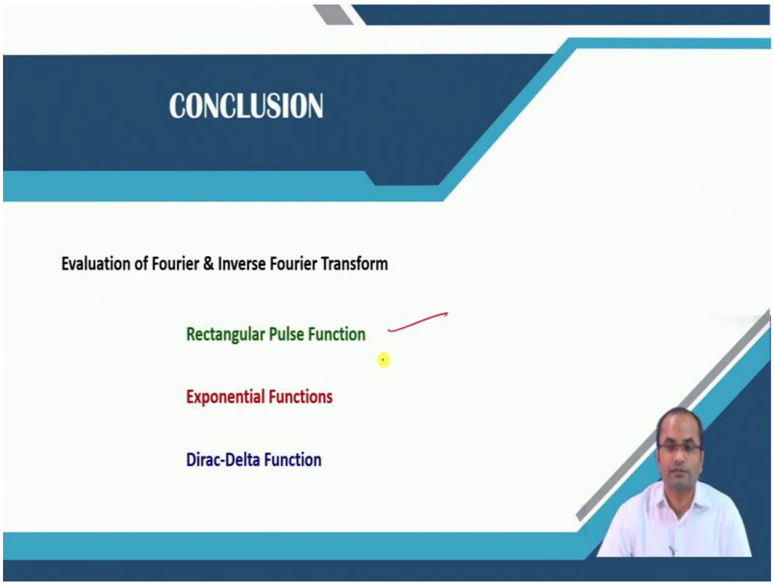

Well, so these are the references we have used for preparing this lecture. And just to conclude here we have evaluated the Fourier and also inverse Fourier Transforms of various functions, those includes the rectangular pulse function, exponential functions, also the Dirac-Delta functions etc. So that is all for this lecture and I thank you for your attention.