Engineering Mathematics-II
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Lecture 57
Laplace Transform of Special Functions

So welcome back to lectures on Engineering Mathematics 2 and this is lecture number 57 on Laplace transform of Special functions.

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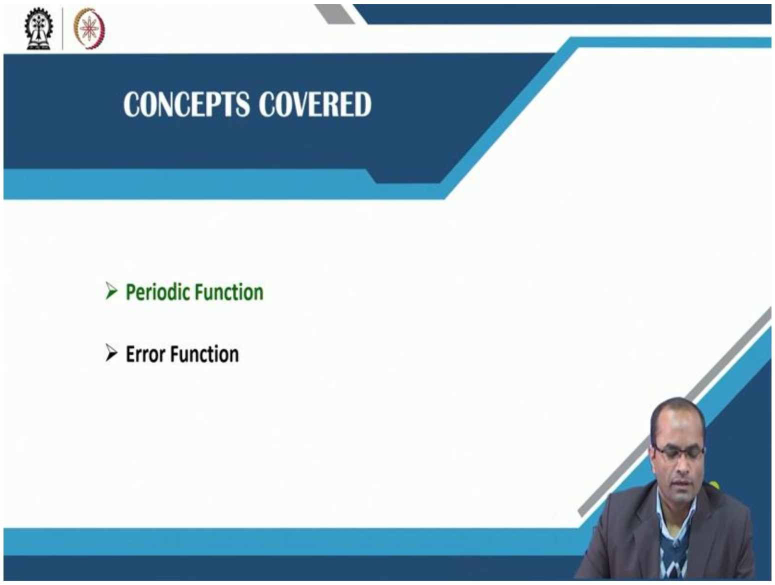

So in this lecture we will cover two types of special function one is the Periodic function and the other one is the Error function.

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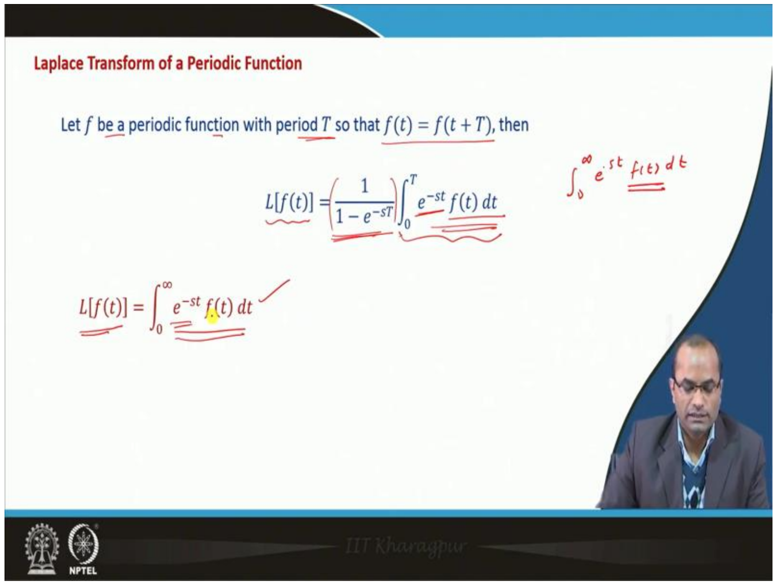

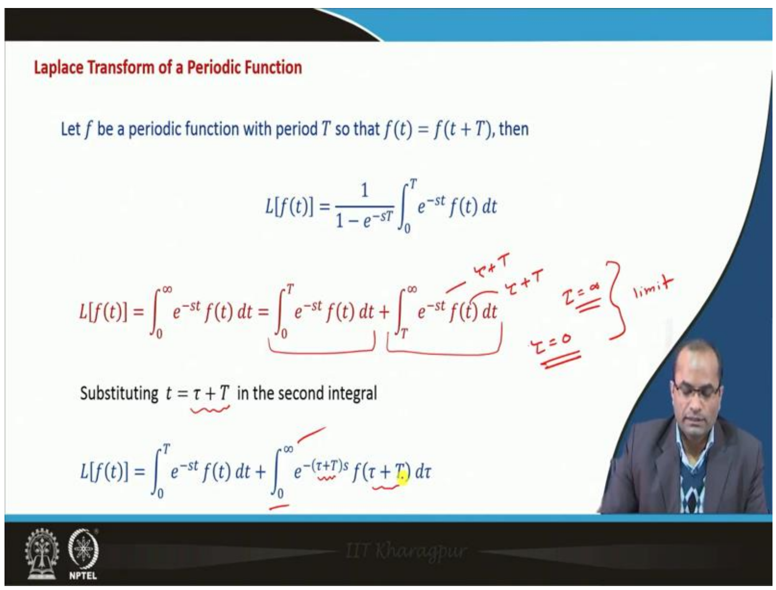

So the Laplace transform of a Periodic function we have already discussed what is the periodic function. So let f be the periodic function with period t that means that ft plus this capital T will be equal to ft. And in that case what we will observe that the Laplace transform of such a periodic function can be computed by this simple formula which says that 1 over 1 minus e power minus st that is outside integral.

And then the integral will be from 0 to capital T e power minus st as usual which comes in the Laplace transform ft dt. So instead of evaluating this 0 to infinity usual this Laplace integral which says that 0 to infinity e power minus st and ft dt. What we can do if we know that this function is a periodic function we can just do this integration from 0 to capital T which is obviously much easier. And then, we have to multiply by this factor.

So how do we get this formula that we can take a look here, so if we compute the Laplace transform of ft that is the definition or the formula for the computation which the where the integral will go from 0 to infinity with this e power minus st dt. So this integral we will break into 2 so first is 0 to this capital T and then this capital T to infinity.

So that is the usual property of the of a integral. And then we will substitute t is equal to tau plus capital T in this second integral that means here this tau will be replaced t will be replaced by tau plus capital T in this case also this t will be replaced by tau plus capital T.

And this dt will be d tau and the limits so when this t was tau t was capital T then the tau will become 0 and when t is infinity in that case tau will be also infinity. So these are the limits here for this new variable. So what we will get now the limits will be now 0 to infinity and e power minus this t is tau plus t and here also t is tau plus t and d tau.

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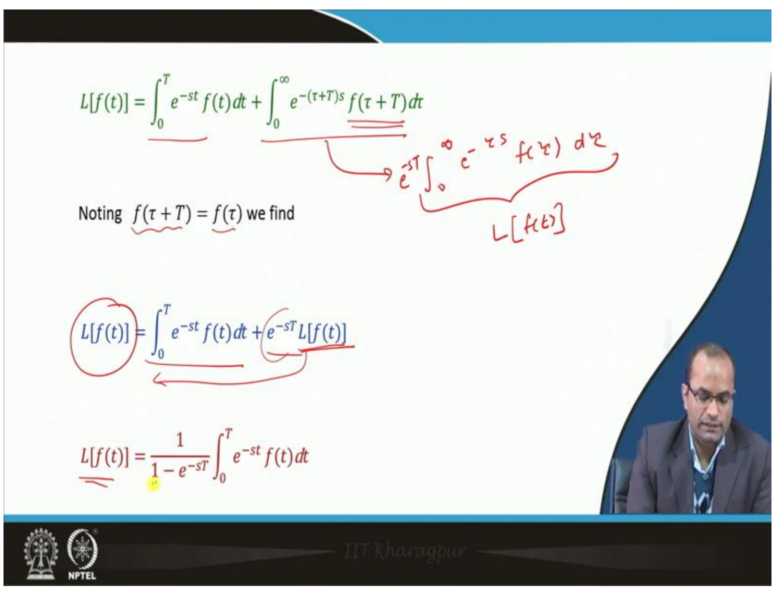

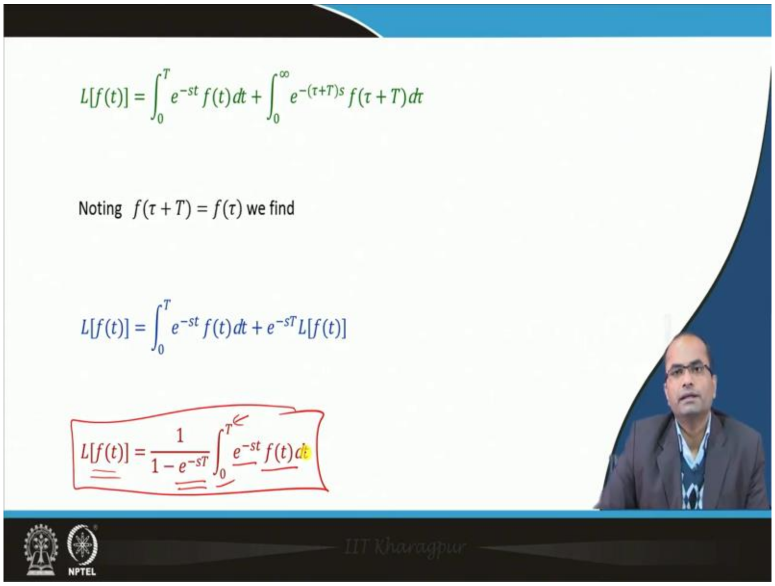

So we have this integral now and we can use this property of the periodic function which says that f tau plus capital T is f tau. So this can be replaced again with this f tau so we have the Laplace transform of ft because after replacing the second integral will become 0 to infinity. And e power minus s e power minus s capital t we can take out because it this term is not depend on tau. And then we have e power minus this tau s and then f tau and d tau and this is again Laplace transform of ft.

So this is what written here so we have the Laplace transform we have t equals to this term plus e power minus st again the Laplace transform of ft. So we can bring this term to the left hand side and then we can divide this term 1 minus e power minus st. And we got this formula which says that the Laplace transform of ft when ft is the periodic function of period capital T. So it can be evaluated by this 1 over 1 minus e power minus s capital T 0 to this capital T period e power minus st ft dt.

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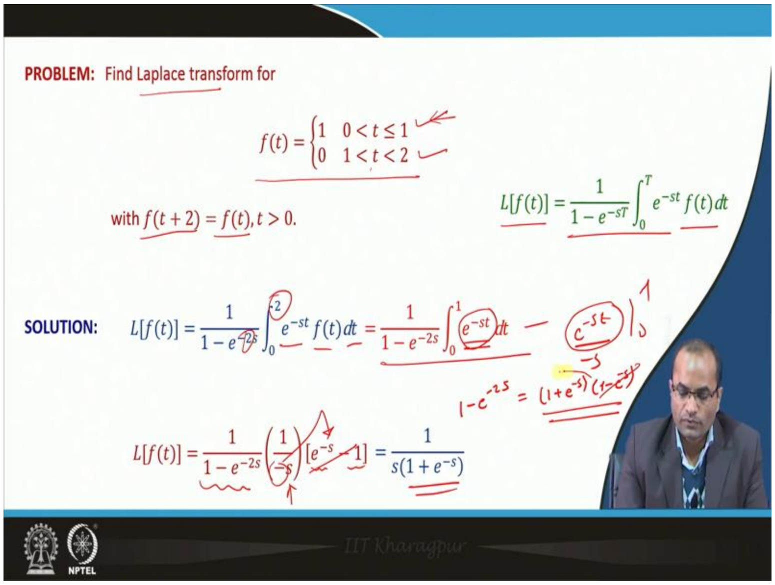

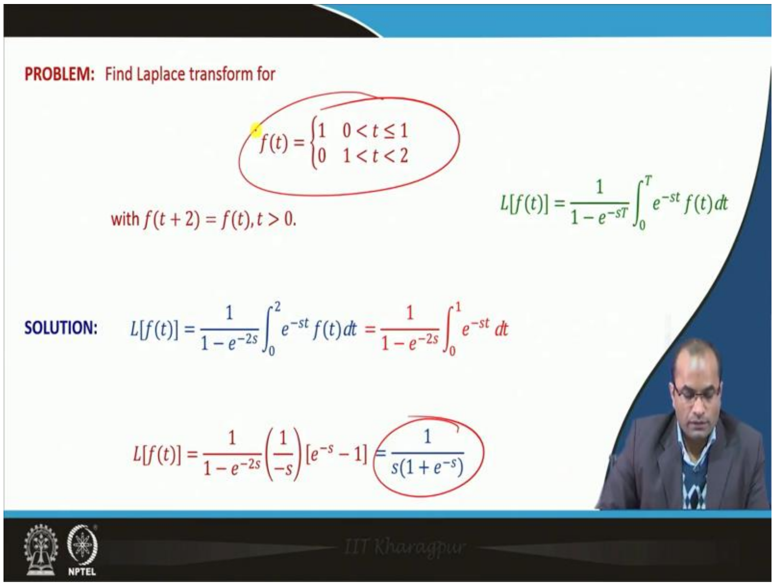

We can go through some of the examples using this formula. So if you want to find the Laplace transform of this function ft which is define from 0 to 2. So in the range here 0 to 1 it is 1 and 1 to 2 it is 0 with this ft plus 2 is equal ft for t positive. Coming to the solution – So, we know for the formula for periodic function because this function is periodic function with period 2. So this capital T will be replaced by this 2 here and then e power minus st ft dt.

And that to now from 0 to 2 it is defined as 1 in 0 to 1 otherwise it is 0. So basically we are remained with this integral 0 to 1 e power minus st the ft is 1 here and then we need to integrate over t this. So e power minus st the integration is e power minus st over minus s so this minus 1 over s and then e power minus st we have to substitute these limits. So we have e power minus s and then minus 1.

Here we can do some more simplifications so here 1 minus this e power minus 2s so 1 minus e power minus 2s it is a square minus this b square which can be written as 1 plus e power minus s and 1 minus e power minus s. So having this there and with this minus sign this is again 1 minus e power minus s so this term will get cancel with 1 minus e power minus s. And then we will get 1 over s this s here and this term 1 plus e power minus s. So this is the Laplace transform of this given periodic function.

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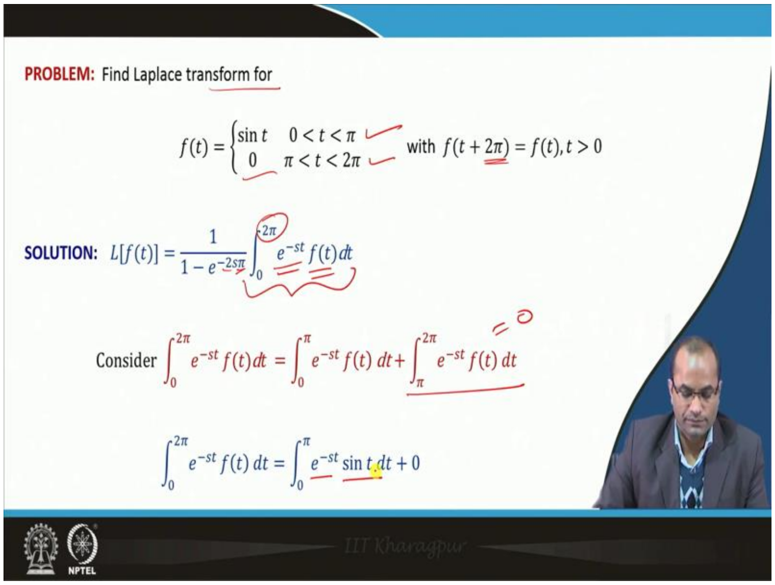

Next problem we want to find the Laplace transform for this function again here we have the periodic function with period 2 pie and it is defined as that in the range 0 to pie it is sin t otherwise it is 0. So again we will apply this formula 1 over 1 minus e power minus this 2 pie is the period and again here also 2 pie e power minus st and then this ft the function dt. So again this is define for 0 to pie only so this integral here 0 to 2 pie we can break from 0 to pie and then pie to 2 pie. And this portion will be 0 because the function is 0 and in the first case we have e power minus st and then sin t dt.

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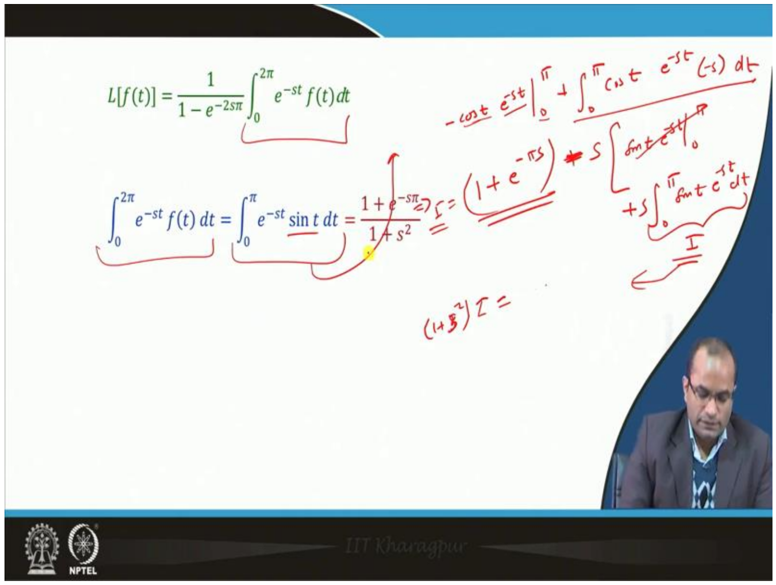

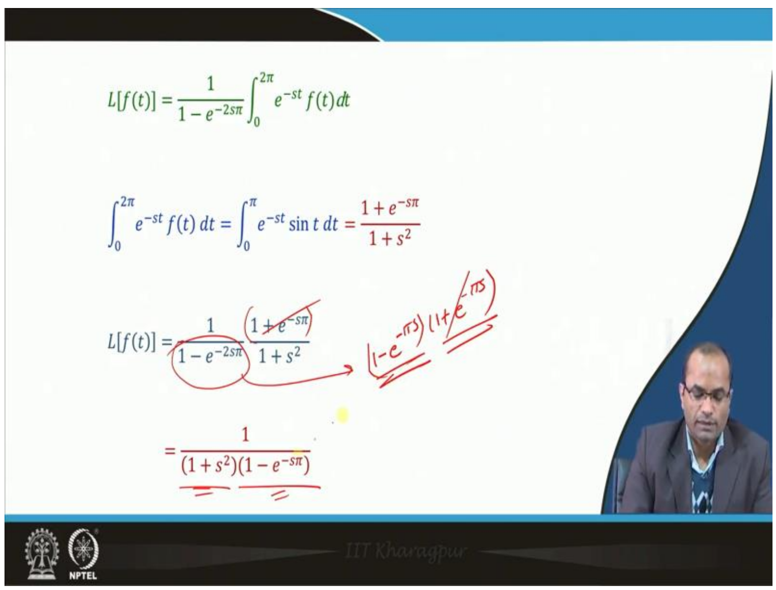

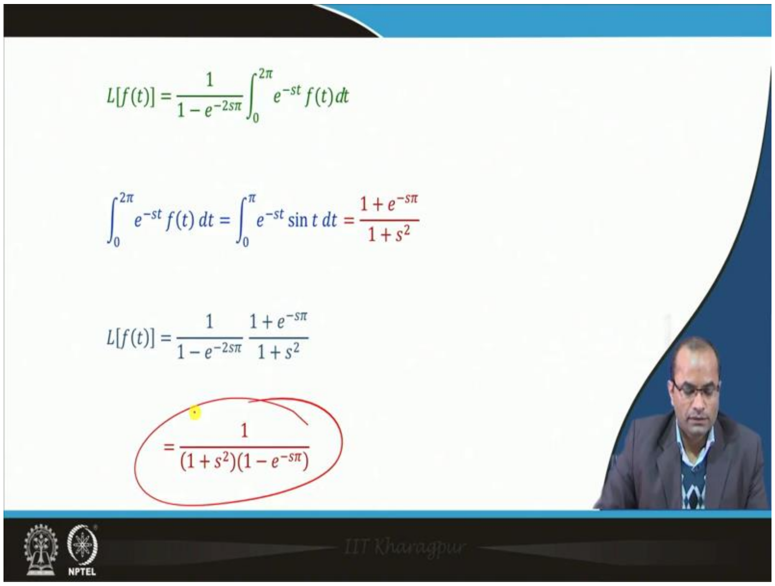

So now we can so have this situation here that this integral 0 to 2 pie ft dt is 0 to pie e power minus st sin t dt which can be again integrated by parts. So here this sin t when we do the integration this will be cos t and then we have e power minus st the limit 0 to pie. And then we have minus 0 to pie the minus minus will be plus for this sin when we write this cos t and e power minus st with the minus s. And then we have dt.

So we can do this integration by parts here indeed we will have 1 so this cos 0 is 1 and e power 0 is also 1. And while writing this pie there and so we will get this cos pie which is minus 1 and then e power so that will become e power minus pie s the first term. And here we need to again do the order of we need to again perform integration by parts. So we have already this minus s there and then again this cos will be sin t e power minus st the limit 0 to pie minus and then here 0 to pie.

And we will have sin t e power minus st dt with minus s which will become the plus s there. So this term will vanish because pie n 0 and here we have this minus s square so this because this is again the integral which we have started with. So here let us say this is again I and it was I equal to so this I the value of this integral we can bring to the left hand side. So we will have 1 plus s square 1 plus s square and this integral I and the right hand side this term.

So this integral gives us exactly 1 plus e power minus pie s over 1 plus s square and then we have this ft the Laplace of ft 1 over 1 minus e power minus 2 pie s. And then, this term coming from here 1 plus e power minus pie s and 1 plus s square. So 1 over 1 plus s square and then we have 1 minus e power minus pie s because again this term will be cancelled out with this term and because this can be written as 1 minus e power minus pie s and 1 plus e power minus pie s. So this 1 plus pie s will get cancel with this is 1 and we will get 1 plus s square and 1 minus this pie s term. So this is the Laplace transform of this desired function.

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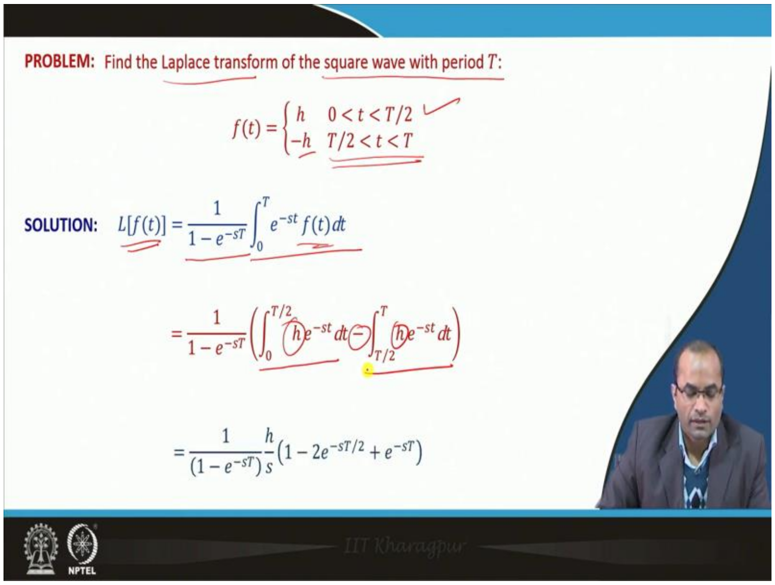

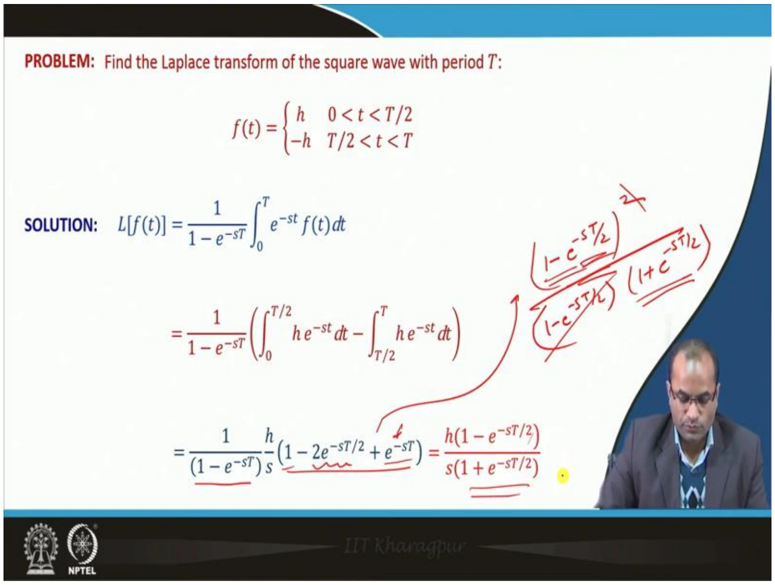

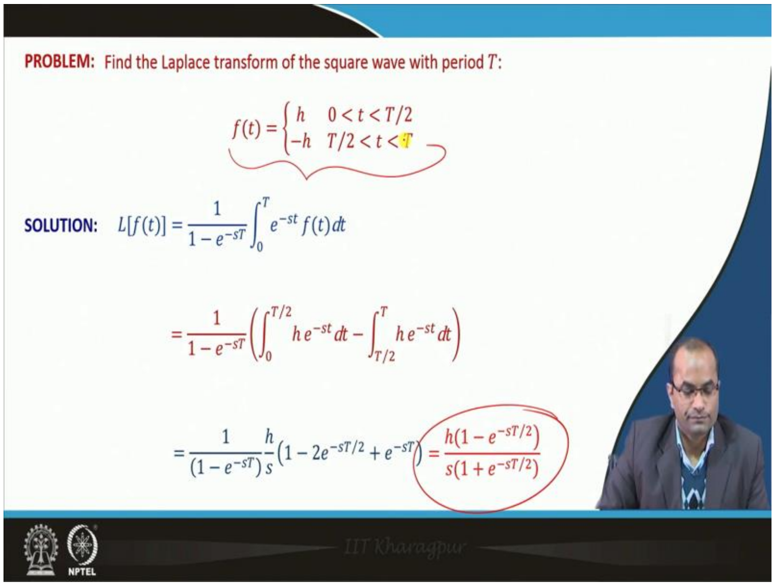

We want to find for instance this Laplace transform of this square wave with period this t which is defined as 0 to t by 2 as h. And then t by 2 to this capital T it is defined as minus h and then this is a periodic function. So we can apply again the same idea that Laplace transform of a periodic function is 1 over 1 minus e power minus st and 0 to this capital T e power minus st ft dt.

So here we can break again into two parts 0 to t by 2 where the function is defined as h and t 2 to t where the function is defined as this minus h. So this can be integrated because here the h is not depending on t so we have e power minus st the integration will be e power minus st over s where we will get 1 minus e power minus st by 2.

When we substitute lower and upper limit. Similarly here we can bring this minus h there and again we can integrate when we combine the 2 we will get this term. Which can be further simplified this h over s naturally h over s is there and this term here in this bracket which is 1 e power minus s t by 2 whole square because this 1 square is 1 there and the e power minus st by 2 square is given here.

And this comes when minus 2 times the product. So this is the whole square of this and then here we have 1 minus so in the division we have 1 minus e power minus st by 2 and 1 plus e power minus st by 2. So the again this term will get cancel with this term. And we have e power minus s t by 2 and then we have 1 plus e power minus s t by 2. So this is the final answer for the Laplace transform of such a square wave function with period capital T.

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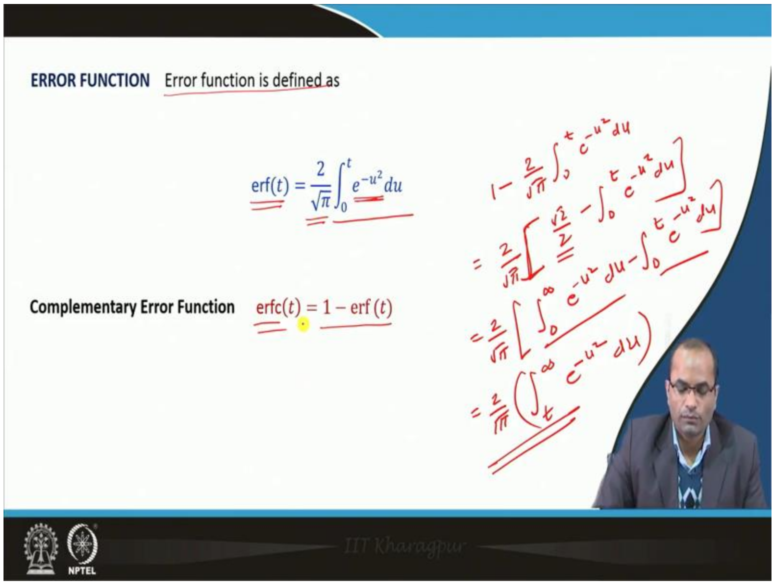

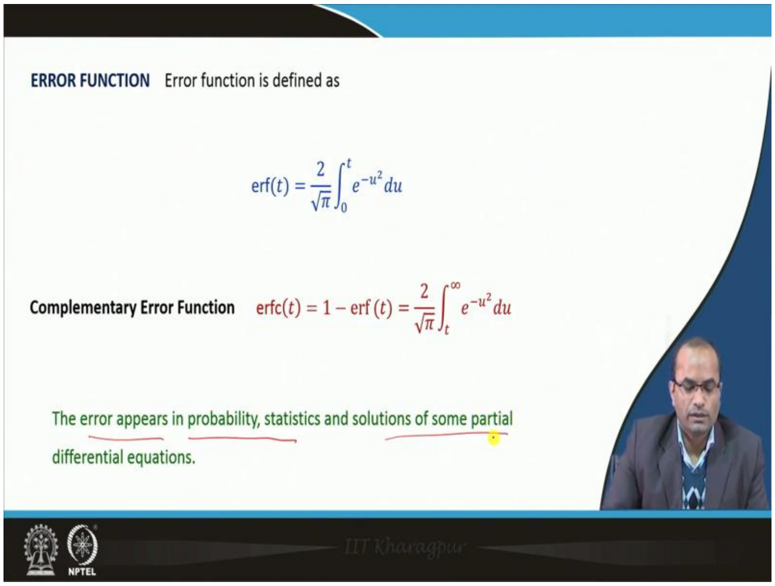

Now we will move to the error function which is which has applications in many areas of science and engineering so this error function is defined as error function of t 2 over square root pie and the integral 0 to t integral of e power minus u square du. There is another function which is called complimentary error function and it is defined as 1 minus this error function of t.

So if we have this 1 minus and if we write this 2 over square root pie 0 to t e power minus u square du in that case. So if we take this common here 2 over square root pie then it will become so the bracket. So square root pie by 2 for this 1 minus this 0 to t e power minus u square du.

And this we can replace Laplacian integral which is 0 to half e power minus u square du and then minus 0 to t e power minus u square du.

So this can be again written as 2 over square root pie and this integral can go now from t to infinity e power minus u square du. So that is the error function the complimentary error function which is exactly given here so 2 over square root pie and this t to infinity. So in the error function we have 0 to t and here instead we have the complimentary part so t to infinity e power minus u square du. And as I said before the error function appears in probability statistics for example also in the solution of some partial differential equations which we may be observe in last lectures.

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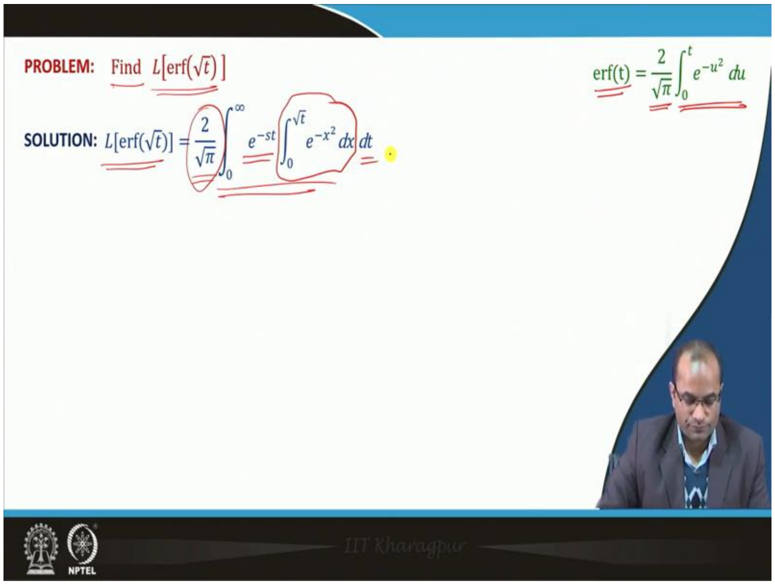

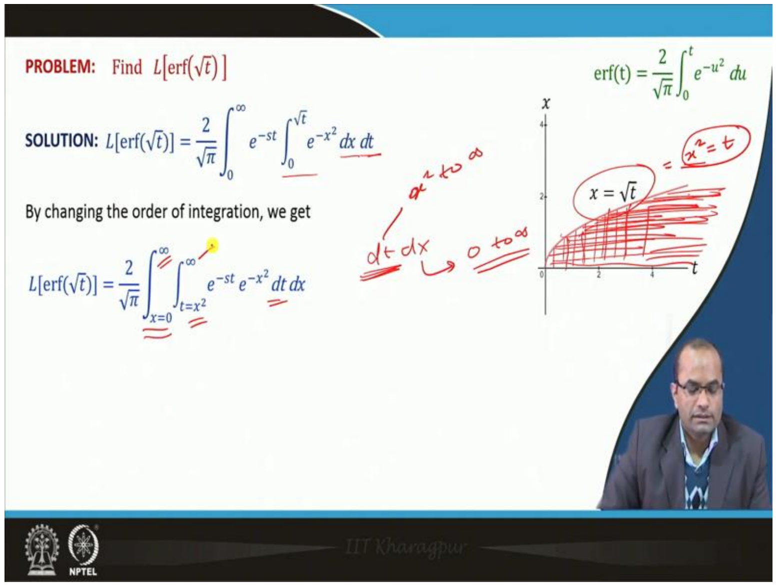
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$\mathrm{erf}(t) = \frac{2}{\sqrt{\pi}}\int_0^t e^{-u^2}\,du$
SOLUTION: $L[\mathrm{erf}(\sqrt{t})] = \frac{2}{\sqrt{\pi}}\int_0^\infty e^{-st}\int_0^{\sqrt{t}} e^{-x^2}\,dx\,dt$
By changing the order of integration, we get
$L[\mathrm{erf}(\sqrt{t})] = \frac{2}{\sqrt{\pi}}\int_{x=0}^\infty\int_{t=x^2}^\infty e^{-st}e^{-x^2}\,dt\,dx$
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t
$x = \sqrt{t}$
$x^2 = t$
x^2 to ∞
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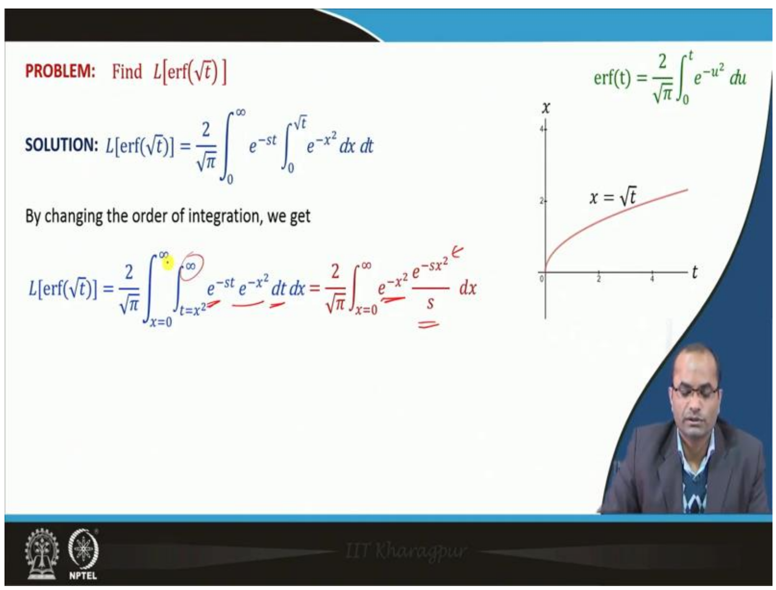
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By changing the order of integration, we get
$L[\mathrm{erf}(\sqrt{t})] = \frac{2}{\sqrt{\pi}}\int_{x=0}^\infty\int_{t=x^2}^\infty e^{-st}e^{-x^2}\,dt\,dx = \frac{2}{\sqrt{\pi}}\int_{x=0}^\infty e^{-x^2}\frac{e^{-sx^2}}{s}\,dx$
x
t
$x = \sqrt{t}$
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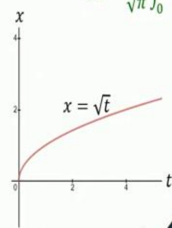
PROBLEM: Find $L[\text{erf}(\sqrt{t})]$

$$\text{erf}(t) = \frac{2}{\sqrt{\pi}}\int_0^t e^{-u^2}\,du$$

SOLUTION: $L[\text{erf}(\sqrt{t})] = \frac{2}{\sqrt{\pi}}\int_0^\infty e^{-st}\int_0^{\sqrt{t}} e^{-x^2}\,dx\,dt$



By changing the order of integration, we get

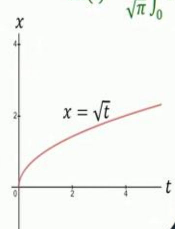
$$L[\text{erf}(\sqrt{t})] = \frac{2}{\sqrt{\pi}}\int_{x=0}^{\infty}\int_{t=x^2}^{\infty} e^{-st}e^{-x^2}\,dt\,dx = \frac{2}{\sqrt{\pi}}\int_{x=0}^{\infty} e^{-x^2}\frac{e^{-sx^2}}{s}\,dx$$

$$= \frac{2}{\sqrt{\pi}}\int_{x=0}^{\infty}\frac{e^{-(s+1)x^2}}{s}\,dx$$

PROBLEM: Find $L[\text{erf}(\sqrt{t})]$

$$\text{erf}(t) = \frac{2}{\sqrt{\pi}}\int_0^t e^{-u^2}\,du$$

SOLUTION: $L[\text{erf}(\sqrt{t})] = \frac{2}{\sqrt{\pi}}\int_0^\infty e^{-st}\int_0^{\sqrt{t}} e^{-x^2}\,dx\,dt$



By changing the order of integration, we get

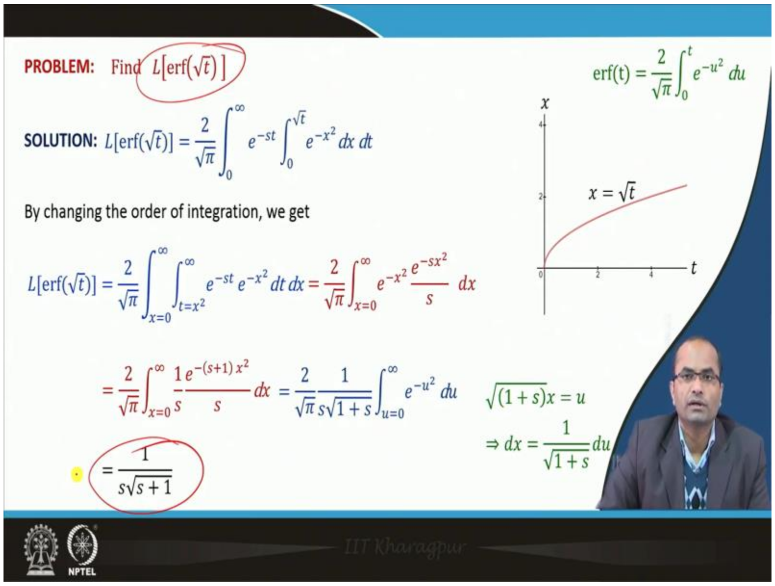
$$L[\text{erf}(\sqrt{t})] = \frac{2}{\sqrt{\pi}}\int_{x=0}^{\infty}\int_{t=x^2}^{\infty} e^{-st}e^{-x^2}\,dt\,dx = \frac{2}{\sqrt{\pi}}\int_{x=0}^{\infty} e^{-x^2}\frac{e^{-sx^2}}{s}\,dx$$

$$= \frac{2}{\sqrt{\pi}}\int_{x=0}^{\infty}\frac{e^{-(s+1)x^2}}{s}\,dx = \frac{2}{\sqrt{\pi}}\frac{1}{s\sqrt{1+s}}\int_{u=0}^{\infty} e^{-u^2}\,du$$

$\sqrt{(1+s)}\,x = u$

$\Rightarrow dx = \frac{1}{\sqrt{1+s}}\,du$

$$= \frac{1}{s\sqrt{s+1}}$$



So how to find the Laplace transform of error function of square root t for instance. So coming to the solution so we have by definition error function of t 2 over square root pi and then 0 to t e power minus u square du.

So the Laplace transform of error function of square root t is equal to 2 over square root pi and then we have this integration 0 to infinity e power minus st 0 to square root t. Because this is the error function 0 to square root t power minus x square with this 2 over square root pi. So that is Laplace. For Laplace we have e power minus st and dt this integral. So this is the Laplace of the error function of square root t.

Now we will change the order of integration for this these two integrals. So it the order is dx dt and now we will write rewrite it as dt dx. And then we will see some simplifications in this integral. So just note that if we take here the t axis here we take x axis. So this inner integral is x is varying from 0 to square root t, so x is equal to 0 means y axis and then we have x is equal to square root t to which is this line here.

And now we want to see this range what is for x we are varying from 0 to this square root t this graph and then for t it is going from 0 to infinity. So this is going to be our region of integration and we want to change it so keep the same but we have to now change the order. Now we will write in dt dx form that means first with respect to this t let us say. So that will go from this graph to infinity, that means so this x is equal to square root t is nothing but x square is equal to t, that means for the limits of t we will go from this x square to infinity.

And then so this t will go from x square to infinity and this x now will go from 0 to infinity, because this part we have covered already that it will move always from x square to infinity and then the possible values where the x will change that is 0 to infinity. So that will be the outer part. So our integral for x now the limits will be 0 to infinity, and for the dt part it will be x square to this infinity. So having this change of order of integration what we can now see here that e power minus x square because with respect to t we can integrate easily the inner integral.

Because here e power minus st that can be integrated, and that will be e power minus st over s, but then we have to put these limits x square to infinity. So e power this instead of t will have x square there. And when infinity that will become 0. So you will have only this portion here as a result of this integration of e power minus st dt e power minus s x square over this s. So we can combine so we have 1 over s there and then e power minus x square minus s x square.

So if we take x square common, we have this s plus 1. And this s we have any way considered so it is already there. And now so we can substitute the square root s plus 1 x s plus 1 x as u so that we get e power minus u square form. So having this substitution we have dx is equal to 1 over 1 plus square root s and du. So going for this substitution this 1 over 1 plus s and there is a s there. So that factor will come out and then we have e power minus u square du.

Concerning the limits they will remain the same because this is just a factor constant factor there. So 0 to infinity for u. And this integral we known this is square root pi by 2, so here we have 2 over square root pi and then square root pi by 2. So we get 1 over s square root s plus 1. So the Laplace transform of the error function of square root t is written in this form 1 over s square root s plus 1.

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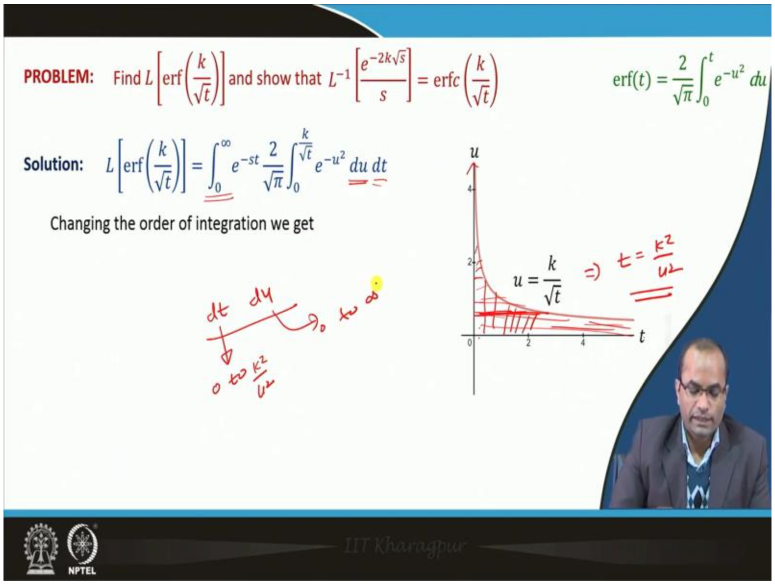

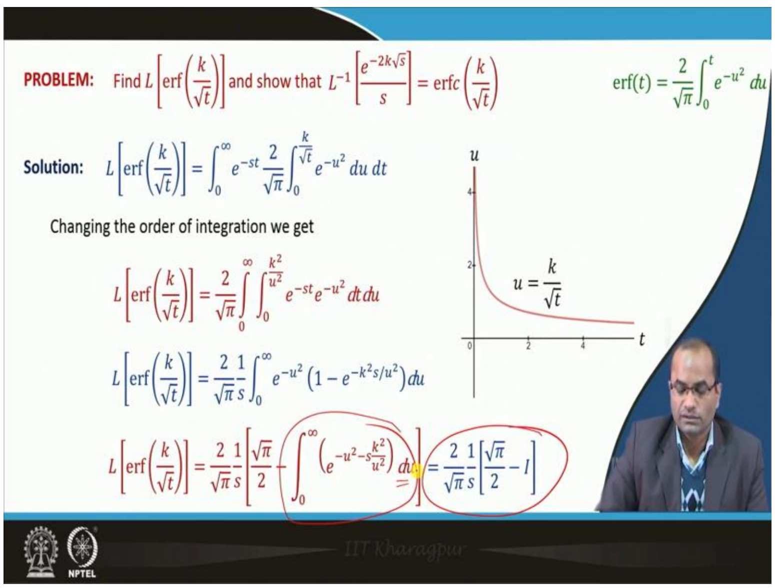

Well so we will find now the Laplace transform of the error function of k over square root t and finally we will show that L inverse of this after this computation the Laplace inverse of e power minus 2 k square root s over s will be error function the complimentary error function of k over is square root t. So coming to the solution now we know that definition of the error function and we will apply this Laplace transform on this error function of k over square root t.

So the Laplace transform of error function k over square root t is e power minus st and the definition we have for the error function 2 over square root pi 0 to now the limit will be this k over square root t and e power minus u square du. And then dt for this Laplace transform. Now we will change the order of integration and what we will get. So changing the order of integration means this is u is going from 0 to k over square root t.

So this u over k root k over square root t that is the graph here. So looking at the region of integration this u in this direction u is going from 0 to always to this graph. And then the overall range for the t is 0 to infinity and then we are covering this region of integration. Now for the change that means we want dt and then du there. So concerning the limits now the inner one that is t, so here this u over k over square rt t can be written as t is equal to k square over u square.

So now for the limits of t will go from 0 to this graph that means t will go from 0 to and this k square over u square. And then the outer one the fix limits for u will be 0 to infinity. So we can change now these limits and we will get for t as 0 to k square over u square and for u we will get 0 to infinity. So this error function k over square root t we have 2 over square root pi, we have 1 over s there because we can integrate now this one it is just depending on t.

So we have e power minus st over minus s which is already there 1 over s, and then we can put the upper limit and then the lower limit. So we will get 1 minus this e power minus k square s k square s divided by u square du. So the error function is 2 over square root pi we have 1 over s and this first integral which is e power minus u square over this u is square root pi by 2 and then with this minus here we have e power minus u square minus s k square over u square and then we have du.

So 2 over square root pi 1 over s we have here square root pi by 2 and then minus this is the integral I. So now we need to evaluate this integral I which is 0 to infinity e power minus u square minus s k square over u square over this u.

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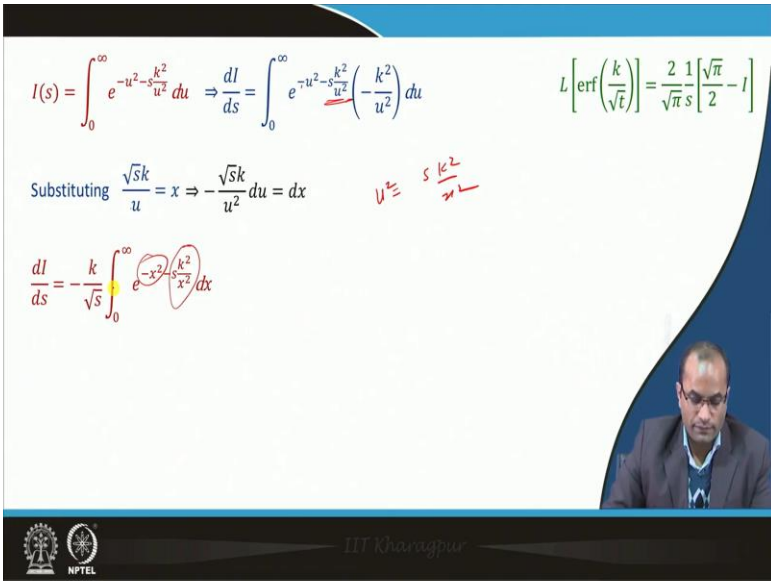

So we have this situation the Laplace transform of error function k over square root t we have rewritten in this form where this I is given by e power minus u square minus s k square over u square du. So we will evaluate this integral and then substitute here we will get the Laplace transform of this error function. So here the trick is that we will get this differential equation out of this integral, so if we differentiate this d so differentiate this Is so we will get dI over ds and here we have to differentiate now so that is e power minus u square minus s k square over u square. And the derivative of this one which is just minus k square over u square.

So that is the derivative we have for I, and then what we observe k square here there is a u square term, so the k square anyway can be taken outside. So we will make a substitution here that s k over u is x square root s k over u is x. So that here we will get a whole square term of x square and let us see what comes out after this substitution. So with this substitution we got that du is equal to dx and there is a factor here square root s k over u square.

So dI over ds is equal to minus k over so the k will appear from here and then we have square root s because of this. So we will get this minus k over square root s the limits will remain they will but with this minus sign again we can get back to the same, otherwise because of this u and x this position we will get infinity to 0. But this minus sign will take care for again getting back to 0 to infinity and this minus is still there.

So we have k over square root s e power minus this u square term, when we substitute this u there, so u square will become s k square over x square. So we are there again s k square over x square with this minus sign. And the second term there will become x square and the we have dx.

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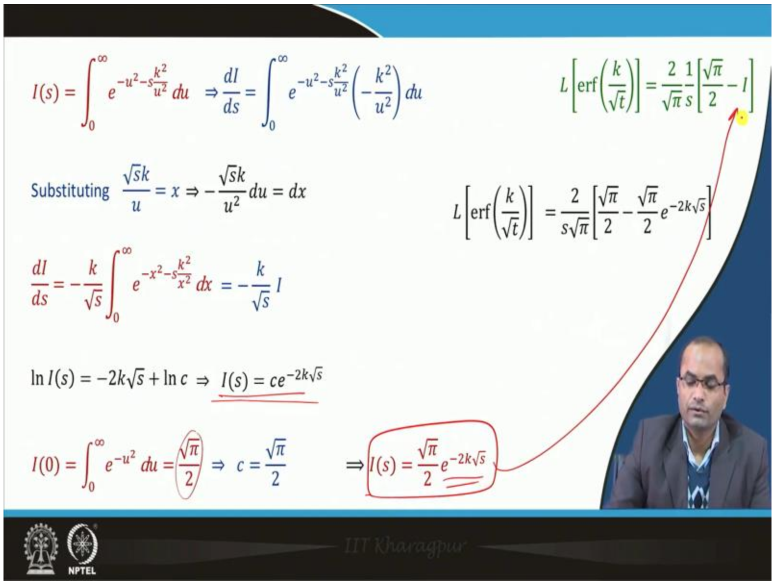

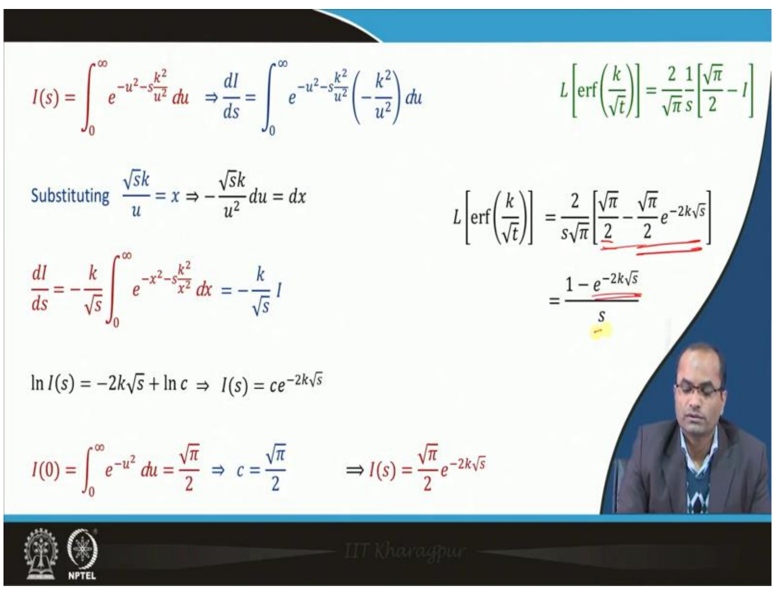

So what is interesting that we are getting back to the same integral which where we have started with e power minus u square minus k square as over u square du instead of u there we have x but the value of the integral is same. That means we got again this I, so we have minus k over square root s into I.

And now we can bring to this I to the left hand side in ds this so the basically separable equation and we can integrate this. So we will get logarithmic of this Is is equal to minus 2 k square root s and this logarithmic c just a arbitrary constant we have added there. So basically we got Is is equal to c e power minus 2 k square root s.

And to compute this constant we have already information that I is 0, so when this s is 0 so this was I so when s is 0 we are getting e power minus u square du well known this integral whose value is square root pi by 2. So with this we can get our constant, so the constant will be just the same value square root pi by 2. After substituting this we got this Is term square root pi by 2 e power minus k 2 k square root s.

So now we can substitute this there and what we will get? We will get square root this pi by 2 and for I we have substituted this square root pi by 2 e power minus 2 k square root s. So this is square root pi by 2 if we bring out we have 1 minus e power minus 2 k square root s over s and that is the Laplace transform of this error function which is written as k over square root t.

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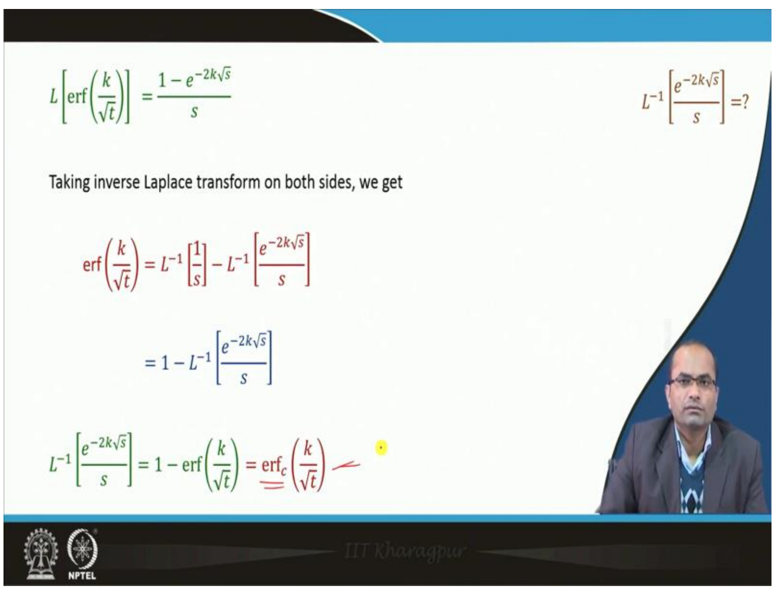

Now, having this result that the Laplace transform of the error function of k over square root t is given by this expression. What we need to know now, what is the Laplace inverse of e power minus 2 k square root s over s. And this is exactly the term sitting there. So what we will do now? If we take the inverse transform of this one, so this was the second question of in this problem that what is the Laplace transform of this function which is actually after this evaluation we do see that there is such a term exist in this Laplace transform of error function of k over square root t.

So now if we take the inverse Laplace transform here we will get error function k over square root t is equal to Laplace inverse 1 over s and Laplace inverse e power minus 2 k square root s over s. So the linearity of this Laplace transform is used here and this value we know Laplace inverse of 1 over s we have 1 there and then minus this Laplace inverse of this function e power minus 2 k square root s over s.

So what we get The Laplace inverse of e power minus 2 k square root s over s? Is 1 minus this error function can go to the right hand side? So we have 1 minus error function of k over square root t. And this we have seen at the beginning that this 1 minus error function k over square root t we can also write as the complimentary error function of this argument k over square root t. So there could be various other combinations of the error function or form of the error function we can evaluate the Laplace transform in this way.

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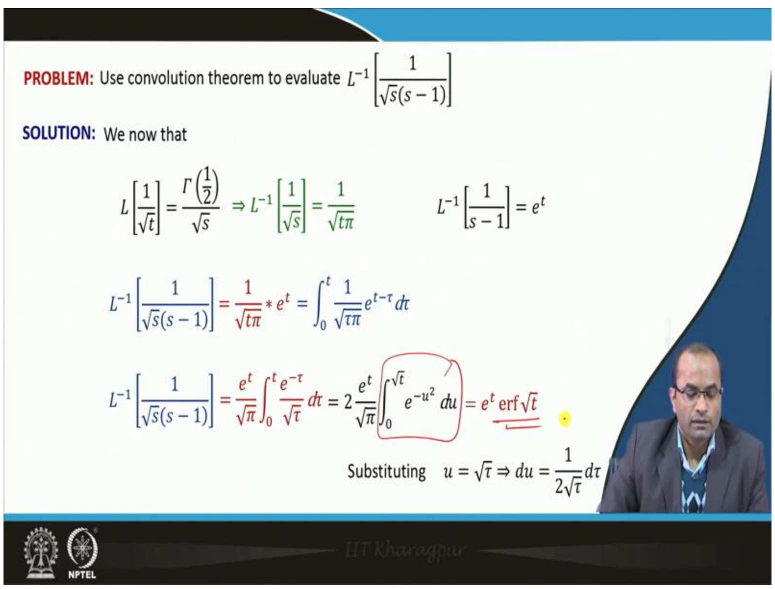

Here we have this using this convolution theorem which we have already discussed in the previous lecture we can evaluate for instance the Laplace inverse of 1 over square root s s minus 1 and the result will be in terms of the error function which we can see in a minute. So we know that Laplace of 1 over square root t is gamma half over square s that means the Laplace inverse of 1 over square root t over 1 over square root s is 1 over square root t pie.

Because gamma is half of square root pie so that go to this side. So we got this Laplace inverse and we also know the Laplace inverse of 1 over s minus 1 that power t. And now this convolution theorem says that the Laplace inverse of the product of these 2 Laplace can be given by the convolution product of the two functions.

So one function is 1 over square root t pi, another one is e power t. So this convolution we have to compute now. That means 0 to t 1 over square root tau pi in e power t minus tau, so e power t can be taken outside, so we have e power t 1 over square root pi. And e power minus tau over square root tau d tau, that is the remaining portion here. So if we substitute here u is equal to square root tau what we will get?

The du will bed d tau this tau tau will get cancel, because 1 over square root tau this d tau will be 2 du. So after substituting this there we will get 2 e power t over square root pi and this 0 to square root t e power minus u square du where we have now we can use the definition of the error function which is 2 over square root pi 0 to square root t e power minus u square du.

So we have this e power t and error function of this square root t. So this is another way where we can get the inverse of some functions and at the end we got this integral which is nothing but the error function of square root of t.

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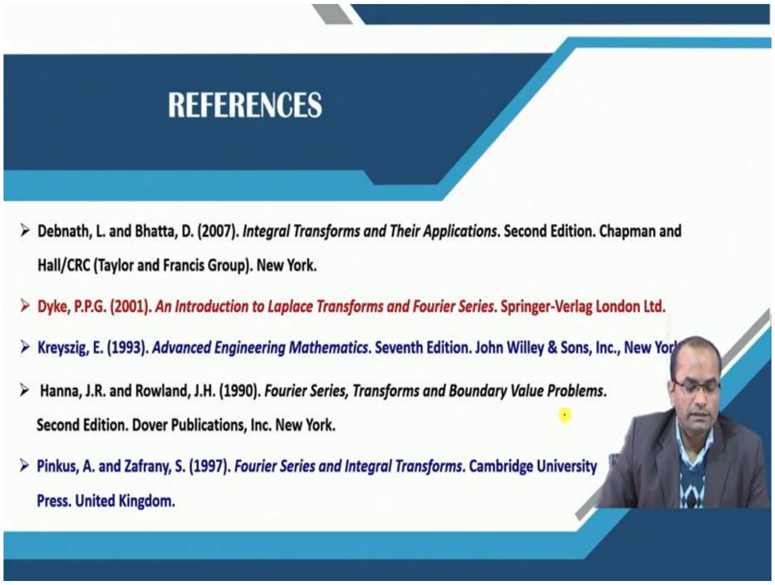

So these are the references w have used for preparing this lecture.

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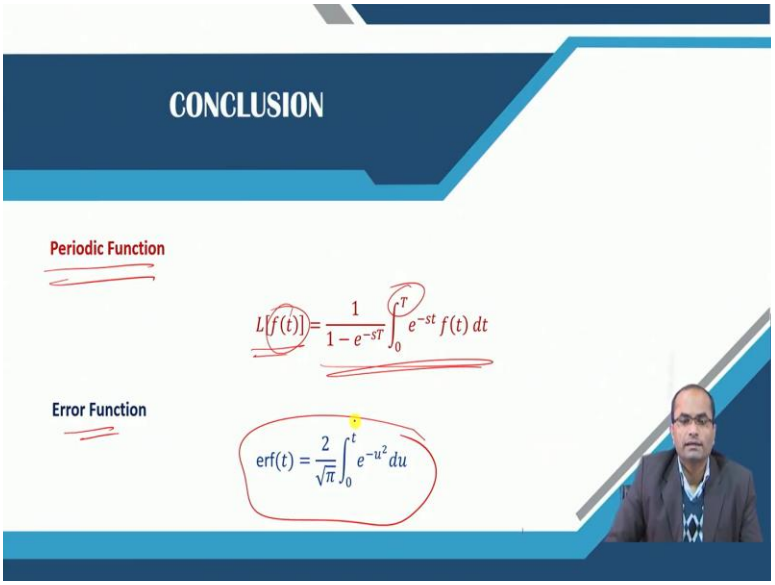

And now, coming to the conclusion, we have discussed two functions here, one is the periodic function, so if f is periodic function its Laplace transform can be evaluated using this integral which is no more the integral of this 0 to infinity. And then the error function which is defined by this integral and we have seen various forms of the error function whose Laplace transform can easily be evaluated. So that is all for this lecture and thank you for your attention.