Advanced Engineering Mathematics
Lecture 18

1 Integral Calculus

Triple Integral(Change of variable).

$$\iiint_D f(x,y,z)\,dx\,dy\,dz = \iiint_{D_1} f(x(u,v,w),y(u,v,w),z(u,v,w))\,|J|\,du\,dv\,dw$$
$$= \iiint_{D_1} f(u,v,w)\,|J|\,du\,dv\,dw,$$

where $|J| = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial z}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$, the Jacobian of the transformation.

Example 1.1. Spherical polar co-ordinate: $x = r\cos\theta\sin\phi, y = r\sin\theta\sin\phi, z = r\cos\theta \Rightarrow |J| = r^2\sin\theta$.

Example 1.2. Cyndrical polar co-ordinate: $x = r\cos\theta, y = r\sin\theta, z = z \Rightarrow |J| = r$

Example 1.3. Evaluate, $I = \iiint xyz\,dx\,dy\,dz$, over the region bounded by $x = 0, y = 0, z = 0, x + y + z = 1$.

Sol.

$$\begin{aligned}
I &= \iiint xyz\,dx\,dy\,dz,\\
&= \int_{x=0}^{1}\int_{y=0}^{1-x}\int_{z=0}^{1-x-y} xyz\,dx\,dy\,dz\\
&= \int_{x=0}^{1} x\,dx \int_{y=0}^{1-x} y\,dy \int_{z=0}^{1-x-y} z\,dz\\
&= \int_{x=0}^{1} x\,dx \int_{y=0}^{1-x} \frac{1}{2}y(1-x-y)^2\,dy\\
&= \frac{1}{2}\int_{x=0}^{1} x\,dx \int_{y=0}^{1-x} y(1-x-y)^2\,dy\\
&= \frac{1}{2}\int_{x=0}^{1} x\,dx \left[-\frac{1}{3}y(1-x-y)^3\Big|_0^{1-x} + \frac{1}{3}\int_0^1 (1-x-y)^3.1.\,dy\right]\\
&= \frac{1}{720}
\end{aligned}$$

Example 1.4. Evaluate, $I = \iiint_D x^2\,dx\,dy\,dz$, where D is the sphere $x^2 + y^2 + z^2 \leq 1$.

Sol. Changing the variables to spherical polar co-ordinate,

$$\begin{aligned}
I &= \int_0^{2\pi}\int_0^{\pi}\int_0^{1} r^2\sin^2\theta\cos^2\phi\,(r^2\sin\theta)\,dr\,d\theta\,d\phi \\
&= \int_0^{2\pi}\cos^2\phi\,d\phi\int_0^{\pi}\sin^2\theta\,d\theta\int_0^{1} r^4\,dr \\
&= \frac{1}{5}\int_0^{2\pi}\cos^2\phi\,d\phi\int_0^{\pi}\sin^2\theta\,d\theta \\
&= \frac{1}{5}\times 4\times 2\int_0^{\frac{\pi}{2}}\cos^2\phi\,d\phi\int_0^{\frac{\pi}{2}}\sin^2\theta\,d\theta \\
&= \frac{4\pi}{15}
\end{aligned}$$

Example 1.5. Evaluate, $I = \iiint_D e^{\sqrt{\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}}}\,dx\,dy\,dz$, where D is the region $\sqrt{\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}} \leq 1$.

Sol. Changing the variables to polar , $\frac{x}{a} = r\sin\theta\cos\phi$, $\frac{y}{b} = r\sin\theta\sin\phi$, $\frac{z}{c} = r\cos\theta$.

$$\begin{aligned}
I &= \int_0^{2\pi}\int_0^{\pi}\int_0^{1} e^r\,(abc\,r^2\sin\theta)\,dr\,d\theta\,d\phi \\
&= abc\int_0^{2\pi}d\phi\int_0^{\pi}\sin\theta\,d\theta\int_0^{1} r^2e^r\,dr \\
&= 4\pi abc\int_0^{1} r^2e^r = 4\pi abc\times(e-2) = 4\pi abc(e-2)
\end{aligned}$$

Volume of a solid by triple integral. $V = \iiint_D dx\,dy\,dz$ It can be shown that on a closed (region) domain D bounded below and above by the surface $z = \phi(x,y), z = \psi(x,y), \psi(x,y) \geq \phi(x,y)$.

$$\begin{aligned}
V &= \iiint_D dx\,dy\,dz \\
&= \iiint_{\phi(x,y)}^{\psi(x,y)} dx\,dy\,dz
\end{aligned}$$

Example 1.6. Find the volume of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} = 1$.

Sol.

$$\begin{aligned}
V &= \iiint_{-c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}}^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} dx\,dy\,dz \\
&= 2c\iint\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}\,dx\,dy \qquad \left(\frac{x}{a} = r\cos\theta,\quad \frac{y}{b} = r\sin\theta,\quad |J| = abr\right) \\
&= 2c\int_0^{\pi}\int_0^{1}\sqrt{1-r^2}\,abr\,dr\,d\theta \\
&= \frac{4}{3}\pi abc
\end{aligned}$$