Advanced Engineering Mathematics
Lecture 2

1 Mean Value Theorem

Theorem 1.1. *(Rolle's Theorem) If a function f, defined on $[a, b]$, is*
i) continuous on $[a, b]$,
ii) derivable on (a, b),
iii) $f(a) = f(b)$.
Then, $\exists$ at least one real number $c \in (a, b)$ such that $f'(c) = 0$.

Example 1.1. Prove that between any two real roots of $e^x \sin x = 1$, there exists a real root or at least one real root of $e^x \cos x + 1 = 0$.

Sol: Let us consider $f(x) = e^{-x} - \sin x$. Let α and β be two real roots of $f(x) = 0$. Then, $f(\alpha) = f(\beta) = 0$. Both e^{-x} and $\sin x$ are continuous for all real x and derivable everywhere. f satisfies all the properties of *Rolle's Theorem* on $[\alpha, \beta]$, therefore there exists a $\tau \in (\alpha, \beta)$ such that

$$\begin{aligned} f'(\tau) &= 0, \quad \alpha < \tau < \beta \\ \Rightarrow -e^{-\tau} - \cos\tau &= 0 \\ \Rightarrow \quad e^{\tau}\cos\tau + 1 &= 0 \end{aligned}$$

Thus, there exists at least one real root $\tau \in (\alpha, \beta)$ of the equation $e^x \cos x + 1 = 0$.

Example 1.2. Verify Rolle's theorem for $f(x) = x^3$ on $[-1, 1]$.

Sol.f is continuous on $[-1, 1]$. f is differentiable on $(-1, 1)$. $f(-1) \neq f(1) \Rightarrow$ The last criteria in Rolle's Theorem is not satisfied. Here $f'(x) = 3x^2$ $\forall x \in (-1, 1)$. Obviously $x = 0 \in (-1, 1)$ such that $f'(0) = 0$. This example shows that the condition in Rolle's theorem is not necessary.

Theorem 1.2. *(Lagrange's Mean Value Theorem) Let a real valued function f defined on $[a, b]$ be*
i) continuous on $[a, b]$,
ii) differentiable in (a, b).
Then there exists at least one point $c \in (a, b)$ such that $\frac{f(b)-f(a)}{b-a} = f'(c)$.

Example 1.3. Apply MVT to prove that

$$\frac{x}{1+x} < \log_e(1+x) < x, \quad \forall x > 0$$

Sol. Let $f(x) = \log_e(1+x)$. then the function f is continuous and differentiable on $[0, x]$. By MVT

$$\begin{aligned} f(x) - f(0) &= x f'(\theta x), \quad \text{where } 0 < \theta < 1 \\ \Rightarrow \log(1+x) &= \frac{x}{1+\theta x} \end{aligned}$$

Now, $0 < \theta < 1$ and $x > 0$ then

$$\begin{aligned} & 0 < \theta x < x \\ \Rightarrow \quad & 1 < 1 + \theta x < 1 + x \\ \Rightarrow \quad & \frac{1}{1+x} < \frac{1}{1+\theta x} < 1 \\ \Rightarrow \quad & \frac{x}{1+x} < \frac{x}{1+\theta x} < x \\ \Rightarrow \quad & \frac{x}{1+x} < \log_e(1+x) < x \quad \forall x > 0 \end{aligned}$$

Alternative Statement. If a real valued function f is continuous in $[a, a+h]$, $h > 0$ and it is differentiable in $(a, a+h)$. The there exists a proper fraction $\theta (0 < \theta < 1)$ such that $f(a+h) - f(a) = hf'(a+\theta h)$.

Example 1.4. By LMVT, prove that $0 < \frac{1}{\log_e(1+x)} - \frac{1}{x} < 1$.

Sol. Choose $f(x) = \log_e(1+x)$. Left as an exercise.