Advanced Engineering Mathematics
Lecture 22

Linear equations. Consider the following linear differential equation

$$\frac{dy}{dx} + P(x)y = Q(x).$$

Here integrating factor is IF=$e^{\int p(x)\,dx}$, then the solution is given by

$$\int d(\phi(x)y) = \int Q(x)\phi(x)\,dx + c$$

Example 0.1. Solve $\frac{dy}{dx} + \frac{4x}{x^2\pi}y = \frac{1}{(x^2+1)^3}$

Sol. Given $P(x) = \frac{4x}{x^2+1}$ and $Q(x) = \frac{1}{(x^2+1)^3}$. Integrating factor, $\phi(x) = e^{\int P(x)\,dx}$.

$$\phi(x) = e^{\int \frac{4x}{x^2+1}\,dx} = e^{2\log(x^2+1)} = (x^2+1)^2.$$

Then,

$$(x^2+1)^2\frac{dy}{dx} + (x^2+1)^2\frac{4x}{x^2+1}y = (x^2+1)^2\frac{1}{(x^2+1)^3}$$
$$\Rightarrow d(y(x^2+1)^2) = \frac{dx}{x^2+1}$$
$$\Rightarrow y(x^2+1)^2 = \tan^{-1} + c$$

Second order differential equation. Let $a(x)y'' + b(x)y' + c(x)y = 0$ be a second order differential equation, where $y(x)$ is the unknown, x is the independent variable, a, b, c are the functions of x. Let $a(x), b(x), c(x)$ are all constants say α, β, γ, respectively and $y = e^{mx}$ be an arbitrary solution, then $\alpha m^2 + \beta m + \gamma = 0$ becomes an auxillary equation.

Case 1. If the roots of the auxillary equations are distinct, then solution $y(x) = C_1e^{m_1x} + C_2e^{m_2x}$.

Case 2. If the roots of the auxillary equations are repeated, then solution $y(x) = (C_1 + C_2x)e^{mx}$.

Case 3. If the roots of the auxillary equations are complex, then solution $y(x) = e^{\alpha x}(C_1 \cos\beta x + C_2 \sin\beta x)$, where $m = \alpha \pm i\beta$.

Example 0.2. Solve the second order differential equation $4y'' + 4y' - 3y = 0$.

Sol. Let $y = e^{mx}$ be the arbitrary solution, then the auxillary equation is

$$4m^2 + 4m - 3 = 0 \Rightarrow (m - \frac{1}{2})(m + \frac{3}{2}) = 0 \Rightarrow m = \frac{1}{2}, -\frac{3}{2}$$

The general solution is $y(x) = C_1e^{\frac{1}{2}x} + C_2e^{-\frac{3}{2}x}$.

Example 0.3. Solve the second order ODE: $y'' - 24y' + 144y = 0$.

Sol.Let $y = e^{mx}$ be arbitrary solution, then the auxillary equation becomes $m^2 - 24m + 144 = 0 \Rightarrow (m-12)^2 = 0 \Rightarrow m = 12, 12$. The general solution of the given equation is $y(x) = (C_1 + C_2x)e^{12x}$.

Example 0.4. Solve the second order ODE: $y'' + 8y' + 25y = 0$.

Sol. Let $y = e^{mx}$ be the arbitrary solution. The auxillary equation is $m^2 + 8m + 25 = 0$. The roots are given by, $m = \frac{-8 \pm 6i}{2} = -4 \pm 3i$. The required solution is $y(x) = e^{-4x}(C_1 \cos 3x + C_2 \sin 3x)$.

Example 0.5. Solve the ODE: $2y''' - 7y'' + 7y' - 2y = 0$

Sol. Let $y = e^{mx}$ be the arbitrary solution. The auxillary equation is $2m^3 - 7m^2 + 7m - 2 = 0 \Rightarrow (m-1)(m-2)(m-3) = 0 \Rightarrow m = 1, 2, \frac{1}{2}$. The required general solution is $y(x) = C_1e^x + C_2e^{2x} + C_3e^{\frac{x}{2}}$.

Example 0.6. Solve the ODE: $y'''' - y''' - 9y'' - 11y' - 4y = 0$.

Sol. Let $y = e^{mx}$ be the arbitrary solution. The auxillary equation is $m^4 - m^3 - 9m^2 - 11m - 4 = 0 \Rightarrow (m+1)^3(m-4) = 0 \Rightarrow m = -1, -1, -1, 4$. The required general solution is $y(x) = (C_1 + C_2x + C_3x^2)e^{-x} + C_4e^{4x}$.

Example 0.7. Solve the ODE: $y'''' + a^4y = 0$.

Let $y = e^{mx}$ be the arbitrary solution. The auxiliary equation is $m^4 + a^4 = 0 \Rightarrow m = -\frac{a}{\sqrt{2}} \pm i\frac{a}{\sqrt{2}}$ and $m = -\frac{a}{\sqrt{2}} \pm i\frac{a}{\sqrt{2}}$. The required general solution is $y(x) = e^{-\frac{a}{\sqrt{2}}}(C_1 \cos \frac{a}{\sqrt{2}}x + C_2 \sin \frac{a}{\sqrt{2}}x) + e^{\frac{a}{\sqrt{2}}}(C_3 \cos \frac{a}{\sqrt{2}}x + C_4 \sin \frac{a}{\sqrt{2}}x)$.