Advanced Engineering Mathematics
Lecture 25

Finite Difference

Let $f(x)$ be a given function in x and $y_0, y_1, \ldots, y_n$ be its corresponding values at $x_0, x_1, \ldots, x_n$, respectively. The x-values are called 'Arguments', and y-values are called 'Entries'. In general, the difference between any two consecutive arguments may not be the same. We can write the given data in the below tabular form for the computational convenience.

Table 1.1					
x	x_0	x_1	x_2	$\ldots$	x_n
$f(x)$	$f(x_0)$	$f(x_1)$	$f(x_2)$	$\ldots$	$f(x_n)$

Then, the values $y_1 - y_0 = f(x_1) - f(x_0)$, $y_2 - y_1$, $y_3 - y_2, \ldots$ are called the first order difference and it is denoted by Δy.

Forward difference operator. The operator Δ is called the Forward Difference Operator, where $\Delta y_n = y_{n+1} - y_n$, for examples, $\Delta y_0 = y_1 - y_0$, $\Delta y_1 = y_2 - y_1, \ldots$. Also, $\Delta^2 y_0 = \Delta(\Delta y_0) = \Delta(y_1 - y_0) = y_2 - 2y_1 + y_0$ and higher powers of the operator, when required, can be obtained using the same procedure.

Backward difference operator. The operator ∇ is called the Backward Difference Operator, where $\nabla y_n = y_n - y_{n-1}$, for examples, $\nabla y_1 = y_1 - y_0$, $\nabla y_2 = y_2 - y_1, \ldots$. Also, $\nabla^2 y_2 = \nabla(\nabla y_1) = \nabla(y_1 - y_0) = y_2 - 2y_1 + y_0$ and higher powers of the operator can be obtained following the same steps, when required. In general, $\nabla^2 y_n = y_n - 2y_{n-1} + y_{n-2}$.

For the given data in table 1.1, the k-th term of a sequence is given by

$$y_k = (1+\Delta)^k y_0 = \left[1 + {}^kC_1\Delta + {}^kC_2\Delta^2 + \ldots\right] y_0 = y_0 + {}^kC_1\Delta y_0 + {}^kC_2\Delta^2 y_0 + \ldots \qquad (1)$$

Example 1. For the given data, find the values of $\Delta^4 y_0$ and $\Delta^5 y_0$, by constructing a Forward Difference Table.

Table 1.2						
x	1	2	3	4	5	6
$f(x)$	4	15	40	85	156	259

Solution. Using the table 1.3 and the formula we can easily see that $\Delta^4 y_0 = 0$ and $\Delta^5 y_0 = 0$.

Table 1.3						
Forward Difference Table						
x	$f(x)$	Δf	$\Delta^2 f$	$\Delta^3 f$	$\Delta^4 f$	$\Delta^5 f$
1	4					
		11				
2	15		14			
		25		6		
3	40		20		0	
		45		6		0
4	85		26		0	
		71		6		
5	156		32			
		103				
6	259					

Example 2. Find the 7th term of the sequence 2, 9, 28, 65, 126, 217 for the nodal points 0, 1, 2, 3, 4, 5, respectively.
Solution. Using the formula (1) and table 1.4, we get

$$\begin{aligned} y_6 &= (1+\Delta)^6\, y_0 \\ &= 1 + {}^6C_1\Delta y_0 + {}^6C_2\Delta^2 y_0 + {}^6C_3\Delta^3 y_0 + {}^6C_4\Delta^4 y_0 + {}^6C_5\Delta^5 y_0 + {}^6C_6\Delta^6 y_0 \\ &= 2 + 6\cdot 7 + 15\cdot 12 + 20\cdot 6 + 15\cdot 0 + 6\cdot 0 + 1\cdot 0 = 344. \end{aligned}$$

Table 1.4							
Forward Difference Table							
x	$y = f(x)$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$	$\Delta^6 y$
0	2						
		7					
1	9		12				
		19		6			
2	28		18		0		
		37		6		0	0
3	65		24		0		
		61		6			
4	126		30				
		91					
5	217						

Interpolation

Let $x_0, x_1, \ldots, x_n$ be the nodal points such that $x_1 - x_0 = x_2 - x_1 = x_n - x_{n-1}$, i.e., the points are equidistant. Let $y = f(x)$ be a function, whose values are $y_0, y_1, \ldots, y_n$ at $x_0, x_1, \ldots, x_n$, respectively.
Our goal is to find a function $f(x)$ such that $y_i = f(x_i)$ must be satisfied. From the sequence

of functions $\{\phi(x)\}$, there is unique nth degree polynomial $P_n(x)$ such that $y_i = P_n(x_i), \forall i = 0, 1, \ldots, n$. This $P_n(x)$ is called 'Interpolating Polynomial' such that $|f(x) - P_n(x)| < \varepsilon$.