Advanced Engineering Mathematics
Lecture 27

Lagrange Interpolation

Let us consider the data table 1.1. Unlike the above cases, here we have, $x_1 - x_0 \neq x_2 - x_1 \neq x_3 - x_2 \neq \dots$ and so on.
The interpolating polynomial is

$$\begin{aligned} f(x) &= P_n(x) \\ &= a_0(x-x_1)(x-x_2)\dots(x-x_n) + a_1(x-x_0)(x-x_2)\dots(x-x_n) \\ &\quad + \dots + a_n(x-x_0)(x-x_2)\dots(x-x_{n-1}) \end{aligned}$$

where,

$$a_0 = \frac{f(x_0)}{(x_0-x_1)(x_0-x_2)\dots(x_0-x_n)}, \qquad a_n = \frac{f(x_n)}{(x_n-x_0)(x_n-x_2)\dots(x_n-x_{n-1})}.$$

Therefore,

$$\begin{aligned} f(x) &= P_n(x) \\ &= \frac{(x-x_1)(x-x_2)\dots(x-x_n)}{(x_0-x_1)(x_0-x_2)\dots(x_0-x_n)} f(x_0) + \frac{(x-x_0)(x-x_2)\dots(x-x_n)}{(x_1-x_0)(x_1-x_2)\dots(x_1-x_n)} f(x_1) \\ &\quad + \dots + \frac{(x-x_0)(x-x_2)\dots(x-x_{n-1})}{(x_n-x_0)(x_n-x_2)\dots(x_n-x_{n-1})} f(x_n). \end{aligned} \tag{1}$$

Example 1. Given the below data, using Lagrange Interpolation formula, find the value of $y(10)$.

Table 2.5				
x	5	6	9	11
$f(x)$	12	13	14	16

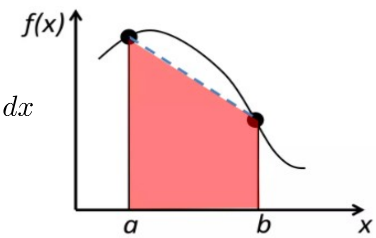
Solution. We have the required interpolating polynomial as follows:

$$\begin{aligned} y &= P_n(x) = P_3(x) \\ &= \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} f(x_0) + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} f(x_1) \\ &\quad + \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} f(x_2) + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} f(x_3) \\ \implies y(10) &= \frac{(10-6)(10-9)(10-11)}{(5-6)(5-9)(5-11)} 12 + \frac{(10-5)(10-9)(10-11)}{(6-5)(6-9)(6-11)} 13 \\ &\quad + \frac{(10-5)(10-6)(10-11)}{(9-5)(9-6)(9-11)} 14 + \frac{(10-5)(10-6)(10-9)}{(11-5)(11-6)(11-9)} 16 \\ &= 14.6666. \end{aligned}$$

Numerical Integration

Trapezoidal Rule

Let $x_0, x_1, \ldots, x_n$ are equidistant nodal points. Also, say, the step length is $h = x_n - x_{n-1}$, then



$$
\begin{aligned}
I &= \int_a^b f(x)\,dx = \int_{x_0}^{x_n = x_0 + nh} f(x)\,dx \\
&= \int_{x_0}^{x_0+h} f(x)\,dx + \int_{x_0+h}^{x_0+2h} f(x)\,dx + \ldots + \int_{x_0+(n-1)h}^{x_0+nh} f(x)\,dx \\
&= \frac{y_0 + y_1}{2}h + \frac{y_1 + y_2}{2}h + \ldots + \frac{y_n + y_{n-1}}{2}h \\
&= \frac{y_0 + y_n}{2}h + h\left(y_1 + y_2 + \ldots + y_{n-1}\right) \\
&= \frac{h}{2}\left[\left(y_0 + y_n\right) + 2\left(y_1 + y_2 + \ldots + y_{n-1}\right)\right]. \qquad (2)
\end{aligned}
$$

The equation (2) is called the Trapezoidal Rule.

Example 1. Find the value of $\int_{-3}^{3} x^4\,dx$ by Trapezoidal rule.
Solution. Here, $y = f(x) = x^4$. The given interval is $[-3, 3]$. Then, $h = \frac{3-(-3)}{6} = 1$ as we have considered $n = 6$ sub-intervals.

x	-3	-2	-1	0	1	2	3
y	81	16	1	0	1	16	81

$$
\begin{aligned}
I &= \int_{-3}^{3} x^4\,dx \\
&= \frac{1}{2}\left[(81 + 81) + 2\left(16 + 1 + 0 + 1 + 16\right)\right] \\
&= \frac{1}{2}\left[162 + 68\right] \\
&= 115.
\end{aligned}
$$

The exact vale of the integration: $I = \int_{-3}^{3} x^4\,dx = \left[\frac{x^5}{5}\right]_{-3}^{3} = 97.2.$

Simpson's $\frac{1}{3}$rd Rule

Lastly, we briefly give the formula for the Simpson's $\frac{1}{3}$rd Rule as follows:

$$
\int_{x_0}^{x_n} f(x)\,dx = \frac{h}{3}\left[\left(y_0 + y_n\right) + 2\left(y_2 + y_4 + \ldots\right) + 4\left(y_1 + y_3 + \ldots\right)\right]. \qquad (3)
$$

Example 1. Proceeding with same example and using Simpson's $\frac{1}{3}$rd rule.

Solution. In short, we get

$$
\begin{aligned}
I &= \int_{-3}^{3} x^4 \, dx \\
&= \frac{h}{3} \left[(y_0 + y_6) + 2\,(y_2 + y_4) + 4\,(y_1 + y_3) \right] \\
&= \frac{1}{3} \left[(81 + 81) + 2\,(1 + 1) + 4\,(16 + 0 + 16) \right] \\
&= \frac{h}{3} \left[162 + 4 + 128 \right] \\
&= 98.
\end{aligned}
$$

Which is clearly more closer to the exact value of the integration than the Trapezoidal rule's outcome.