Advanced Engineering Mathematics
Lecture 29

Linear Combination

Let V be a vector space. Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be the elements in V. A vector $\beta \in V$ is said to be a linear combination of $\alpha_1, \alpha_2, \ldots, \alpha_n$, if β can ne expressed as

$$\beta = c_1\alpha_1 + c_2\alpha_2 + \ldots + c_n\alpha_n = \sum_{i=1}^{n} c_i\alpha_i, \tag{1}$$

where $c_1, c_2, \ldots, c_n$ are constants in $\mathbb{F}$.

Example 1. $\beta = (1,1)$, $\alpha_1 = \left(\frac{1}{2}, \frac{1}{2}\right)$ and $\beta_1 = \left(\frac{1}{2}, \frac{1}{2}\right) \implies \beta = \alpha_1 + \alpha_2$ OR $\alpha_1 = \left(\frac{1}{3}, \frac{3}{4}\right)$ and $\beta_1 = \left(\frac{3}{4}, \frac{1}{4}\right) \implies \beta = \alpha_1 + \alpha_2$.

Linear Dependence and Independence of Vectors

Let $\alpha_1, \alpha_2, \ldots, \alpha_n \in V$ and $c_1, c_2, \ldots, c_n \in \mathbb{F}$ such that $c_1\alpha_1 + c_2\alpha_2 + \ldots + c_n\alpha_n = \theta \in V$ with not all c_i's zero. Such a set $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ is said to be linearly dependent.
If $c_1\alpha_1 + c_2\alpha_2 + \ldots + c_n\alpha_n = \theta \in V$, only when $c_1 = c_2 = \ldots = c_n = 0$, then the set $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ is said to be linearly independent.

Example 1. Check whether the set $\left\{(1,1), \left(\frac{1}{2}, \frac{1}{2}\right)\right\}$ is linearly dependent or not.
Solution. Let $\alpha_1 = (1,1)$ and $\alpha_2 = \left(\frac{1}{2}, \frac{1}{2}\right)$ such that

$$\begin{aligned} & c_1\alpha_1 + c_2\alpha_2 = \theta \\ \implies & (c_1, c_1) + \left(\frac{c_2}{2}, \frac{c_2}{2}\right) = (0,0) \\ \implies & c_1 + \frac{c_2}{2} = 0, \quad c_1 + \frac{c_2}{2} = 0 \\ \implies & c_1 + \frac{c_2}{2} = 0 \implies c_1 = -\frac{c_2}{2}. \end{aligned}$$

Taking $c_1 = 1$, $c_2 = -2$, i.e., $\alpha_1 = -c_2\alpha_2 = 2\alpha_2$, which implies that $\left\{(1,1), \left(\frac{1}{2}, \frac{1}{2}\right)\right\}$ is linearly dependent.

Example 2. Check whether the set $\{(1,0), (0,1)\}$ is linearly dependent or not.
Solution. Let $\alpha_1 = (1,0)$ and $\alpha_2 = (0,1)$ such that

$$\begin{aligned} & c_1\alpha_1 + c_2\alpha_2 = \theta \\ \implies & (c_1, 0) + (0, c_2) = (0,0) \\ \implies & (c_1, c_2) = (0,0) \\ \implies & c_1 = 0, \quad c_2 = 0. \end{aligned}$$

This implies that $\{(1,0), (0,1)\}$ is linearly independent.

Example 3. Check whether the set $\{(1,0,0), (0,1,0), (0,0,1)\}$ is linearly dependent or not.

Example 4. Check whether the set $\{(1,2,4), (2,4,8)\}$ is linearly dependent or not.

Linear Span of a Set

Let $V \neq \emptyset$ be a vector space and let $S = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be a set of vectors/elements in V. If the set S is such that every element of V is expressed as a linear combination of elements in S, then S is called the linear span of V.

Basis

Let $V \neq \emptyset$ be a vector space. A set $S \subset V$ is said to be a basis of V if

(i) S is a linearly independent set.

(ii) S generates V.

Example 1. Let $V = \mathbb{R}^3$ be the vector space with respect to the usual operations over $\mathbb{R}$. Let $S = \{(1,0,0), (0,1,0), (0,0,1)\} \subset \mathbb{R}^3$. Verify whether S is a basis or not.

Solution. Let $c_1, c_2, c_3 \in \mathbb{F}$ such that $c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3 = \theta = (0,0,0)$, which implies $c_1 = c_2 = c_3 = 0$. Hence, S is a linearly independent set.
Now let $\xi = (a,b,c) \in \mathbb{R}^3$, then we can write

$$\xi = (a,b,c) = a\,(1,0,0) + b\,(0,1,0) + c\,(0,0,1) = a\alpha_1 + b\alpha_2 + c\alpha_3.$$

This shows that S is a basis of V.

Example 2. Let $V = \mathbb{R}^3$ be the vector space with respect to the usual operations over $\mathbb{R}$. Let $S = \{(1,2,1), (2,1,1), (1,1,2)\} \subset \mathbb{R}^3$. Verify whether S is a basis of $\mathbb{R}^3$ or not.

Solution. Let $c_1, c_2, c_3 \in \mathbb{F}$ such that $c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3 = \theta = (0,0,0)$, which implies $c_1 = c_2 = c_3 = 0$

$$\begin{aligned} c_1 + 2c_2 + c_3 &= 0, \\ 2c_1 + c_2 + c_3 &= 0, \\ c_1 + c_2 + 2c_3 &= 0, \\ \implies c_1 = c_2 = c_3 &= 0. \end{aligned}$$

Hence, S is a linearly independent set.

Dimension

The number of elements in a basis of a vector space is called the dimension of V (or rank of V), and it is denoted by $\dim V$ or $\dim(V)$.

Example 1. Note that the dimension of the vector space in the previous example is: $\dim(V) = 3$. Similarly, $\dim(\mathbb{R}^n) = n$.

Example 2. Let $W \subset \mathbb{R}^3$ given by $W = \left\{(x,y,z) \in \mathbb{R}^3 : x + 2y + z = 0, 2x + y + 3z = 0\right\}$. then, find the basis and dimension of W.

Solution. Let $\xi = (a, b, c) \in W^3$ such that

$$\begin{aligned} a + 2b + c &= 0, \\ 2a + b + 3c &= 0, \\ \implies \frac{a}{5} = \frac{b}{-1} = \frac{c}{-3} &= k(\text{say}) \\ \implies a = 5k, b = -k, c &= -3k \\ \implies \xi = k(5, -1, -3), k &\in \mathbb{R}. \end{aligned}$$

This means that $\xi = (a, b, c) = k(5, -1, -3), k \in \mathbb{R}$ belongs to W. In other words, $W = L(\alpha), \alpha(5, -1, -3)$. The set $S = \{\alpha\} \subset W$ is a linearly independent set, i.e., S is also a basis of W. Hence, $\dim(W) = 1$.

Example 3. Let $W = \{(x, y, z) \in \mathbb{R}^3 : x + 3y + 4z = 0\}$. Find the basis and dimension of V.