Advanced Engineering Mathematics
Lecture 32

Linear Transformation

Let V and W be two vector spaces over a field $\mathbb{F}$. A mapping $T : V \to W$ is said to be a linear mapping/transformation, if it satisfies the following conditions:

(i) $T(\alpha + \beta) = T(\alpha) + T(\beta)$, $\forall \alpha, \beta \in V$.

(ii) $T(c\alpha) = cT(\alpha)$, $\forall \alpha \in V$.

In short, $T(c\alpha + \beta) = cT(\alpha) + T(\beta)$, $\forall c \in \mathbb{F},\ \alpha, \beta \in V$.

Example 1. The identity mapping $T : V \to V$ defined by $T(x) = x$, $\forall x \in V$, which clearly satisfies, $T(x + y) = x + y = T(x) + T(y)$ and $T(cx) = cx = cT(x)$.

Example 2. The zero mapping $T : V \to W$ defined by $T(x) = \theta'$, $\forall x \in V,\ \theta' \in W$. Which follows as the previous example.

Example 3. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ such that $T(x_1, x_2, x_3) = (x_1, x_2, 0),$ where $(x_1, x_2, x_3) \in \mathbb{R}^3$.
Solution. Let $\alpha = (x_1, x_2, x_3), \beta = (y_1, y_2, y_3) \in \mathbb{R}^3$, then we can easily check that

$$\begin{aligned} T(\alpha + \beta) &= T(x_1 + y_1, x_2 + y_2, x_3 + y_3) \\ &= (x_1 + y_1, x_2 + y_2, 0) \\ &= (x_1, x_2, 0) + (y_1, y_2, 0) \\ &= T(\alpha) + T(\beta), \\ T(c\alpha) &= T(cx_1, cx_2, cx_3) \\ &= (cx_1, cx_2, 0) \\ &= c(x_1, x_2, 0) \\ &= cT(\alpha). \end{aligned}$$

Hence, T is a linear transformation.

Example 4. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ such that $T(x_1, x_2, x_3) = (x_1+1, x_2+1, x_3+1),$ where $(x_1, x_2, x_3) \in \mathbb{R}^3$. Verify whether T is linear. (Answer: No)

Kernel of a Linear Transformation

Let V and W be two vector spaces and $T : V \to W$ be a linear mapping. The set of all $\alpha \in V$ such that $T(\alpha) = \theta'$ in W, where θ' is the null element in W, is said to be the Kernel of T and it is defined by $\ker(T)$. In other words,

$$\ker(T) = \left\{\alpha \in V : T(\alpha) = \theta'\right\}. \tag{1}$$

The dimension of $\ker(T)$ is called the Nullity of T. Also, $\ker(T) \subset V$ is a subspace.

Image of a Linear Transformation

Let $T : V \to W$ be a linear map. The image of the elements of V under the mapping T forms a subset of W. This subset is called the image of T and it is denoted by $\mathrm{Im}(T)$. In other words,

$$\mathrm{Im}(T) = \{T(\alpha) : \alpha \in V\}.$$

Note that $\mathrm{Im}(T) \subset W$ is a subspace. The dimension of $\mathrm{Im}(T)$ is called rank of T, i.e., $\mathrm{rank}(T)$.

Example 1. Verify whether the below transformation is linear or not.

$$T(x_1, x_2, x_3) = (x_1 + x_2 + x_3, 2x_1 + x_2 + 2x_3, x_1 + 2x_2 + x_3), \quad \text{where } (x_1, x_2, x_3) \in \mathbb{R}^3$$

If yes, find $\ker(T), \mathrm{Im}(T)$.

Solution. Let $\alpha = (x_1, x_2, x_3)$ and $\beta = (y_1, y_2, y_3)$. Then,

$$\begin{aligned} T(\alpha + \beta) &= T(x_1 + y_1, x_2 + y_2, x_3 + y_3) \\ &= (x_1 + y_1 + x_2 + y_2 + x_3 + y_3, 2x_1 + 2y_1 + x_2 + y_2 + 2x_3 + 2y_3, x_1 + y_1 + 2x_2 + 2y_2 + x_3 + y_3) \\ &= (x_1 + x_2 + x_3 + y_1 + y_2 + y_3, 2x_1 + x_2 + 2x_3 + 2y_1 + y_2 + 2y_3, x_1 + 2x_2 + x_3 + y_1 + 2y_2 + y_3) \\ &= (x_1 + x_2 + x_3, 2x_1 + x_2 + 2x_3, x_1 + 2x_2 + x_3) + (y_1 + y_2 + y_3, 2y_1 + y_2 + 2y_3, y_1 + 2y_2 + y_3) \\ &= T(x_1, x_2, x_3) + T(y_1, y_2, y_3) = T(\alpha) + T(\beta) \end{aligned}$$

For $c \in \mathbb{F}$ and $\alpha = (x_1, x_2, x_3) \in V$

$$T(c\alpha) = T(cx_1, cx_2, cx_3) = (cx_1 + cx_2 + cx_3, 2cx_1 + cx_2 + 2cx_3, cx_1 + 2cx_2 + cx_3) = cT(\alpha)$$

$\implies T$ is a linear map.
Now let $\alpha = (x_1, x_2, x_3) \in \mathbb{R}^3$. Then, by the definition of Kernel, we write

$$\begin{aligned} T(\alpha) = \theta \implies T(x_1, x_2, x_3) &= (0, 0, 0) \\ \implies (x_1 + x_2 + x_3, 2x_1 + x_2 + 2x_3, x_1 + 2x_2 + x_3) &= (0, 0, 0) \end{aligned}$$

which leads to the system of linear equations

$$\begin{aligned} x_1 + x_2 + x_3 &= 0 \\ 2x_1 + x_2 + 2x_3 &= 0 \\ x_1 + 2x_2 + x_3 &= 0. \end{aligned}$$

which further reduces to: $x_1 + x_3 = 0, x_2 = 0$.
The solution to the above equations is $k(1, 0, -1),\ k \in \mathbb{R}$. Hence, $\ker(T) = \{(1, 0, -1)\}$ and $\dim(\ker(T)) = 1$, also known as Nullity of T.