Advanced Engineering Mathematics
Lecture 35

Example 4. Consider $A = \begin{bmatrix} 2 & 1 \\ 3 & 5 \end{bmatrix}$. Find the characteristic equation.

Solution. Let $\lambda \in \mathbb{F}$. Then, the characteristic equation is $\det(A - \lambda I_2) = 0 \implies \begin{vmatrix} 2-\lambda & 1 \\ 3 & 5-\lambda \end{vmatrix} = 0 \implies \lambda^2 - 7\lambda + 7 = 0$.

Cayley−Hamilton Theorem. *Every square matrix A of order n satisfies its own characteristic equation.*

Example 5. Verify the Cayley-Hamilton theorem for the matrix $A = \begin{bmatrix} 2 & 1 \\ 3 & 5 \end{bmatrix}$.

Solution. The characteristic equation for A is given by $\lambda^2 - 7\lambda + 7 = 0$ as we found out previously.
Goal: $A^2 - 7A + 7I = \vec{0}$

$$\begin{aligned}
\text{L.H.S.} &\implies A \cdot A - 7A + 7I \\
&\implies \begin{bmatrix} 2 & 1 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 3 & 5 \end{bmatrix} - 7\begin{bmatrix} 2 & 1 \\ 3 & 5 \end{bmatrix} + 7\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\
&\implies \begin{bmatrix} 7 & 7 \\ 14 & 28 \end{bmatrix} - \begin{bmatrix} 14 & 7 \\ 21 & 35 \end{bmatrix} + \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix} \\
&\implies \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \implies \mathbf{0}_{2\times 2} \implies \text{R.H.S.}
\end{aligned}$$

Example 6. Using the Cayley-Hamilton theorem, find the inverse of $\begin{bmatrix} 2 & 1 \\ 3 & 5 \end{bmatrix}$.

Solution. We know $\begin{vmatrix} 2 & 1 \\ 3 & 5 \end{vmatrix} = 7 > 0$. Again,

$$\begin{aligned}
& A^2 - 7A + 7I = 0 \\
\implies & A(A - 7I) = -7I \\
\implies & A^{-1} = -\frac{1}{7}(A - 7I) \\
\implies & A^{-1} = \frac{1}{7}\left(\begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix} - \begin{bmatrix} 2 & 1 \\ 3 & 5 \end{bmatrix}\right) \\
\implies & A^{-1} = \frac{1}{7}\begin{bmatrix} 5 & -1 \\ -3 & 2 \end{bmatrix}.
\end{aligned}$$

Proposition 1. *If A is a singular matrix, then 0 is an eigenvalue of A.*

Proposition 2. *The eigenvalues in a diagonal matrix are its diagonal entries.*

Proposition 3. *If λ is an eigenvalue of a non-singular matrix, then $\frac{1}{\lambda}$ is an eigenvalue of A^{-1}.*

Eigenvector. Let A be an $n \times n$ matrix over a field $\mathbb{F}$. A non-null vector X belonging to $V_n(k)$ is said to be an eigenvector of A, if there exists a scalar $\lambda \in \mathbb{F}$ such that $AX = \lambda X$ holds.

Example 7. Let $A = \begin{bmatrix} 1 & 3 \\ 4 & 5 \end{bmatrix}$. Find the eigenvalues and eigenvectors of A.

Solution. Let $\lambda \in \mathbb{F}$. Then, $\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 3 \\ 4 & 5-\lambda \end{vmatrix} = 0 \implies \lambda = -1, 7.$

Let $X \in \mathbb{R}^2$ be the eigenvector corresponding to the eigenvalue -1, then

$$\begin{aligned} AX &= \lambda X \\ \implies \begin{bmatrix} 1 & 3 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= -1 \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ \implies \begin{bmatrix} x_1 + 3x_2 \\ 4x_1 + 5x_2 \end{bmatrix} &= - \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ \implies \begin{bmatrix} 2x_1 + 3x_2 \\ 4x_1 + 6x_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned}$$

We get $x_1 = -\frac{3}{2}c$, where $x_2 = c \in \mathbb{R}$. Then, the required eigenvector is $X = \begin{bmatrix} -\frac{3}{2} \\ 1 \end{bmatrix} c \ ; \ c \in \mathbb{F}$.

For $\lambda = 7$, we have

$$\begin{aligned} AX &= \lambda X \\ \implies \begin{bmatrix} 1 & 3 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= 7 \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ \implies \begin{bmatrix} x_1 + 3x_2 \\ 4x_1 + 5x_2 \end{bmatrix} &= \begin{bmatrix} 7x_1 \\ 7x_2 \end{bmatrix} \\ \implies \begin{bmatrix} -6x_1 + 3x_2 \\ 4x_1 - 2x_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned}$$

We get $x_1 = \frac{1}{2}c$, where $x_2 = c \in \mathbb{R}$. Then, the required eigenvector is $X = \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} c \ ; \ c \in \mathbb{F}$.