Advanced Engineering Mathematics
Lecture 44

Directional derivative: Let $f(x, y, z)$ be a scalar function defined on a region $\mathcal{R}$. Let P be any arbitrary point in $\mathcal{R}$ and suppose Q is a point in the region $\mathcal{R}$ which is a neighboring point of P in the direction of a given unit vector $\hat{a}$. Then the limit $\lim_{Q\to P} \dfrac{f(Q) - f(P)}{PQ}$, if it exists, is called directional derivative of f at the point P in the direction of $\hat{a}$.

Remark 0.1. Let $P = (x, y, z)$ and $Q = (x + \delta x, y + \delta y, z + \delta z)$ be two points in the region $\mathcal{R}$. Let $\hat{a}$ be the unit vector and $\delta f = f(Q) - f(P)$. Then $\frac{\delta f}{\delta s}$ represents the average rate of change of f per unit distance in the direction of $\hat{a}$. Now, the directional derivative of at P in the direction of $\hat{a}$ is

$$\lim_{Q\to P} \frac{f(Q) - f(P)}{PQ} = \lim_{\delta s\to 0} \frac{\delta f}{\delta s} = \frac{df}{ds}.$$

Theorem 0.2. *The directional derivative of a scalar function f at a point $P(x, y, z)$ in the direction of a unit vector $\hat{a}$ is given by*

$$\frac{df}{ds} = \vec{\nabla} f \cdot \hat{a}.$$

Example 1. Let $f(x, y, z) = x^3yz + 4xz^2$ be a scalar function. The directional derivative of f in the direction of the vector $2\hat{i} - \hat{j} - 2\hat{k}$ at a point $(1, -2, -1)$ is given by

$$\left.\frac{df}{ds}\right|_{(1,-2,-1)} = \vec{\nabla} f(1, -2, -1) \cdot \frac{2\hat{i} - \hat{j} - 2\hat{k}}{\sqrt{2^2 + 1^2 + 2^2}}.$$

$\vec{\nabla} f(x, y, z) = (3x^2yz+4z^2)\,\hat{i}+x^3z\,\hat{j}+(x^3y+8xz)\,\hat{k}$. Therefore, $\vec{\nabla} f(1, -2, -1) = 10\hat{i}-\hat{j}-10\hat{k}$, and

$$\frac{df}{ds} = \frac{20 + 1 + 20}{3} = \frac{41}{3}.$$

Let $f(x, y, z) = c$ be the equation of the level surface. Let $\vec{r} = x\,\hat{i} + y\,\hat{j} + z\,\hat{k}$ be a position vector of any point P on this surface. Then $\vec{\nabla} f = \frac{\partial f}{\partial x}\,\hat{i} + \frac{\partial f}{\partial y}\,\hat{j} + \frac{\partial f}{\partial z}\,\hat{k}$ is a vector along the normal to the surface at the point P, i.e., $\vec{\nabla} f$ is perpendicular to the tangent plane at the point P.

Let $Q = (X, Y, Z)$ then $\vec{PQ} = \vec{R} - \vec{r} = (X - x)\,\hat{i} + (Y - y)\,\hat{j} + (Z - z)\,\hat{k}$ lies on the tangent plane at P to the surface. But since, $\vec{\nabla} f$ is perpendicular to the tangent plane, i.e., $\vec{\nabla} f \perp \vec{PQ}$ implies

$$\vec{\nabla} f \cdot \vec{PQ} = 0$$
$$(X - x)\frac{\partial f}{\partial x} + (Y - y)\frac{\partial f}{\partial y} + (Z - z)\frac{\partial f}{\partial z} = 0.$$

This is the required equation of a tangent plane at P.

Example 2. Let $x^2y + 2xz = 4$ be the level surface. Unit normal to the level surface at $(2, -2, 3)$ is $\widehat{\nabla} f(2, -2, 3)$.

$$\vec{\nabla} f(x, y, z) = (2xy + 2z)\,\hat{i} + x^2\,\hat{j} + 2x\,\hat{k}$$
$$\vec{\nabla} f(2, -2, 3) = -2\hat{i} + 4\hat{j} + 4\hat{k}$$

Therefore unit normal to the given level surface at $(2, -2, 3)$ is $\frac{1}{3}(-\hat{i} + 2\hat{j} + 2\hat{k})$.