Advanced Engineering Mathematics
Lecture 46

Divergence: Let $\vec{V}$ be any given differentiable vector function. Then the divergence of $\vec{V}$ is given by

$$\begin{aligned}\vec{\nabla}\cdot\vec{V} = \operatorname{div}(\vec{V}) =& \operatorname{div}\vec{V}\\ =&\Big(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\Big)\cdot\big(V_1,V_2,V_3\big)\\ =&\frac{\partial V_1}{\partial x}+\frac{\partial V_2}{\partial y}+\frac{\partial V_3}{\partial z}.\end{aligned}$$

Curl of a vector function: Let $\vec{V}$ be any given differentiable vector function. Then the curl of $\vec{V}$ is given by

$$\begin{aligned}\vec{\nabla}\times\vec{V} = \operatorname{curl}(\vec{V}) = \operatorname{curl}\vec{V} =& \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ V_1 & V_2 & V_3\end{vmatrix}\\ =&\Big(\frac{\partial V_3}{\partial y}-\frac{\partial V_2}{\partial z}\Big)\hat{i}+\Big(\frac{\partial V_1}{\partial z}-\frac{\partial V_3}{\partial x}\Big)\hat{j}+\Big(\frac{\partial V_2}{\partial x}-\frac{\partial V_1}{\partial y}\Big)\hat{k}.\end{aligned}$$

Irrotational vector: A vector function $\vec{V}$ is said to be irrotational if $\operatorname{curl}\vec{V}=0$. In other words, there exists a scalar function φ such that $\vec{V}=-\vec{\nabla}\varphi$.

Solenoidal vector: A vector function $\vec{V}$ is said to be solenoidal if $\operatorname{div}\vec{V}=0$.

Laplacian of a scalar: The Laplacian operator ∇^2 is defined as

$$\Delta := \nabla^2 = \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}.$$

Laplace equation: $\operatorname{div}\big(\vec{\nabla}u\big)=\nabla^2 u=0$.

Example 1. If $\vec{r}=x\,\hat{i}+y\,\hat{j}+z\,\hat{k}$, then

$$\begin{aligned}\operatorname{div}\vec{r} =& \frac{\partial x}{\partial x}+\frac{\partial y}{\partial y}+\frac{\partial z}{\partial z}=3.\\ \operatorname{curl}\vec{r} =& \Big(\frac{\partial z}{\partial y}-\frac{\partial y}{\partial z}\Big)\hat{i}+\Big(\frac{\partial x}{\partial z}-\frac{\partial z}{\partial x}\Big)\hat{j}+\Big(\frac{\partial y}{\partial x}-\frac{\partial x}{\partial y}\Big)\hat{k}=\vec{0}.\end{aligned}$$

Example 2. Let $\vec{V}=(x+3y)\hat{i}+(y-2z)\hat{j}+(x+az)\hat{k}$. Then $\operatorname{div}\vec{V}=1+1+a$. Therefore, $\vec{V}$ is solenoidal if $a+2=0$, i.e., $a=-2$.

Example 3. Let $\vec{V}=(\sin y+z)\hat{i}+(x\cos y-z)\hat{j}+(x-y)\hat{k}$. Then

$$\begin{aligned}\operatorname{curl}\vec{V} =& \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ \sin y+z & x\cos y-z & x-y\end{vmatrix}\\ =&(-1+1)\hat{i}+(1-1)\hat{j}+(\cos y-\cos y)\hat{k}\\ =&\vec{0}.\end{aligned}$$

Therefore, $\vec{V}$ is a irrotational vector.

Vector identities in terms of div & curl : Let $\vec{V}$ be a vector function and φ be a scalar function.

(i) $\operatorname{div} \varphi\vec{V} = \vec{\nabla}\varphi \cdot \vec{V} + \varphi \operatorname{div} \vec{V}$.

(ii) $\operatorname{curl} \varphi\vec{V} = \vec{\nabla}\varphi \times \vec{V} + \varphi \operatorname{curl} \vec{V}$.

(iii) $\vec{\nabla}(\vec{f} \cdot \vec{g}) = (\vec{g} \cdot \vec{\nabla})\vec{f} + (\vec{f} \cdot \vec{\nabla})\vec{g} + \vec{g} \times \operatorname{curl} \vec{f} + \vec{f} \times \operatorname{curl} \vec{g}$.

Property: For any vector function $\vec{V} = V_1\,\hat{\imath} + V_2\,\hat{\jmath} + V_3\,\hat{k}$, $\operatorname{div}\big(\operatorname{curl} \vec{V}\big) = 0$.

Proof.

$$\begin{aligned}\operatorname{div}\big(\operatorname{curl} \vec{V}\big) =& \vec{\nabla} \cdot \left(\Big(\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z}\Big)\hat{\imath} + \Big(\frac{\partial V_1}{\partial z} - \frac{\partial V_3}{\partial x}\Big)\hat{\jmath} + \Big(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y}\Big)\hat{k}\right) \\ =& \frac{\partial^2 V_3}{\partial x \partial y} - \frac{\partial^2 V_2}{\partial x \partial z} + \frac{\partial^2 V_1}{\partial y \partial z} - \frac{\partial^2 V_3}{\partial y \partial x} + \frac{\partial^2 V_2}{\partial z \partial x} - \frac{\partial^2 V_1}{\partial z \partial y} \\ =& 0.\end{aligned}$$

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