Advanced Engineering Mathematics
Lecture 49

The three vectors $\hat{t}$, $\hat{n}$, and $\hat{b}$ are perpendicular to each other. The plane corresponds to each pair forms three mutually perpendicular planes.

(i) **Osculating Plane:** The osculating plane to a curve at a point P is the plane containing the tangent and principle normal at P. The equation of osculating plane is

$$(\vec{R} - \vec{r}) \cdot \hat{b} = 0,$$

where $\vec{R}$ is the position of any point on the plane, and $\vec{r}$ is the position vector of the point P.

(ii) **Normal plane:** It is a plane to a curve at a point P containing $\hat{n}$ and $\hat{b}$, and perpendicular to $\hat{t}$. The equation of the normal plane is

$$(\vec{R} - \vec{r}) \cdot \hat{t} = 0.$$

(iii) **Rectifying plane:** It is a plane to a curve at a point P containing $\hat{b}$ and $\hat{t}$, and perpendicular to $\hat{n}$. The equation of the rectifying plane is

$$(\vec{R} - \vec{r}) \cdot \hat{n} = 0.$$

Example 1. Find $\hat{t}$, $\hat{b}$, $\hat{n}$, κ and τ for the curve $x = 2t$, $y = t^2$ and $z = \frac{1}{3}t^3$ at $t = 1$. Also find the equation of osculating plane, normal plane, and rectifying plane.

Solution: The given equation of the curve can be put into the vector form as

$$\begin{aligned}\vec{r} =& x\,\hat{i} + y\,\hat{j} + z\,\hat{k}\\ =& 2t\,\hat{i} + t^2\,\hat{j} + \frac{1}{3}t^3\,\hat{k}\end{aligned}$$

$$\frac{d\vec{r}}{dt} = 2\,\hat{i} + 2t\,\hat{j} + t^2\,\hat{k}, \qquad \frac{d^2\vec{r}}{dt^2} = 2\,\hat{j} + 2t\,\hat{k}, \qquad \frac{d^3\vec{r}}{dt^3} = 2\,\hat{k}.$$

At $t = 1$:

$$\vec{r} = 2\hat{i} + \hat{j} + \frac{1}{3}\hat{k}, \qquad \frac{d\vec{r}}{dt} = 2\hat{i} + 2\hat{j} + \hat{k}, \qquad \frac{d^2\vec{r}}{dt^2} = 2\hat{j} + 2\hat{k}, \qquad \frac{d^3\vec{r}}{dt^3} = 2\hat{k}.$$

Also, $\dot{\vec{r}} \times \ddot{\vec{r}} = (2\hat{i} + 2\hat{j} + \hat{k}) \times (2\hat{j} + 2\hat{k}) = 2\hat{i} - 4\hat{j} + 4\hat{k}$, and $(\dot{\vec{r}} \times \ddot{\vec{r}}) \cdot \dddot{\vec{r}} = 8$.

(i) $\hat{t} = \frac{d\vec{r}}{dt} / \left|\frac{d\vec{r}}{dt}\right| = \frac{2\hat{i}+2\hat{j}+\hat{k}}{\sqrt{2^2+2^2+1}} = \frac{1}{3}(2\hat{i} + 2\hat{j} + \hat{k})$.

(ii) As $\ddot{\vec{r}} = \kappa\hat{n}\dot{s}^2 + \hat{t}\ddot{s}$, and $\dddot{\vec{r}} = (\dddot{s} - \kappa^2\dot{s})\hat{t} + (3\dot{s}\ddot{s}\kappa + \dot{s}^2\kappa)\hat{n} + \dot{s}^3\kappa\tau\hat{b}$, we obtain

$$\begin{aligned}\dot{\vec{r}} \times \ddot{\vec{r}} =& \hat{t}\dot{s} \times (\kappa\hat{n}\dot{s}^2 + \hat{t}\ddot{s}) = \kappa\hat{b}\dot{s}^3\\ \kappa\hat{b} =& \frac{\dot{\vec{r}} \times \ddot{\vec{r}}}{\dot{s}^3}\end{aligned}$$

The binormal vector can be chosen as $\hat{b} = \frac{1}{3}(\hat{i} - 2\hat{j} + 2\hat{k})$ such that $\hat{b} \cdot \hat{t} = 0$.

(iii) $\hat{n} = \hat{b} \times \hat{t} = \frac{1}{3}(\hat{i} - 2\hat{j} + 2\hat{k}) \times \frac{1}{3}(2\hat{i} + 2\hat{j} + \hat{k}) = \frac{1}{3}(-2\hat{i} + \hat{j} + 2\hat{k})$.

(iv)

$$
\begin{aligned}
\kappa\hat{b} =& \frac{\dot{\vec{r}} \times \ddot{\vec{r}}}{\dot{s}^3} \\
\kappa =& \frac{|\dot{\vec{r}} \times \ddot{\vec{r}}|}{|\dot{s}^3|} = \frac{|\dot{\vec{r}} \times \ddot{\vec{r}}|}{|\dot{\vec{r}}|^3} \\
=& \frac{|2\hat{i} - 4\hat{j} + 4\hat{k}|}{|2\hat{i} + 2\hat{j} + \hat{k}|^3} = \frac{2}{9}.
\end{aligned}
$$

(v) $\tau = \frac{(\dot{\vec{r}}\times\ddot{\vec{r}})\cdot\dddot{\vec{r}}}{|\dot{\vec{r}}\times\ddot{\vec{r}}|^2} = \frac{8}{|2\hat{i}-4\hat{j}+4\hat{k}|^2} = \frac{2}{9}.$

Let $\vec{R} = X\,\hat{i} + Y\,\hat{j} + Z\,\hat{k}$ be the position of any point on the plane.

(i) The equation of osculating plane:

$$
\begin{aligned}
(\vec{R} - \vec{r}) \cdot \hat{b} &= 0 \\
\big((X\,\hat{i} + Y\,\hat{j} + Z\,\hat{k}) - 2\hat{i} + \hat{j} + \frac{1}{3}\hat{k}\big) \cdot \frac{1}{3}(\hat{i} - 2\hat{j} + 2\hat{k}) &= 0 \\
(X-2) - 2(Y-1) + 2\big(Z - \frac{1}{3}\big) &= 0 \\
X - 2Y + 2Z &= \frac{2}{3}
\end{aligned}
$$

(ii) The equation of normal plane:

$$
\begin{aligned}
(\vec{R} - \vec{r}) \cdot \hat{t} &= 0 \\
\big((X\,\hat{i} + Y\,\hat{j} + Z\,\hat{k}) - 2\hat{i} + \hat{j} + \frac{1}{3}\hat{k}\big) \cdot \frac{1}{3}(2\hat{i} + 2\hat{j} + \hat{k}) &= 0 \\
2(X-2) + 2(Y-1) + \big(Z - \frac{1}{3}\big) &= 0 \\
2X + 2Y + Z &= \frac{19}{3}
\end{aligned}
$$

(iii) The equation of rectifying plane:

$$
\begin{aligned}
(\vec{R} - \vec{r}) \cdot \hat{n} &= 0 \\
\big((X\,\hat{i} + Y\,\hat{j} + Z\,\hat{k}) - 2\hat{i} + \hat{j} + \frac{1}{3}\hat{k}\big) \cdot \frac{1}{3}(-2\hat{i} + \hat{j} + 2\hat{k}) &= 0 \\
-2(X-2) + (Y-1) + 2\big(Z - \frac{1}{3}\big) &= 0 \\
2X - Y - 2Z &= \frac{7}{3}
\end{aligned}
$$