Advanced Engineering Mathematics
Lecture 53

Volume integrals: Suppose V is a volume bounded b surface S. Suppose $f(x, y, z)$ is a scalar function defined on V. Then the volume integral of $f(x, y, z)$ is denoted by

$$I_V = \iiint_V f(x, y, z)\, dv,$$

where $dv = dx\, dy\, dz$. If f is a vector function, say $\vec{f}$, then $\iiint_V \vec{f} \cdot d\vec{v}$ is an example of volume integral.

Example. Evaluate $\iiint_V f(x, y, z)\, dv$, where $f(x, y, z) = 45x^2y$ and V is the volume of the closed region bounded by $4x + 2y + z = 8$, $x = 0$, $y = 0$, and $z = 0$.

Solution: The region V is the following:

$$\begin{aligned}
&z\text{-varies in the range:} \quad 0 \le z \le 8 - 4x - 2y,\\
&y\text{-varies in the range:} \quad 0 \le y \le 4 - 2x,\\
&x\text{-varies in the range:} \quad 0 \le x \le 2.
\end{aligned}$$

$$\begin{aligned}
I_V = \iiint_V f(x, y, z)\, dv = &\int_{x=0}^{2} \int_{y=0}^{4-2x} \int_{z=0}^{8-4x-2y} 45x^2y\, dx\, dy\, dz\\
=&45 \int_{x=0}^{2} \int_{y=0}^{4-2x} x^2y(8 - 4x - 2y)\, dx\, dy\\
=&45 \int_{x=0}^{2} \frac{x^2}{3}(4 - 2x)^3\, dx = 128.
\end{aligned}$$

Example. Evaluate the volume integral $\iiint_V f(x, y, z)\, dv$, where $f(x, y, z) = xyz$ and V is the volume of the region bounded by the sphere $x^2 + y^2 + z^2 = 1$ in the first octant.

Solution: The region V is the following:

$$\begin{aligned}
&z\text{-varies in the range:} \quad 0 \le z \le \sqrt{1 - x^{-}y^2},\\
&y\text{-varies in the range:} \quad 0 \le y \le \sqrt{1 - x^2},\\
&x\text{-varies in the range:} \quad 0 \le x \le 1.
\end{aligned}$$

$$\begin{aligned}
I_V = \iiint_V f(x, y, z)\, dv = &\int_{x=0}^{1} \int_{y=0}^{\sqrt{1-x^2}} \int_{z=0}^{\sqrt{1-x^{-}y^2}} xyz\, dx\, dy\, dz\\
=&\frac{1}{2} \int_{x=0}^{1} \int_{y=0}^{\sqrt{1-x^2}} xy(1 - x^2 - y^2)\, dx\, dy\\
=&\frac{1}{8} \int_{x=0}^{1} x(1 - x^2)^2\, dx = \frac{1}{48}.
\end{aligned}$$

Gauss divergence theorem: Suppose V is the volume bounded by a closed piecewise smooth surface S. Let $\vec{f}(x, y, z)$ be a vector function of position which is continuous and has continuous first order partial derivative on V. Then,

$$\iiint_V \operatorname{div} \vec{f}\, ds = \iint_S \vec{f} \cdot \hat{n}\, ds,$$

where $\hat{n}$ is the unit outward drawn normal to S.

Example. If $\vec{f}(x, y, z) = ax\,\hat{i} + by\,\hat{j} + cz\,\hat{k}$, a, b, c are constant, then find the value of $\iint_S \vec{f} \cdot \hat{n}\, ds$, where S is the surface of unit sphere.

Solution: By divergence theorem,

$$\begin{aligned}\iint_S \vec{f} \cdot \hat{n}\, ds &= \iiint_V \operatorname{div} \vec{f}\, dv \\ &= \iiint_V (a + b + c)\, dv = \frac{4\pi}{3}(a + b + c).\end{aligned}$$